

## BUILDING SPECIFICATIONS IN THE EVENT-B INSTITUTION

MARIE FARRELL , ROSEMARY MONAHAN , AND JAMES F. POWER 

Department of Computer Science and Hamilton Institute, Maynooth University, Ireland  
*e-mail address:* marie.farrell@mu.ie

**ABSTRACT.** This paper describes a formal semantics for the Event-B specification language using the theory of institutions. We define an institution for Event-B,  $\mathcal{EVT}$ , and prove that it meets the validity requirements for satisfaction preservation and model amalgamation. We also present a series of functions that show how the constructs of the Event-B specification language can be mapped into our institution. Our semantics sheds new light on the structure of the Event-B language, allowing us to clearly delineate three constituent sub-languages: the superstructure, infrastructure and mathematical languages. One of the principal goals of our semantics is to provide access to the generic modularisation constructs available in institutions, including specification-building operators for parameterisation and refinement. We demonstrate how these features subsume and enhance the corresponding features already present in Event-B through a detailed study of their use in a worked example. We have implemented our approach via a parser and translator for Event-B specifications, EB2EVT, which also provides a gateway to the HETS toolkit for heterogeneous specification.

### 1. INTRODUCTION

When formal methods are used in software development, formal specification languages are used to model software systems and to provide a basis for their verification. Event-B is an industrial-strength formal specification language that originally evolved from the B-Method [Sch01], and has been used in a multitude of domains including air-traffic control systems and medical devices [Abr10]. Even though it is a mature formal specification language, Event-B has a number of drawbacks, particularly its lack of standardised modularisation constructs. In addition, while Event-B has been provided with a semantics in terms of proof obligations, the abstractness of this approach can make it difficult to deal formally with modularisation, or to provide a concrete basis for interoperability with other formalisms.

---

*Key words and phrases:* Event-B, Formal Semantics, Institutions, Refinement, Modularisation.

This work was initially funded by a Government of Ireland Postgraduate Grant from the Irish Research Council, and subsequently supported by EPSRC Hubs for Robotics and AI in Hazardous Environments: EP/R026092 (FAIR-SPACE).

Farrell and Monahan dedicate this paper to the memory of Dr. James F. Power who passed away before he could see this work accepted for publication. We thank him for his contributions and encouragement throughout this project.

In this paper, we describe the formal constructions that allow the Event-B language to be embedded into the theory of institutions. Institutions are mathematical structures that make use of category theory and provide a generic framework for formalising a language or logic [GB92, ST12]. Having built an institution to support the core concepts in Event-B, we can then use this institution as a target for the semantics of the full Event-B specification language. This in turn gives access to a range of generic modularisation constructs and provides a foundation for interoperability between Event-B and other formalisms.

**1.1. The structure of this paper.** Sections 2 and 3 review the background material and basic definitions used in this paper. The semantics of programming languages is a well-understood concept that has been studied for many years and we use it as a starting point for our research. It has become evident from our investigations that defining a semantics of a specification language, such as Event-B in our case, is a slightly different endeavour. We provide some background on the semantics of specification languages in Section 2, and we give the basic definitions from the theory of institutions in this context. In Section 3 we give an overview of the Event-B specification language, describe its abstract syntax and introduce an example that we will use throughout the paper.

The core of our semantics is presented in Sections 4 and 5. In Section 4 we define our institution for Event-B,  $\mathcal{EVT}$ , that was originally developed in our previous work [FMP17b]. The  $\mathcal{EVT}$  institution provides the mathematical structures that will act as a target for the semantics of constructs from the Event-B language. The proofs of the satisfaction relation and the comorphism between  $\mathcal{FOPEQ}$  and  $\mathcal{EVT}$  are contained in the paper. All other necessary proofs are contained in the appendix.

In Section 5 we present a series of definitions of the functions that map the abstract syntax of the Event-B language into the corresponding constructs in the  $\mathcal{EVT}$  institution. A key step here is splitting the Event-B language into a three-layer model, by decomposing it into its superstructure, infrastructure and mathematical sub-languages. As well as aiding the presentation, this also helps to expose the important features of the Event-B language. In particular, once we have delineated these sub-languages, we can then employ the specification-building operators made available to us in the theory of institutions in our definition.

In the final sections of our paper, Sections 6 and 7, we explore some of the consequences of our semantics. We have validated our semantics by building a Haskell translator tool that we call EB2EVT, and in Section 6 we describe the correspondence between this tool and our semantics. To illustrate the use of this tool in practice we apply it to our running example from Section 3.2.

Formal refinement is core to the Event-B development process and, as such, any formalisation of Event-B should take this into account. In Section 7 we discuss what refinement means in the theory of institutions and show how our formalisation of  $\mathcal{EVT}$  supports Event-B refinement. Again, this is illustrated via our running example. The weaknesses that Event-B has in terms of modularisation are also addressed in Section 7 where we show how the generic specification-building operators from the theory of institutions can provide a set of well-defined modularisation constructs for Event-B.

One of the major benefits of these operators is that they are designed to be used in an institution-, and thus formalism-independent fashion. In this way, we provide scope for interoperability between Event-B and other formalisms that have been incorporated into the institution-theoretic framework. Our semantics and its associated EB2EVT tool

are instrumental to this, since they allow us to bridge the gap between the Rodin tool for Event-B and the HETS tool-set used for institutions.

**1.2. Contributions of this paper.** The principal contributions of this paper are summarised as follows:

- (1) A formal semantics for the Event-B formal specification language, defined as a series of functions from Event-B constructs to specifications over  $\mathcal{EVT}$ . This provides clarity on the meaning of the language elements and their interaction, and allows us to distinguish their differing levels of responsibility using the three-layer model presented in Figure 2.
- (2) A well-defined set of modularisation constructs using the specification-building operators available through the theory of institutions. These are built-in to our semantics, they subsume and extend the existing Event-B modularisation constructs, and they provide a standardised approach to exploring new modularisation possibilities.
- (3) An explication of Event-B refinement in the context of the  $\mathcal{EVT}$  institution. Refinement in  $\mathcal{EVT}$  incorporates and extends the refinement constructs of Event-B, and our semantics provides a more general context for exploring other options.

In addition, we have implemented an EB2EVT translator tool that transforms Event-B specifications that have been developed using the Rodin Platform into specifications over the  $\mathcal{EVT}$  institution. We have used this to validate our semantic definitions, and to provide interoperability with the existing large corpus of specifications available in this format [FMP17c].

## 2. THE SEMANTICS OF SPECIFICATION LANGUAGES

The study of the semantics of specification languages is closely related to the semantics of programming languages, since both are species of formal languages and, in many cases, have closely-related origins. For example, the specification language at the heart of the Vienna Development Method (VDM) emerged from a line of research on the semantics of programming languages starting with PL/1 and Algol [Jon01, AJ16]. Providing a semantics has many practical advantages, since the semantics can facilitate an increased understanding of the language, can provide a basis for standardisation documents (e.g., the ISO standard for Z [ISO02]), can act as a specification for tools that process the language, and can provide a basis for comparison and interoperability with similar languages.

There are many approaches to defining a semantics for a programming language, including denotational, operational and axiomatic approaches among others [Wat91, Gun92, Win93, GGM96]. A denotational semantics shows how to map the language constructs to mathematical structures [SS71, Sch86], and much of the focus of this approach is on the correct selection of these structures. In contrast, the axiomatic approach characterises the language in terms of the logical properties that must hold [Hoa69], and much of the focus here is on proof obligations and theorem-proving. While not exclusive, these perspectives can often be reflected in the specification language itself, with a contrast between *model-oriented* and *property-oriented* approaches [Bj06].

Some of the best-known model-oriented specification languages are Z and VDM, both of which use typed first-order logic and set theory overlaid with a set of language constructs to build specification units. The Z specification language has been given a denotational semantics by Spivey [Spi88]. This semantics defines a mapping from the basic syntactic

unit of specification, a *schema*, to a *variety*, which consists of a signature and set of models. These models map names used in the specification to appropriate values from the underlying logic, such as sets, relations and functions. In a similar vein, the semantics of VDM presented by Larsen and Pawlowski [LP95] shows how to map a *document* to a set of *environments* that model it; these environments map names to their denotations.

In order to specify the behaviour of software systems, many specification languages are based on models that include a concept of *state*. This has some similarity to the operational semantics used for programming languages [Hen90], though the overall approach is still essentially denotational. An early example was the COLD specification language, and its semantics mapped the basic unit of specification, a *class description*, to a set of *states* along with a *transition relation* for each procedure [Fei94]. The COLD specification-building operators are based on a class algebra, closely related to the structuring operators found in algebraic specification. There are now many formalisms and languages which have semantics defined in terms of states and transitions including, for example, sequences of events in Object-Z [Smi12, Smi95], the *trace*-based semantics of CSP [Hoa78, Ros98] and the *behaviours* of TLA+ [Lam02, Mer12].

Along a similar vein, Unified Theories of Programming (UTP) is a framework where specifications are interpreted in terms of relations between their before and after states [HJ98]. This work, which is rooted in lattice theory, somewhat parallels the approach using institutions, which is rooted in category theory. More recent work uses UTP to define an algebra of programs and an associated refinement relation to derive denotational and operational models of the Guarded Command Language (GCL) and CSP [HL18].

Translational approaches to formal semantics appear throughout the literature, in particular, the denotational UTP semantics for Circus (combination of CSP and Z) which translates the CSP component of the specification into Z [OCW09]. Another related translational approach is the semantics of a Dafny program which is given in terms of its translation to the Boogie intermediate verification language which generates the verification conditions to be proven with the Z3 SMT solver [Lei10].

The semantics of Clear, the first algebraic specification language, is a mix of denotational semantics and category-theoretic elements [BG80]. Subsequently, the Common Algebraic Specification Language, CASL, was devised as a basis for future work on algebraic specifications using the unifying framework supplied by the theory of institutions [Mos04]. The semantics of CASL follows a similar approach to Clear and is thus founded in the theory of institutions. In particular, CASL's two step semantics involves translating CASL constructs to their underlying mathematical concepts. In this way, the CASL static semantics determines signatures and its model semantics determines model-classes over the underlying institution. The Heterogeneous Toolset, HETS, provides tool support for CASL specifications and facilitates the use of heterogeneous specifications in an institution-theoretic fashion [MML07].

**2.1. Modularisation of specifications.** In his semantics for Z, Spivey notes that while ordinary mathematical reasoning is similar to software specification, the latter has the additional problem of scale. In this context, dealing with modularisation, including naming and structuring constructs, assumes increased importance. In modular software engineering, a system is decomposed into multiple independent *modules* that can be recomposed at a later stage to form the entire system [GJM02]. There are a multitude of benefits to this

approach including ease of code maintenance, code reuse and enabling multiple engineers to work on different components of the same system in parallel.

Although intended to provide support for provably correct stepwise development, formal refinement has been often used as a modularisation technique in formal specification languages. In these languages, support for other modularisation features is typically added after the language has been defined [EMCO92, KP98]. The Z specification language has a *schema calculus*, whose structuring operations are deliberately designed to look like logical operators, but this does not have to be the case. One of the purposes of re-engineering the Z specification language in the development of Object-Z was to enhance the support for modular developments [Wor92, Smi12]. Similarly, the VVSL language was developed as an approach to modularisation for VDM [Mid89].

When Burstall and Goguen presented the semantics of the Clear specification language [BG80], one of their key observations was that the specification-building operators could be given a meaning in a very abstract way, almost independently of the notions of the underlying formalism. They characterised the meaning of a Clear specification as an inter-connected collection of *theories*, where a theory is a closed set of properties derivable from the sentences of the specification. In this context, the semantics of the specification-structuring operators become theory transformations, which Burstall and Goguen describe as “the mathematical essence of Clear semantics”.

The Clear specification language shares many concepts with other algebraic specification languages [BHK89, EMCO92], most notably culminating in the Common Algebraic Specification Language (CASL) [Mos04] and the Heterogeneous Tool Set (HETS) [MML07]. These ideas were generalised in the theory of institutions, which describes these features in terms of category theory [GB92]. It has been observed that most of the formal languages used in Computer Science can be captured as institutions. One notable example is a category-theoretic formalisation of Z and its structuring mechanisms which captures the structure of modular Z specifications in a natural way [CAPT15]. The semantics of Event-B that we present in this paper follows a similar school of thought by first defining an institution for Event-B, called  $\mathcal{EVT}$ , in Section 4 and then translating Event-B specifications into structured specifications in the  $\mathcal{EVT}$  institution in Section 5.

**2.2. Institutions: basic definitions.** The theory of institutions, originally developed by Goguen and Burstall in a series of papers originating from their work on algebraic specification, provides a general framework for defining a logical system [GB83, BG77, GB92]. Institutions are generally used for representing logics in terms of their vocabulary (signatures), syntax (sentences) and semantics (models and satisfaction condition). The basic maxim of institutions, inspired by Tarski’s theory of truth [Tar44], is that

*“Truth is invariant under change of notation”.*

The use of institutions facilitates the development of specification language concepts, such as structuring of specifications, parameterisation, and refinement. This is achieved independently of the underlying logical system resulting in an environment for heterogeneous specification and combination of logics [ST88, BG77]. The theory of institutions is rooted in category theory and our work follows the notation and terminology used by Sannella and Tarlecki [ST12]. Thus, we define an institution as follows [GB92]:

**Definition 2.1 (Institution).** An **institution**  $\mathcal{INS}$  for some given logic consists of:

- A category  $\mathbf{Sign}_{\mathcal{I}NS}$  whose objects are called *signatures* and whose arrows are called *signature morphisms*.
- A functor  $\mathbf{Sen}_{\mathcal{I}NS} : \mathbf{Sign}_{\mathcal{I}NS} \rightarrow \mathbf{Set}$  yielding a set  $\mathbf{Sen}_{\mathcal{I}NS}(\Sigma)$  of  $\Sigma$ -sentences for each signature  $\Sigma \in |\mathbf{Sign}_{\mathcal{I}NS}|$  and a function  $\mathbf{Sen}_{\mathcal{I}NS}(\sigma) : \mathbf{Sen}_{\mathcal{I}NS}(\Sigma) \rightarrow \mathbf{Sen}_{\mathcal{I}NS}(\Sigma')$  for each signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$ .
- A functor  $\mathbf{Mod}_{\mathcal{I}NS} : \mathbf{Sign}_{\mathcal{I}NS}^{op} \rightarrow \mathbf{Cat}$  yielding a category  $\mathbf{Mod}_{\mathcal{I}NS}(\Sigma)$  of  $\Sigma$ -models for each signature  $\Sigma \in |\mathbf{Sign}_{\mathcal{I}NS}|$  and a functor  $\mathbf{Mod}_{\mathcal{I}NS}(\sigma) : \mathbf{Mod}_{\mathcal{I}NS}(\Sigma') \rightarrow \mathbf{Mod}_{\mathcal{I}NS}(\Sigma)$  for each signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$ .
- For every signature  $\Sigma$ , a *satisfaction relation*  $\models_{\mathcal{I}NS, \Sigma}$  determining the satisfaction of  $\Sigma$ -sentences by  $\Sigma$ -models.
- An institution must uphold the **satisfaction condition**: for any signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$ ,  $\phi \in \mathbf{Sen}_{\mathcal{I}NS}(\Sigma)$  and  $M' \in |\mathbf{Mod}_{\mathcal{I}NS}(\Sigma')|$ , the translations  $\mathbf{Mod}_{\mathcal{I}NS}(\sigma)$  of models and  $\mathbf{Sen}_{\mathcal{I}NS}(\sigma)$  of sentences preserve the following satisfaction condition

$$M' \models_{\mathcal{I}NS, \Sigma'} \mathbf{Sen}_{\mathcal{I}NS}(\sigma)(\phi) \iff \mathbf{Mod}_{\mathcal{I}NS}(\sigma)(M') \models_{\mathcal{I}NS, \Sigma} \phi$$

where  $\mathbf{Sen}_{\mathcal{I}NS}(\sigma)(\phi)$  (resp.  $\mathbf{Mod}_{\mathcal{I}NS}(\sigma)(M')$ ) indicates the translation of  $\phi$  (resp. reduction of  $M'$ ) along the signature morphism  $\sigma$ .

Note that we often denote  $\mathbf{Mod}_{\mathcal{I}NS}(\sigma) : \mathbf{Mod}_{\mathcal{I}NS}(\Sigma') \rightarrow \mathbf{Mod}_{\mathcal{I}NS}(\Sigma)$  by

$$-|_{\sigma} : \mathbf{Mod}_{\mathcal{I}NS}(\Sigma') \rightarrow \mathbf{Mod}_{\mathcal{I}NS}(\Sigma)$$

to indicate the model reduct along a signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$ . Model reducts are central to the definition of an institution as they allow us to consider a model over one signature as a model over another via a signature morphism.

Institutions have been defined for many different types of logic [ST12, §4.1]. One example is the institution of first-order predicate logic with equality,  $\mathcal{FOPEQ}$ :

**Definition 2.2** ( $\mathcal{FOPEQ}$ ). The institution for first-order predicate logic with equality,  $\mathcal{FOPEQ}$ , is composed of the following [ST12, §4.1] [FMP17b, Far17]:

**Signatures:** Tuples of the form,  $\Sigma_{\mathcal{FOPEQ}} = \langle S, \Omega, \Pi \rangle$ , where  $S$  is a set of sort names,  $\Omega$  is a set of operation names indexed by arity and sort, and  $\Pi$  is a set of predicate names indexed by arity. Signature morphisms are sort/arity-preserving functions that rename sorts, operations and predicates.

**Sentences:** For any  $\Sigma_{\mathcal{FOPEQ}} = \langle S, \Omega, \Pi \rangle$ ,  $\Sigma_{\mathcal{FOPEQ}}$ -sentences are closed first-order formulae built out of atomic formulae using  $\wedge, \vee, \neg, \Rightarrow, \iff, \exists, \forall$ . Atomic formulae are equalities between  $\langle S, \Omega \rangle$ -terms, predicate formulae of the form  $p(t_1, \dots, t_n)$  where  $p \in \Pi$  and  $t_1, \dots, t_n$  are terms (with variables), and the logical constants **true** and **false**.

**Models:** Given a signature  $\Sigma_{\mathcal{FOPEQ}} = \langle S, \Omega, \Pi \rangle$ , a model over  $\mathcal{FOPEQ}$  consists of a carrier set  $|A|_s$  for each sort name  $s \in S$ , a function  $f_A : |A|_{s_1} \times \dots \times |A|_{s_n} \rightarrow |A|_s$  for each operation name  $f \in \Omega_{s_1 \dots s_n, s}$  and a relation  $p_A \subseteq |A|_{s_1} \times \dots \times |A|_{s_n}$  for each predicate name  $p \in \Pi_{s_1 \dots s_n}$ , where  $s_1, \dots, s_n$ , and  $s$  are sort names.

**Satisfaction Relation:** The satisfaction relation in  $\mathcal{FOPEQ}$  is the usual satisfaction of first-order sentences by first-order structures. This amounts to evaluating the truth of the  $\mathcal{FOPEQ}$ -formula using the values in the carrier sets and the interpretation of operation and predicate names supplied by the models.

We will use  $\mathcal{FOPEQ}$  in Section 4 as part of the definition of our Event-B institution.

**2.3. Writing Specifications over an Institution.** Formally, a specification document written in a specification language is a set of sentences from that language, known as a *presentation*, which is the basic unit of specification over an institution.

**Definition 2.3** (Presentation). For any signature  $\Sigma$ , a  $\Sigma$ -presentation (sometimes called a *flat specification*) is a pair  $\langle \Sigma, \Phi \rangle$  where  $\Phi \subseteq \mathbf{Sen}(\Sigma)$ .  $M \in |\mathbf{Mod}(\Sigma)|$  is a model of a  $\Sigma$ -presentation  $\langle \Sigma, \Phi \rangle$  if  $M \models \Phi$  [ST12, §4.2].

To support modularisation, *structured specifications* are then built up from flat specifications using the specification-building operators that are available in the theory of institutions [ST12, §5]. These generic specification-building operators are summarised briefly in Table 1, and we will demonstrate their use in Event-B in Section 7. Note that the intended semantics of a specification  $SP$  is given by its signature, denoted  $Sig[SP]$ , and its class of models,  $Mod[SP] \subseteq Mod(Sig[SP])$ .

**2.4. Heterogeneous specification: connecting institutions.** Just as we can combine flat and structured specifications within an institution using the specification-building operators, we can combine specifications from different specification languages by defining a mapping between their institutions. This supports the software engineering notion of separation of concerns, allowing us to model different aspects of a specification using different formalisms, each of which is more suited to specifying particular aspects of the system. An institution *comorphism* allows the embedding of a primitive institution into a more complex one, and we will use it in Section 4.5 to define a relationship between  $\mathcal{FOPEQ}$  and  $\mathcal{EVT}$ .

**Definition 2.4** (Institution Comorphism). An institution comorphism  $\rho : \mathbf{INS} \rightarrow \mathbf{INS}'$  has three components:

- A functor  $\rho^{Sign} : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ .
- A natural transformation  $\rho^{Sen} : \mathbf{Sen} \rightarrow \rho^{Sign}; \mathbf{Sen}'$ , that is, for each  $\Sigma \in |\mathbf{Sign}|$ , a function  $\rho_{\Sigma}^{Sen} : \mathbf{Sen}(\Sigma) \rightarrow \mathbf{Sen}'(\rho^{Sign}(\Sigma))$ .
- A natural transformation  $\rho^{Mod} : (\rho^{Sign})^{op}; \mathbf{Mod}' \rightarrow \mathbf{Mod}$ , that is, for each  $\Sigma \in |\mathbf{Sign}|$ , a functor  $\rho_{\Sigma}^{Mod} : \mathbf{Mod}'(\rho^{Sign}(\Sigma)) \rightarrow \mathbf{Mod}(\Sigma)$ .

Further, an institution comorphism must ensure that for any signature  $\Sigma \in |\mathbf{Sign}|$ , the translations  $\rho_{\Sigma}^{Sen}$  of sentences and  $\rho_{\Sigma}^{Mod}$  of models preserve the satisfaction relation, that is, for any  $\psi \in \mathbf{Sen}(\Sigma)$  and  $M' \in |\mathbf{Mod}'(\rho^{Sign}(\Sigma))|$  we have

$$\rho_{\Sigma}^{Mod}(M') \models_{\Sigma} \psi \iff M' \models'_{\rho^{Sign}(\Sigma)} \rho_{\Sigma}^{Sen}(\psi)$$

and the following diagrams in  $\mathbf{Sen}$  and  $\mathbf{Mod}$  commute for each signature morphism in  $\mathbf{Sign}$  [ST12, §10.4]:

$$\begin{array}{ccc} \Sigma_1 & \mathbf{Sen}(\Sigma_1) \xrightarrow{\rho_{\Sigma_1}^{Sen}} \mathbf{Sen}'(\rho^{Sign}(\Sigma_1)) & \mathbf{Mod}'(\rho^{Sign}(\Sigma_2)) \xrightarrow{\rho_{\Sigma_2}^{Mod}} \mathbf{Mod}(\Sigma_2) \\ \downarrow \sigma & \mathbf{Sen}(\sigma) \downarrow & \mathbf{Mod}'(\rho^{Sign}(\sigma)) \downarrow & \mathbf{Mod}(\sigma) \downarrow \\ \Sigma_2 & \mathbf{Sen}(\Sigma_2) \xrightarrow{\rho_{\Sigma_2}^{Sen}} \mathbf{Sen}'(\rho^{Sign}(\Sigma_2)) & \mathbf{Mod}'(\rho^{Sign}(\Sigma_2)) \xrightarrow{\rho_{\Sigma_1}^{Mod}} \mathbf{Mod}(\Sigma_1) \end{array}$$

In summary, this section has outlined the basic definitions that will be needed to understand our construction of the Event-B institution that will be presented in Section 4.

Operation	Format	Description
Translation	$SP_1$ <b>with</b> $\sigma$	<p>Renames the signature components of <math>SP_1</math> (e.g., sort, operation and predicate names in <math>\mathcal{FOP}\mathcal{EQ}</math>) using the signature morphism <math>\sigma : \Sigma_{SP_1} \rightarrow \Sigma'</math>. The resulting specification is the same as the original with the appropriate renamings carried out when <math>\sigma</math> is bijective. In the case where <math>\sigma</math> is not surjective then translation adds new symbols. If <math>\sigma</math> is not injective then the signature morphism requires that the interpretations of the symbols identified by <math>\sigma</math> coincide.</p> $\begin{aligned} \text{Sig}[SP_1 \text{ with } \sigma] &= \Sigma' \\ \text{Mod}[SP_1 \text{ with } \sigma] &= \{M' \in  \mathbf{Mod}(\Sigma')  \mid M' _{\sigma} \in \text{Mod}[SP_1]\}. \end{aligned}$
Sum	$SP_1$ <b>and</b> $SP_2$	<p>Combines the specifications <math>SP_1</math> and <math>SP_2</math>. It is the most straightforward way of combining specifications with different signatures. This is achieved by forming a specification with a signature corresponding to the union of the signatures of <math>SP_1</math> and <math>SP_2</math>. Translation should be used before sum to avoid any unintended name clashes.</p> $SP_1 \text{ and } SP_2 = (SP_1 \text{ with } \iota) \cup (SP_2 \text{ with } \iota')$ <p>where <math>\text{Sig}[SP_1] = \Sigma</math>, <math>\text{Sig}[SP_2] = \Sigma'</math>, <math>\iota : \Sigma \hookrightarrow \Sigma \cup \Sigma'</math>, <math>\iota' : \Sigma' \hookrightarrow \Sigma \cup \Sigma'</math> and <math>\cup</math> is applied to specifications (<math>SP_3</math> and <math>SP_4</math>) over the same signature (<math>\Sigma''</math>) as follows</p> $\begin{aligned} \text{Sig}[SP_3 \cup SP_4] &= \Sigma'' \\ \text{Mod}[SP_3 \cup SP_4] &= \text{Mod}[SP_3] \cap \text{Mod}[SP_4]. \end{aligned}$
Enrichment	$SP_1$ <b>then</b> ...	<p>Extends the specification <math>SP_1</math> by adding new sentences after the <b>then</b> specification-building operator. This operator can be used to represent superposition refinement of Event-B specifications by adding new variables and events. The resulting signature and model construction is similar to <b>and</b>.</p>
Hiding	$SP_1$ <b>hide via</b> $\sigma$	<p>Hiding via the signature morphism <math>\sigma</math> allows viewing a specification, <math>SP_1</math>, as a specification containing only the signature components of another specified by the signature morphism <math>\sigma : \Sigma \rightarrow \Sigma_{SP_1}</math>. This is useful for representing refinement as it allows a concrete specification to be viewed as one written using only those signature items supported by its corresponding abstract specification.</p> $\begin{aligned} \text{Sig}[SP_1 \text{ hide via } \sigma] &= \Sigma \\ \text{Mod}[SP_1 \text{ hide via } \sigma] &= \{M _{\sigma} \mid M \in \text{Mod}[SP_1]\}. \end{aligned}$

Table 1: The basic institution-theoretic specification-building operators that can be used to modularise specifications in a formalism-independent manner. Here  $SP_1$  and  $SP_2$  denote specifications written over some institution, and  $\sigma$  is a signature morphism in the same institution.



### 3. AN OVERVIEW OF THE EVENT-B SPECIFICATION LANGUAGE

In this section, we describe the prerequisite background material that is pertinent to this paper. In particular, Section 3.1 briefly outlines the Event-B formal specification language and Section 3.2 introduces a simple example that we will refer to throughout this document. Finally, Section 3.3 discusses existing approaches to defining a semantics for Event-B.

**3.1. The Syntax for Event-B.** Event-B evolved from the B Method, often termed Classical-B, as a formal method for the specification and verification of systems using a set-theoretic notation [Sch01]. The abstract syntax for Event-B is described briefly in [Abr10] and we provide a more detailed version in Figure 1. An Event-B specification consists of a set of *machine* and *context* definitions.

- *Machines* encapsulate the dynamic behaviour of a system in terms of a state space, represented by a set of variables and constrained by the machine-level invariants. A machine contains a set of events, which can be parameterised by values, and are further constrained by the predicates in the guards, witnesses and actions. Guards are predicates representing the conditions under which a given event may be triggered. Witnesses are used to relate abstract and refined event parameters. Actions are before-after predicates describing the updates to the state that are caused by the corresponding event.

Events are also equipped with a status which is either **ordinary**, **anticipated** or **convergent** and this is used for proving termination properties with respect to a variant expression defined in the machine. Specifically, **anticipated** events must not increase the variant expression, whilst **convergent** events must strictly decrease the variant expression.

- *Contexts* model the static behaviour of a system using sets and constants which can be constrained by axioms.

Both machines and contexts allow the user to specify *theorems* which are used to generate a specific set of proof obligations. Since these must be consequences of the specification and do not add any additional constraints, we omit them from further discussion here.

In Figure 1 we do not give a definition for the non-terminals *predicate* and *expression*, which we intend to be understood as representing the corresponding concepts from first-order logic. We will ultimately delegate the semantics of these constructs in Section 5.1 to  $\mathcal{FOPEQ}$ , the institution for first-order predicate logic with equality. Thus we immediately distinguish two basic sub-languages in Event-B: the Event-B *mathematical language*, consisting of predicate logic, set theory and arithmetic, and the Event-B *modelling language* which uses these in the definition of events, machines and contexts.

In the spirit of the institutional approach, we can further decompose the Event-B modelling language into two sub-languages which (inspired by an early version of the UML specification [OMG11a, OMG11b]) we call the *infrastructure language* and the *superstructure language*. The infrastructure language covers those concepts which are internal to a given context or machine, while the superstructure language includes the specification-building operators in Event-B. These constructs allow one machine to *refine* another, or to *see* a context, and allow a context to *extend* other contexts. It is our thesis that these Event-B operators can be represented and substantially extended using the generic specification-building operators that are part of the institutional framework, and we show how this can be achieved in Section 5.4.

In summary, we divide the constructs of the Event-B language into three layers, each layer corresponding to one of its three constituent sub-languages, as illustrated in Figure 2.

```

Specification ::= (Machine | Context)+
  Machine ::= machine MachineName
    [refines MachineName]
    [sees ContextName+]
    MachineBody
  end

  MachineBody ::= variables VarName+
    invariants LabelledPred*
    [theorems LabelledPred+]
    [variant Expression]
    events InitEvent Event*

  LabelledPred ::= LabelName ':' Predicate
  InitEvent ::= event Initialisation
    status ordinary
    [then LabelledPred]
  end

  Event ::= event EventName
    status Stat
    [refines EventName+]
    EventBody
  end

  EventBody ::= [any VarName+]
    [where LabelledPred]
    [with LabelledPred]
    [then LabelledPred]

  Stat ::= ordinary | convergent | anticipated
  Context ::= context ContextName
    [extends ContextName+]
    ContextBody
  end

  ContextBody ::= [sets SetName+]
    [constants ConstName+]
    [axioms LabelledPred+]
    [theorems LabelledPred+]

  MachineName, ContextName ::= String
  SetName, ConstName, VarName ::= String
  LabelName ::= String

```

Figure 1: The Event-B syntax, consisting of the definition of machines, contexts and events. This syntax does not define the non-terminals *Predicate* and *Expression*, since these will be mapped to *FOPÉQ*-formulae and terms respectively in our semantics.

The right hand side of Figure 2 shows the institutional constructs that we will use to define the semantics of each of these constituent sub-languages. This three-layer model plays a key role in structuring the definitions of the functions given in Section 5.

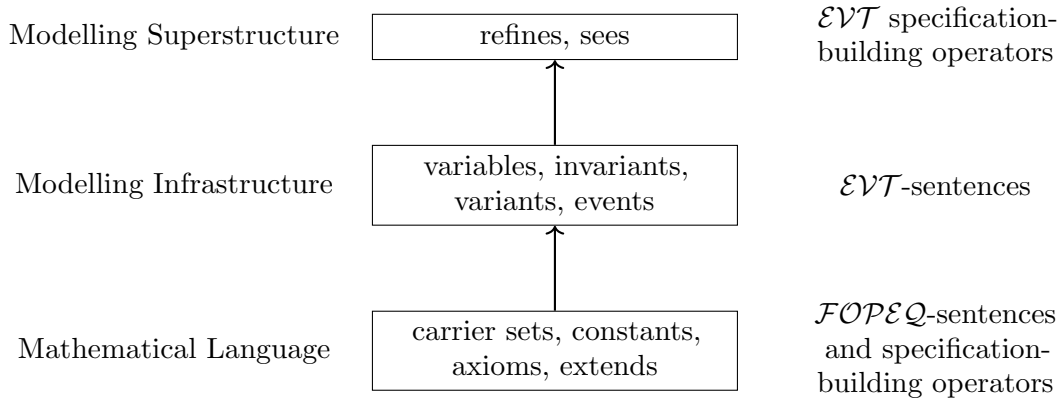


Figure 2: We split the Event-B syntax, defined in Figure 1, into three sub-languages: superstructure, infrastructure and a mathematical language. This figure shows the corresponding Event-B constructs, and their representation in our semantics.

**3.2. A running example illustrating Event-B specification and refinement.** We illustrate the use of Event-B via a running example of *cars on a bridge* that was originally presented in Abrial’s book on modelling in Event-B [Abr10, Ch. 2]. Figure 3 contains an Event-B specification corresponding to the first abstract machine in this development. It describes the behaviour of cars entering and leaving the mainland. This abstract model is comprised of a context (lines 1–7) specifying a natural number constant  $d$  representing the total number of cars, and a basic machine description (lines 8–29) which has a counter  $n$  for the number of cars on the island or on the bridge, along with events to initialise the number of cars, and to increase or decrease the counter as cars leave or enter. In this example, the bridge is only wide enough for one car so it can only be used by cars travelling in one direction at any given time.

**Refinement.** Event-B supports the process of refinement allowing the system specification to be gradually constructed from an abstract specification into a concrete specification [AH07]. The first refinement step is represented by giving a specification of a concrete machine  $m_1$ , shown in Figure 4, which is declared to refine  $m_0$  (line 2). New variables are added to record the number of cars on the bridge going to the island, the number of cars on the island and the number of cars on the bridge going to the mainland ( $a$ ,  $b$  and  $c$  respectively on line 5) and the invariants on lines 7–15 ensure that these function as expected.

The abstract machine,  $m_0$ , is refined by  $m_1$  in two ways. First, new events are added to describe cars entering and leaving the island ( $IL\_out$  and  $IL\_in$  on lines 30–42). Both of these events are defined to be **convergent**, meaning that they must decrease the value of variant expression defined on line 16. Second, the events  $ML\_in$  and  $ML\_out$  from  $m_0$  are refined to utilise the new variables,  $a$ ,  $b$  and  $c$ ; their relationship with the abstract events from  $m_0$  is noted explicitly on lines 24 and 44.

**3.3. Existing approaches to Event-B semantics.** As noted earlier, the only semantic definitions that exist for Event-B have been given in terms of proof obligations which, due

```

1 CONTEXT cd
2   CONSTANTS
3     d
4   AXIOMS
5     axm1:  $d \in \mathbb{N}$ 
6     axm2:  $d > 0$ 
7 END

8 MACHINE m0
9   SEES cd
10  VARIABLES
11    n
12  INVARIANTS
13    inv1:  $n \in \mathbb{N}$ 
14    inv2:  $n \leq d$ 
15  EVENTS
16    Initialisation
17    then
18      act1:  $n := 0$ 
19    Event ML_out  $\hat{=}$  ordinary
20    when
21      grd1:  $n < d$ 
22    then
23      act1:  $n := n + 1$ 
24    Event ML_in  $\hat{=}$  ordinary
25    when
26      grd1:  $n > 0$ 
27    then
28      act1:  $n := n - 1$ 
29 END

```

Figure 3: An Event-B specification of an abstract machine `m0` and context `cd`. This machine specification consists of the events `ML_out` and `ML_in` that model the behaviour of cars leaving and entering the mainland respectively.

to their abstract nature, can inhibit the formal definition of modularisation constructs and interoperability. We discuss these approaches below.

Proof support for Event-B is provided by its Eclipse-based IDE, the Rodin Platform [ABH<sup>+</sup>10]. For each of the specification components listed in Figure 1, the Rodin Platform generates a series of proof obligations which can then be proved automatically and/or interactively using its proving interface. Rodin supports refinement by reusing proofs carried out in an abstract model to prove properties about the concrete model.

This notion of associating a specification with the set of proof obligations it generates can be seen as a form of semantics for those specifications, and this is the approach taken by Hallerstede in his semantics for Event-B [Hal08]. Note that this is not a property-oriented specification, since proof obligations should be theorems of the underlying mathematical context, not axioms that could be used to specify a set of models or define a theory. In essence, the proof obligations are used to characterise rather than fix the semantics of Event-B models. By not associating Event-B with any behavioural model, this approach has the advantage of making an Event-B specification independent of any model of computation. Hallerstede asserts that this facilitates interoperability, since there is no predetermined semantic model to complicate the mapping.

```

1 MACHINE m1
2 REFINES m0
3 SEES cd
4 VARIABLES
5   a, b, c
6 INVARIANTS
7   inv1:  $a \in \mathbb{N}$  //on bridge to island
8   inv2:  $b \in \mathbb{N}$  //on island
9   inv3:  $c \in \mathbb{N}$  //on bridge to mainland
10  inv4:  $n = a + b + c$ 
11  inv5:  $a = 0 \vee c = 0$ 
12  thm1:  $a + b + c \in \mathbb{N}$  theorem
13  thm2:  $c > 0 \vee a > 0$ 
14          $\vee (a + b < d \wedge c = 0)$ 
15          $\vee (0 < b \wedge a = 0)$  theorem
16 VARIANT 2*a + b
17 EVENTS
18   Initialisation
19   then
20     act2:  $a := 0$ 
21     act3:  $b := 0$ 
22     act4:  $c := 0$ 
23   Event ML.out  $\hat{=}$  ordinary
24   REFINES ML.out
25   when
26     grd1:  $a + b < d$ 
27     grd2:  $c = 0$ 
28   then
29     act1:  $a := a + 1$ 
30   Event IL.in  $\hat{=}$  convergent
31   when
32     grd1:  $a > 0$ 
33   then
34     act1:  $a := a - 1$ 
35     act2:  $b := b + 1$ 
36   Event IL.out  $\hat{=}$  convergent
37   when
38     grd1:  $0 < b$ 
39     grd2:  $a = 0$ 
40   then
41     act1:  $b := b - 1$ 
42     act2:  $c := c + 1$ 
43   Event ML.in  $\hat{=}$  ordinary
44   REFINES ML.in
45   when
46     grd1:  $c > 0$ 
47   then
48     act2:  $c := c - 1$ 
49 END

```

Figure 4: An Event-B specification of a machine m1 with additional events IL\_in and IL\_out to model the behaviour of cars entering and leaving the island. The variables a, b, and c keep track of the number of cars on the bridge going to the island, the number of cars on the island and the number of cars on the bridge going to the mainland, respectively.

However, this approach also has some disadvantages. For larger specifications it offers no explicit name-management, and has no formal concept of modularisation. For example, the semantics of an Event-B machine at the end of a long refinement chain would be a large series of proof obligations collected along the way from different machines and contexts, and with no explicit mention of named events. In a previous empirical study of a large corpus of Event-B specifications we have found a tendency for monolithic development, with repeated sub-specifications analogous to software code clones [FMP17c]. We hypothesise that a more fully developed set of specification-building operators would facilitate the construction of large Event-B specifications in a more modular manner.

Abrial presented a model of Event-B machines in terms of states (variable-to-value mappings) and traces [Abr10, Ch. 14]. While Abrial intends these as a justification for the proof obligations, we have used a version of these models in our construction of the Event-B institution in Section 4.3, and we thus define the semantics in this context. Since the resulting proof obligations will then be part of the theory associated with the model, we anticipate that our approach will be complementary to that of Hallerstede.

There have been other approaches to defining a semantics for elements of the Event-B language, principally in the context of mapping it to other formalisms. It has been shown that a shallow embedding of the Event-B logic in Isabelle/HOL provides a sound semantics for the mathematical language of Event-B but not for the full modelling language [Sch10]. Moreover, Event-B is a special form of Back’s Action systems which has been formally defined using Dijkstra’s weakest precondition predicate transformers [Bac90]. This accounts for the refinement calculus used but not for a full semantic definition of the Event-B modelling language itself.

Other related work includes a CSP semantics for Event-B refinement [STW14]. This behavioural semantics defines the relationship between the first and last machines in a refinement chain but does not give a semantics for the internal components of these machines. In previous work, we presented an institution-theoretic approach to interoperability between Event-B (using the  $\mathcal{EVT}$  institution) and CSP (using the  $\mathcal{CSPCASL}$  institution) [FMP17a]. This work supported heterogeneous specification by constraining the models of  $\mathcal{EVT}$ -specifications by the models of  $\mathcal{CSPCASL}$  ones.

The approach presented in this paper treats Event-B as a complete language rather than as a calculus whose semantics can be given in terms of proof obligations. Our semantics commits to the theory of institutions, and gets in return a standard set of specification-building operators, along with a generic framework for interoperability with other formalisms, albeit at the price of including them in the institutional framework.

#### 4. $\mathcal{EVT}$ : THE INSTITUTION FOR EVENT-B

This section introduces the  $\mathcal{EVT}$  institution, which will play the role of the semantic domains for our semantics: that is, our semantics will map Event-B specifications into structured specifications in the  $\mathcal{EVT}$  institution. The definitions in this section then ensure that these structured specifications have a meaningful semantics.

To define the  $\mathcal{EVT}$  institution, Sections 4.1 through 4.4 give definitions for its basic elements corresponding to Definition 2.1, namely  $\mathbf{Sign}_{\mathcal{EVT}}$ ,  $\mathbf{Sen}_{\mathcal{EVT}}$ ,  $\mathbf{Mod}_{\mathcal{EVT}}$  and the satisfaction relation,  $\models_{\mathcal{EVT}}$ . We then prove that the institution-theoretic satisfaction condition holds for  $\mathcal{EVT}$ .

A structured specification in  $\mathcal{EVT}$  represents an Event-B machine. We handle the Event-B contexts using the institution for first-order predicate logic with equality ( $\mathcal{FOPEQ}$  in Definition 2.2), and in Section 4.5 we define an institution comorphism from  $\mathcal{FOPEQ}$  to  $\mathcal{EVT}$  to enable this. Finally, in Section 4.6 we show that  $\mathcal{EVT}$  exhibits good behaviour with respect to modularisation by proving properties relating to pushouts and amalgamation in  $\mathcal{EVT}$ . This allows us to use the generic specification-building operators of institutions in  $\mathcal{EVT}$ , and we discuss the pragmatics of this in Section 4.7.

This work was originally presented in [FMP17b] and it forms the basis of our definition of a translational semantics in the subsequent sections.

**4.1.  $\text{Sign}_{\mathcal{EVT}}$ , the Category of  $\mathcal{EVT}$ -signatures.** Signatures describe the vocabulary that specifications written in a particular institution can have. Here, we define what it means to be an  $\mathcal{EVT}$ -signature (Definition 4.1) and an  $\mathcal{EVT}$ -signature morphism (Definition 4.2). We show that  $\text{Sign}_{\mathcal{EVT}}$  is indeed a category (Lemma 4.3) which is a necessary requirement for the definition of the  $\mathcal{EVT}$  institution.

**Definition 4.1 ( $\mathcal{EVT}$ -Signature).** A **signature** in  $\mathcal{EVT}$  is a five-tuple of sets  $\Sigma_{\mathcal{EVT}} = \langle S, \Omega, \Pi, E, V \rangle$  where

- The first three components form a standard  $\mathcal{FOPEQ}$ -signature,  $\langle S, \Omega, \Pi \rangle$ .
- $E$  is a function from event names to their status. We write elements of this function as tuples of the form  $(e \mapsto s)$ , for an event named  $e$  with a status  $s$  which always belongs to the poset  $\{\text{ordinary} < \text{anticipated} < \text{convergent}\}$ .
- $V$  is a set of sort-indexed variables. We write elements of this set as tuples of the form  $(v : s)$ , for a variable named  $v$  with sort  $s \in S$ .

In theory it is possible for two variables of different sorts to share the same name, but we advise against this as it may lead to confusing specifications. We assume that  $V$  does not contain any *primed* variable names and that such names are not introduced via any signature morphisms.

Following the convention for Event-B [Abr10], we assume that every signature has an initial event, called `Init`, whose status is always `ordinary`.

*Notation:* We write  $\Sigma$  in place of  $\Sigma_{\mathcal{EVT}}$  when describing a signature over our institution for Event-B. For signatures over institutions other than  $\mathcal{EVT}$  we will use the subscript notation where necessary; e.g., a signature over  $\mathcal{FOPEQ}$  is denoted by  $\Sigma_{\mathcal{FOPEQ}}$ . For a given signature  $\Sigma$ , we access its individual components using a dot-notation, e.g.  $\Sigma.S$  for the set  $S$  in the tuple  $\Sigma$ . For any signature,  $\Sigma.E$  is a function from event names to their status, and we will access the domain of this function,  $\text{dom}(\Sigma.E)$  when we wish to deal with event names only.

**Definition 4.2 ( $\mathcal{EVT}$ -Signature Morphism).** A **signature morphism**  $\sigma : \Sigma \rightarrow \Sigma'$  is a five-tuple containing  $\sigma_S, \sigma_\Omega, \sigma_\Pi, \sigma_E$  and  $\sigma_V$ . Here  $\sigma_S, \sigma_\Omega, \sigma_\Pi$  are the mappings taken from the corresponding signature morphism in  $\mathcal{FOPEQ}$ , as follows

- $\sigma_S : \Sigma.S \rightarrow \Sigma'.S$  is a function mapping sort names to sort names.
- $\sigma_\Omega : \Sigma.\Omega \rightarrow \Sigma'.\Omega$  is a family of functions mapping operation names in  $\Sigma.\Omega$ , respecting the arities and result sorts.
- $\sigma_\Pi : \Sigma.\Pi \rightarrow \Sigma'.\Pi$  is a family of functions mapping the predicate names in  $\Sigma.\Pi$ , respecting the arities.

and

- $\sigma_E : \Sigma.E \rightarrow \Sigma'.E$  is a function on event names and their status with the condition that for the initial event  $\sigma_E(\text{Init} \mapsto \text{ordinary}) = (\text{Init} \mapsto \text{ordinary})$ , while for non-initial events  $\sigma_E$  preserves the status ordering given by the poset  $\{\text{ordinary} < \text{anticipated} < \text{convergent}\}$ .
- $\sigma_V : \Sigma.V \rightarrow \Sigma'.V$  is a function on sets of variable names whose sorts change according to  $\sigma_S$ . Thus if some variable  $v$  is renamed to  $w$ , we must have that  $\sigma(v : s) = (w : \sigma_S(s))$ , where  $\sigma_S$  is the sort mapping as described above.

**Lemma 4.3.**  *$\mathcal{EVT}$ -signatures and signature morphisms define a category  $\mathbf{Sign}_{\mathcal{EVT}}$ . The objects are signatures and the arrows are signature morphisms. Composition of arrows is performed componentwise.*

In this section, we have defined the category of  $\mathcal{EVT}$ -signatures,  $\mathbf{Sign}_{\mathcal{EVT}}$ . Next, we define the functor  $\mathbf{Sen}_{\mathcal{EVT}}$  that yields  $\mathcal{EVT}$ -sentences.

**4.2. The Functor  $\mathbf{Sen}_{\mathcal{EVT}}$ , yielding  $\mathcal{EVT}$ -sentences.** The second component of an institution is a functor called  $\mathbf{Sen}$  that generates a set of sentences over a particular signature (Definition 2.1). Here, we define what is meant by a  $\Sigma_{\mathcal{EVT}}$ -Sentence (Definition 4.4) and we prove that  $\mathbf{Sen}_{\mathcal{EVT}}$  is indeed a functor (Lemma 4.5).

**Definition 4.4 ( $\Sigma_{\mathcal{EVT}}$ -Sentence).** Sentences over  $\mathcal{EVT}$  are of the form  $\langle e, \phi(\bar{x}, \bar{x}') \rangle$ . Here,  $e$  is an event name in the domain of  $\Sigma.E$  (recall that elements of  $\Sigma.E$  are functions containing tuples of the form  $(e \mapsto \text{status})$ ) and  $\phi(\bar{x}, \bar{x}')$  is an open  $\mathcal{FOPEQ}$ -formula over these variables  $\bar{x}$  from  $\Sigma.V$  and the primed versions,  $\bar{x}'$ , of the variables.

In the Rodin Platform, Event-B machines are presented (syntactically sugared) as can be seen in abstract form on the left of Figure 5. The set of variables in a machine is denoted by  $\bar{x}$  (line 2) and the invariants are denoted by  $I(\bar{x})$  on line 3. The  $\mathcal{EVT}$ -signature derived from this machine would look like:

$$\Sigma = \langle S, \Omega, \Pi, \{(\text{Init} \mapsto \text{ordinary}), \dots, (e_i \mapsto \text{convergent}), \dots\}, \{\bar{x}\} \rangle$$

where the  $\mathcal{FOPEQ}$ -component of the signature,  $\langle S, \Omega, \Pi \rangle$ , is drawn from the “seen” context(s) on line 1. The variant expression, denoted by  $n(\bar{x})$  on line 4, is used for proving termination properties. As described in Section 3.1, events that have a status of **anticipated** or **convergent** must not increase or must strictly decrease the variant expression respectively [Abr10]. Each machine has an **Initialisation** event (lines 6–7) whose action is interpreted as a predicate  $BA(\bar{x})$ . Events can have parameter(s) as given by the list of identifiers  $\bar{p}$  on line 10.  $G(\bar{x}, \bar{p})$  and  $W(\bar{x}, \bar{p})$  are predicates that represent the guard(s) and witness(es) respectively over the variables and parameter(s) (lines 11–12). Actions are interpreted as before-after predicates i.e.  $\mathbf{x} := \mathbf{x} + 1$  is interpreted as  $\mathbf{x}' = \mathbf{x} + 1$ . Thus, the predicate  $BA(\bar{x}, \bar{p}, \bar{x}')$  on line 13 represents the action(s) over the parameter(s)  $\bar{p}$  and the sets of variables  $\bar{x}$  and  $\bar{x}'$ .

Formulae written in the mathematical language (such as the axioms that may appear in contexts) are interpreted as sentences over  $\mathcal{FOPEQ}$ . We can include these in specifications over  $\mathcal{EVT}$  using the comorphism defined in Section 4.5. We represent the Event-B invariant, variant and event predicates as sentences over  $\mathcal{EVT}$ . These are illustrated alongside the Event-B predicates that they correspond to in Figure 5.



<pre> 1 MACHINE m REFINES a SEES ctx 2 VARIABLES <math>\bar{x}</math> 3 INVARIANTS <math>I(\bar{x})</math> 4 VARIANT <math>n(\bar{x})</math> 5 EVENTS 6 Initialisation ordinary 7   then act-name: <math>BA(\bar{x}')</math> 8   : 9 Event <math>e_i \hat{=} \text{convergent}</math> 10  any <math>\bar{p}</math> 11  when guard-name: <math>G(\bar{x}, \bar{p})</math> 12  with witness-name: <math>W(\bar{x}, \bar{p})</math> 13  then act-name: <math>BA(\bar{x}, \bar{p}, \bar{x}')</math> 14  : 15 END </pre>	<pre> <math>\{\langle e, I(\bar{x}) \wedge I(\bar{x}') \rangle \mid e \in \text{dom}(\Sigma.E)\}</math>   <math>\langle \text{Init}, BA(\bar{x}') \rangle</math>   <math>\langle e_i, n(\bar{x}') &lt; n(\bar{x}) \rangle</math>   <math>\langle e, \exists \bar{p} \cdot G(\bar{x}, \bar{p}) \wedge W(\bar{x}, \bar{p}) \wedge BA(\bar{x}, \bar{p}, \bar{x}') \rangle</math> </pre>
---	--

Figure 5: The elements of an Event-B machine specification as presented in Rodin (left) and the corresponding sentences in the  $\mathcal{EVT}$  institution (right). The  $\mathcal{EVT}$ -signature is not shown here, but would have to be constructed before the sentences.

**Invariants:** For each Event-B invariant,  $I(\bar{x})$ , we form the open  $\mathcal{FOPEQ}$ -sentence  $I(\bar{x}) \wedge I(\bar{x}')$  and pair it with each event name in  $\Sigma$ , since invariants apply to all events in a machine. Thus, we form the set of  $\mathcal{EVT}$  event sentences  $\{\langle e, I(\bar{x}) \wedge I(\bar{x}') \rangle \mid e \in \text{dom}(\Sigma.E)\}$ .

**Variants:** Firstly, we assume the existence of a suitable type for variant expressions and the usual interpretation of the predicates  $<$  and  $\leq$  over the integers in the signature. In particular, we assume that all specifications include a standard built-in library that defines mathematical operators, such as  $<$  and  $\leq$ , which are axiomatised consistently with their Event-B counterparts. The variant expression applies to specific events, so we pair it with an event name in order to meaningfully evaluate it. This expression can be translated into an open  $\mathcal{FOPEQ}$ -term, which we denote by  $n(\bar{x})$ , and we use this to construct a formula based on the status of the event(s) in the signature  $\Sigma$ .

- For each  $(e \mapsto \text{anticipated}) \in \Sigma.E$  we form the  $\mathcal{EVT}$ -sentence  $\langle e, n(\bar{x}') \leq n(\bar{x}) \rangle$
- For each  $(e \mapsto \text{convergent}) \in \Sigma.E$  we form the  $\mathcal{EVT}$ -sentence  $\langle e, n(\bar{x}') < n(\bar{x}) \rangle$

**Events:** Event guard(s) and witnesses are also labelled predicates that can be translated into open  $\mathcal{FOPEQ}$ -formulae over the variables  $\bar{x}$  in  $V$  and parameters  $\bar{p}$ . These are denoted by  $G(\bar{x}, \bar{p})$  and  $W(\bar{x}, \bar{p})$  respectively. In Event-B, actions are interpreted as before-after predicates, thus they can be translated into open  $\mathcal{FOPEQ}$ -formulae denoted by  $BA(\bar{x}, \bar{p}, \bar{x}')$ . Thus for each event we build the formula

$$\phi(\bar{x}, \bar{x}') = \exists \bar{p} \cdot G(\bar{x}, \bar{p}) \wedge W(\bar{x}, \bar{p}) \wedge BA(\bar{x}, \bar{p}, \bar{x}')$$

where  $\bar{p}$  are the event parameters. This is used to build an  $\mathcal{EVT}$ -sentence of the form  $\langle e, \phi(\bar{x}, \bar{x}') \rangle$ .

The **Init** event, which is an Event-B sentence over only the after variables denoted by  $\bar{x}'$ , is a special case. In this case, we form the  $\mathcal{EVT}$ -sentence  $\langle \text{Init}, \phi(\bar{x}') \rangle$  where  $\phi(\bar{x}')$  is a predicate over the after values of the variables as assigned by the **Init** event.

$$\begin{aligned}
& 1 \{ \langle e, d > 0 \rangle \mid e \in \{\text{Init}, \text{ML\_out}, \text{ML\_in}\} \} \\
& 2 \{ \langle e, n \leq d \wedge n' \leq d \rangle \mid e \in \{\text{Init}, \text{ML\_out}, \text{ML\_in}\} \} \\
& 3 \langle \text{Init}, n' = 0 \rangle \\
& 4 \langle \text{ML\_out}, n < d \wedge n' = n + 1 \rangle \\
& 5 \langle \text{ML\_in}, n > 0 \wedge n' = n - 1 \rangle
\end{aligned}$$

Figure 6: These are the  $\mathcal{EVT}$ -sentences corresponding to the abstract Event-B specification as illustrated on lines 1–29 of Figure 3. Note that here the constant,  $d$ , is interpreted as a 0-ary operator.

As a further example, Figure 6 contains the  $\mathcal{EVT}$ -sentences corresponding to the abstract Event-B specification on lines 1–29 of Figure 3 excluding those invariants which are used for typing purposes.

**Lemma 4.5.** *There is a functor  $\mathbf{Sen}_{\mathcal{EVT}} : \mathbf{Sign}_{\mathcal{EVT}} \rightarrow \mathbf{Set}$  generating for each  $\mathcal{EVT}$ -signature  $\Sigma$  a set of  $\mathcal{EVT}$ -sentences which is an object in the category  $\mathbf{Set}$  and for each  $\mathcal{EVT}$ -signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  (morphisms in the category  $\mathbf{Sign}_{\mathcal{EVT}}$ ) a function  $\mathbf{Sen}(\sigma) : \mathbf{Sen}(\Sigma_1) \rightarrow \mathbf{Sen}(\Sigma_2)$  (morphisms in the category  $\mathbf{Set}$ ) translating  $\mathcal{EVT}$ -sentences.*

In this section, we have defined what is meant by an  $\mathcal{EVT}$ -sentence. Next, we define  $\mathcal{EVT}$ -models and the  $\mathbf{Mod}_{\mathcal{EVT}}$  functor.

**4.3. The Functor  $\mathbf{Mod}_{\mathcal{EVT}}$ , yielding  $\mathcal{EVT}$ -models.** Our construction of  $\mathcal{EVT}$ -models is based on Event-B mathematical models as described by Abrial [Abr10, Ch. 14]. In these models the state is represented as a sequence of variable-values and models are defined over before and after states. We interpret these states as sets of variable-to-value mappings in our definition of  $\mathcal{EVT}$ -models and so we define a  $\Sigma$ -state of an algebra  $A$  in Definition 4.6.

**Definition 4.6** ( $\Sigma$ -*State* $_A$ ). For any given  $\mathcal{EVT}$ -signature  $\Sigma$  we define a  $\Sigma$ -**state** of an algebra  $A$  as a set of (sort appropriate) variable-to-value mappings whose domain is the set of sort-indexed variable names  $\Sigma.V$ . Note that  $A$  is a model for the underlying  $\mathcal{FOPEQ}$  signature associated with  $\Sigma$ . We define the set  $\mathit{State}_A$  as the set of all such  $\Sigma$ -states. By “sort appropriate” we mean that for any variable  $x$  of sort  $s$  in  $V$ , the corresponding value for  $x$  should be drawn from  $|A|_s$ , the carrier set of  $s$  given by a  $\mathcal{FOPEQ}$ -model  $A$ .

Essentially, a  $\Sigma$ -state $_A$  corresponds to valuations in the  $\Sigma_{\mathcal{FOPEQ}}$ -model,  $A$ , of the variables in  $\Sigma.V$ .

**Definition 4.7** ( $\Sigma_{\mathcal{EVT}}$ -*Model*). Given  $\Sigma = \langle S, \Omega, \Pi, E, V \rangle$ ,  $\mathbf{Mod}_{\mathcal{EVT}}(\Sigma)$  provides a category of  $\mathcal{EVT}$ -models, where an  $\mathcal{EVT}$ -**model** over  $\Sigma$  is a tuple  $\langle A, L, R \rangle$  where

- $A$  is a  $\Sigma_{\mathcal{FOPEQ}}$ -model.
- $L \subseteq \mathit{State}_A$  is the non-empty initialising set that provides the states after the **Init** event. We note that trivial models are excluded as the initialising set  $L$  is never empty. We can see this because even in the extreme cases where there are no predicates or variables in the **Init** event,  $L$  is the singleton containing the empty map ( $L = \{\{\}\}$ ).
- For every event name,  $e$ , in the domain of  $E$ , denoted by  $e \in \mathit{dom}(E)$ , other than **Init**, we define  $R.e \subseteq \mathit{State}_A \times \mathit{State}_A$  where for each pair of states  $\langle s, s' \rangle$  in  $R.e$ ,  $s$  provides

$$\begin{array}{l}
1 \quad \text{Event } e \hat{=} \\
2 \quad \text{when } \text{grd1}: x < 2 \\
3 \quad \text{then } \text{act1}: x := x + 1 \\
4 \quad \quad \text{act2}: y := \text{false}
\end{array}
\quad R_e = \left\{ \begin{array}{l} \{x \mapsto 0, y \mapsto \text{false}, x' \mapsto 1, y' \mapsto \text{false}\}, \\ \{x \mapsto 0, y \mapsto \text{true}, x' \mapsto 1, y' \mapsto \text{false}\}, \\ \{x \mapsto 1, y \mapsto \text{false}, x' \mapsto 2, y' \mapsto \text{false}\}, \\ \{x \mapsto 1, y \mapsto \text{true}, x' \mapsto 2, y' \mapsto \text{false}\} \end{array} \right\}$$

Figure 7: The left hand side shows an example of an Event-B event,  $e$ , with natural number variable  $x$  and boolean variable  $y$ . When  $x < 2$ , the event increments the value of  $x$  and changes  $y$  to **false**. The right hand side enumerates  $R_e$ , the corresponding set of variable-to-value mappings for the event on the left.

values for the variables  $x$  in  $V$ , and  $s'$  provides values for their primed versions  $x'$ . Then  $R = \{R.e \mid e \in \text{dom}(E) \text{ and } e \neq \text{Init}\}$ .

Intuitively, a model over  $\Sigma$  maps every event name  $e$  in  $(\Sigma.E)$  to a set of variable-to-value mappings over the carriers corresponding to the sorts of each of the variables  $x \in \Sigma.V$  and their primed versions  $x'$ . For example, given the event  $e$  on the left of Figure 7, with natural number variable  $x$  and boolean variable  $y$ , we construct the variable-to-value mappings on the right of Figure 7. The notation used in this example is interpreted as *variable name*  $\mapsto$  *value* which is a data state where the value is drawn from the carrier set corresponding to the sort of the variable name given in  $\Sigma.V$ .

The proof requirement here is to show that  $\mathbf{Mod}_{\mathcal{EVT}}(\Sigma)$  forms a category for a given  $\mathcal{EVT}$ -signature  $\Sigma$ .

**Lemma 4.8.** *For any  $\mathcal{EVT}$ -signature  $\Sigma$  there is a category of  $\mathcal{EVT}$ -models  $\mathbf{Mod}_{\mathcal{EVT}}(\Sigma)$  where the objects in the category are  $\mathcal{EVT}$ -models and the arrows are  $\mathcal{EVT}$ -model morphisms.*

Model reducts are central to the definition of an institution (Definition 2.1) and in Definition 4.9, we define the  $\mathcal{EVT}$ -model reduct. Then, in Lemma 4.10, we prove that the  $\mathcal{EVT}$ -model reduct is a functor.

**Definition 4.9 ( $\mathcal{EVT}$ -model reduct).** The reduct of an  $\mathcal{EVT}$ -model  $M = \langle A, L, R \rangle$  along an  $\mathcal{EVT}$ -signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$  is given by  $M|_{\sigma} = \langle A|_{\sigma}, L|_{\sigma}, R|_{\sigma} \rangle$ . Here  $A|_{\sigma}$  is the reduct of the  $\mathcal{FOPEQ}$ -component of the  $\mathcal{EVT}$ -model along the  $\mathcal{FOPEQ}$ -components of  $\sigma$ .  $L|_{\sigma}$  and  $R|_{\sigma}$  are based on the reduction of the states of  $A$  along  $\sigma$ , i.e., for every  $\Sigma'$ -state  $s$  of  $A$ , that is for every sorted map  $s : \Sigma'.V \rightarrow |A|$ ,  $s|_{\sigma}$  is the map  $\Sigma'.V \rightarrow |A|$  given by the composition  $\sigma_V; s$ . This extends in the usual manner from states to sets of states and to relations on states.

**Lemma 4.10.** *For each  $\mathcal{EVT}$ -signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  the  $\mathcal{EVT}$ -model reduct is a functor  $\mathbf{Mod}(\sigma)$  from  $\Sigma_2$ -models to  $\Sigma_1$ -models.*

**Lemma 4.11.** *There is a functor  $\mathbf{Mod}_{\mathcal{EVT}}$  yielding a category  $\mathbf{Mod}(\Sigma)$  of  $\mathcal{EVT}$ -models for each  $\mathcal{EVT}$ -signature  $\Sigma$ , and for each  $\mathcal{EVT}$ -signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  a functor  $\mathbf{Mod}(\sigma)$  from  $\Sigma_2$ -models to  $\Sigma_1$ -models.*

In this section, we defined  $\mathcal{EVT}$ -models, model reducts and the  $\mathbf{Mod}_{\mathcal{EVT}}$  functor. Next, we describe the satisfaction relation,  $\models_{\mathcal{EVT}}$ , in  $\mathcal{EVT}$  and prove that  $\mathcal{EVT}$  is a valid institution.

**4.4. The Satisfaction Relation for  $\mathcal{EVT}$ .** All institutions are equipped with a satisfaction relation to evaluate if a particular model satisfies a sentence. Here, we define the satisfaction relation for  $\mathcal{EVT}$ . In order to define the satisfaction relation for  $\mathcal{EVT}$ , we describe an embedding from  $\mathcal{EVT}$ -signatures and models to  $\mathcal{FOPEQ}$ -signatures and models.

Given an  $\mathcal{EVT}$ -signature  $\Sigma = \langle S, \Omega, \Pi, E, V \rangle$  we form the following two  $\mathcal{FOPEQ}$ -signatures:

- $\Sigma_{\mathcal{FOPEQ}}^{(V, V')} = \langle S, \Omega \cup V \cup V', \Pi \rangle$  where  $V$  and  $V'$  are the variables and their primed versions, respectively, that are drawn from the  $\mathcal{EVT}$ -signature, and represented as 0-ary operators with unchanged sort. The intuition here is that the set of variable-to-value mappings for the free variables in an  $\mathcal{EVT}$ -signature  $\Sigma$  are represented as expansions of  $A$  along the signature inclusion obtained by adding a distinguished 0-ary operation symbol to the corresponding  $\mathcal{FOPEQ}$ -signature for each of the variables  $x \in V$  and their primed versions.
- Similarly, for the initial state and its variables, we construct the signature  $\Sigma_{\mathcal{FOPEQ}}^{(V')} = \langle S, \Omega \cup V', \Pi \rangle$ .

Given the  $\mathcal{EVT}$   $\Sigma$ -model  $\langle A, L, R \rangle$ , we construct the  $\mathcal{FOPEQ}$ -models:

- For every pair of states  $\langle s, s' \rangle$ , we form the  $\Sigma_{\mathcal{FOPEQ}}^{(V, V')}$ -model expansion  $A^{(s, s')}$ , which is the  $\mathcal{FOPEQ}$ -component  $A$  of the  $\mathcal{EVT}$ -model, with  $s$  and  $s'$  added as interpretations for the new operators that correspond to the variables from  $V$  and  $V'$  respectively.
- For each initial state  $s' \in L$  we construct the  $\Sigma_{\mathcal{FOPEQ}}^{(V')}$ -model expansion  $A^{(s')}$  analogously.

For any  $\mathcal{EVT}$ -sentence over  $\Sigma$  of the form  $\langle e, \phi(\bar{x}, \bar{x}') \rangle$ , we create a corresponding  $\mathcal{FOPEQ}$ -formula by replacing the free variables with their corresponding operator symbols. We write this (closed) formula as  $\phi(\bar{x}, \bar{x}')$ .

Given these  $\mathcal{FOPEQ}$ -signatures and models, we now define the satisfaction relation for  $\mathcal{EVT}$  in Definition 4.12.

**Definition 4.12 (Satisfaction Relation).** We define the satisfaction relation in  $\mathcal{EVT}$  as follows. We specify satisfaction for initial events over  $L$  and non-initial events over  $R$ . For any  $\mathcal{EVT}$ -model  $\langle A, L, R \rangle$  and  $\mathcal{EVT}$ -sentence  $\langle e, \phi(\bar{x}, \bar{x}') \rangle$ , where  $e$  is an event name other than **Init**, we define:

$$\langle A, L, R \rangle \models_{\Sigma} \langle e, \phi(\bar{x}, \bar{x}') \rangle \iff \forall \langle s, s' \rangle \in R. e \cdot A^{(s, s')} \models_{\Sigma_{\mathcal{FOPEQ}}^{(V, V')}} \phi(\bar{x}, \bar{x}')$$

Similarly, we evaluate the satisfaction relation of  $\mathcal{EVT}$ -sentences of the form  $\langle \mathbf{Init}, \phi(\bar{x}') \rangle$  as follows:

$$\langle A, L, R \rangle \models_{\Sigma} \langle \mathbf{Init}, \phi(\bar{x}') \rangle \iff \forall s' \in L. A^{(s')} \models_{\Sigma_{\mathcal{FOPEQ}}^{(V')}} \phi(\bar{x}')$$

**Theorem 4.13 (Satisfaction Condition).** *Given  $\mathcal{EVT}$  signatures  $\Sigma_1$  and  $\Sigma_2$ , a signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$ , a  $\Sigma_2$ -model  $M_2$  and a  $\Sigma_1$ -sentence  $\psi_1$ , the following satisfaction condition holds:*

$$\mathbf{Mod}(\sigma)(M_2) \models_{\mathcal{EVT}_{\Sigma_1}} \psi_1 \iff M_2 \models_{\mathcal{EVT}_{\Sigma_2}} \mathbf{Sen}(\sigma)(\psi_1)$$

*Proof.* We must prove this for  $\mathcal{EVT}$ -sentences of the form  $\langle e, \phi(\bar{x}, \bar{x}') \rangle$  as defined in Definition 4.4, with the satisfaction relation,  $\models_{\mathcal{EVT}}$ , as defined in Definition 4.12. Let  $M_2$  be the  $\mathcal{EVT}$ -model  $\langle A_2, L_2, R_2 \rangle$ , and let  $\psi_1$  be the  $\mathcal{EVT}$ -sentence  $\langle e, \phi(\bar{x}, \bar{x}') \rangle$ , then the satisfaction condition is equivalent to

$$\forall \langle s, s' \rangle \in R_2 |_{\sigma}. e \cdot (A_2 |_{\sigma})^{(s, s')} |_{\sigma} \models_{\Sigma_{\mathcal{FOPEQ}}^{(V_1, V_1')}} \phi(\bar{x}, \bar{x}')$$

$$\iff \forall \langle s, s' \rangle \in R_2 \cdot \sigma_E(e) \cdot A_2^{(s, s')} \models_{\mathcal{FOPEQ}_{\Sigma_{\mathcal{FOPEQ}}^{(V_2, V_2')}}} \text{Sen}(\sigma)(\phi(\bar{x}, \bar{x}'))$$

Here, validity follows from the validity of satisfaction in  $\mathcal{FOPEQ}$ . We prove the result for initial events in the same way. Since validity follows from that of  $\mathcal{FOPEQ}$ , we thus conclude that  $\mathcal{EVT}$  preserves the satisfaction condition required of an institution.  $\square$

In this section we have presented the satisfaction relation in  $\mathcal{EVT}$  and showed that  $\mathcal{EVT}$  preserves the axiomatic property of an institution (satisfaction condition in Theorem 4.13). As the satisfaction relation in  $\mathcal{EVT}$  (Definition 4.12) relied on a  $\mathcal{FOPEQ}$  embedding of signatures and models, we discuss the relationship between  $\mathcal{FOPEQ}$  and  $\mathcal{EVT}$  in Section 4.5.

**4.5. Specifying Event-B contexts by relating  $\mathcal{FOPEQ}$  and  $\mathcal{EVT}$ .** Initially, we defined the relationship between  $\mathcal{FOPEQ}$  and  $\mathcal{EVT}$  to be a duplex institution formed from a restricted version of  $\mathcal{EVT}$  ( $\mathcal{EVT}_{res}$ ) and  $\mathcal{FOPEQ}$  where  $\mathcal{EVT}_{res}$  is the institution  $\mathcal{EVT}$  but does not contain any  $\mathcal{FOPEQ}$  components. Duplex institutions are constructed by enriching one institution by the sentences of another ( $\mathcal{EVT}_{res}$  is enriched by sentences from  $\mathcal{FOPEQ}$ ) using an institution semi-morphism [GB92] [ST12, §10.1]. This approach would allow us to constrain  $\mathcal{EVT}_{res}$  by  $\mathcal{FOPEQ}$  and thus facilitate the use of  $\mathcal{FOPEQ}$ -sentences in an elegant way. However, duplex institutions are not supported in HETS, the only tool-set currently available for institutions [MML07], and therefore we opt for a comorphism (Definition 2.4) which embeds the simpler institution  $\mathcal{FOPEQ}$  into the more complex institution  $\mathcal{EVT}$  [ST12, §10.4].

**Definition 4.14 (The institution comorphism  $\rho : \mathcal{FOPEQ} \rightarrow \mathcal{EVT}$ ).** We define  $\rho : \mathcal{FOPEQ} \rightarrow \mathcal{EVT}$  to be an institution comorphism composed of:

- The functor  $\rho^{Sign} : \mathbf{Sign}_{\mathcal{FOPEQ}} \rightarrow \mathbf{Sign}_{\mathcal{EVT}}$  which takes as input a  $\mathcal{FOPEQ}$ -signature of the form  $\langle S, \Omega, \Pi \rangle$  and extends it with the set  $E = \{(\text{Init} \mapsto \text{ordinary})\}$  and an empty set of variable names  $V$ . Here,  $\rho^{Sign}(\sigma)$  works as  $\sigma$  on  $S, \Omega$  and  $\Pi$ , it is the identity on the **Init** event and the empty function on the empty set of variable names.
- The natural transformation  $\rho^{Sen} : \mathbf{Sen}_{\mathcal{FOPEQ}} \rightarrow \rho^{Sign}; \mathbf{Sen}_{\mathcal{EVT}}$  which treats any closed  $\mathcal{FOPEQ}$ -sentence (given by  $\phi$ ) as an Event-B invariant sentence, to form the set of  $\mathcal{EVT}$ -sentences  $\{\langle e, \phi \rangle \mid e \in \text{dom}(\Sigma.E)\}$ . As there are no variables in the signature, and  $\phi$  is any closed formula, we do not require  $\phi$  to be over the variables  $\bar{x}$  and  $\bar{x}'$ .
- The natural transformation  $\rho^{Mod} : (\rho^{Sign})^{op}; \mathbf{Mod}_{\mathcal{EVT}} \rightarrow \mathbf{Mod}_{\mathcal{FOPEQ}}$  is such that for any  $\mathcal{FOPEQ}$ -signature  $\Sigma$  and  $\mathcal{EVT}$ -model  $\langle A, L, \emptyset \rangle$  for  $\rho^{Sign}(\Sigma)$  then
 
$$\rho_{\Sigma}^{Mod}(\langle A, L, \emptyset \rangle) = A$$

Next, we prove that  $\rho : \mathcal{FOPEQ} \rightarrow \mathcal{EVT}$  meets the axiomatic requirements of an institution comorphism.

**Theorem 4.15.** *The institution comorphism  $\rho$  is defined such that for any  $\Sigma \in |\mathbf{Sign}_{\mathcal{FOPEQ}}|$ , the translations  $\rho_{\Sigma}^{Sen} : \mathbf{Sen}_{\mathcal{FOPEQ}}(\Sigma) \rightarrow \mathbf{Sen}_{\mathcal{EVT}}(\rho^{Sign}(\Sigma))$  and  $\rho_{\Sigma}^{Mod} : \mathbf{Mod}_{\mathcal{EVT}}(\rho^{Sign}(\Sigma)) \rightarrow \mathbf{Mod}_{\mathcal{FOPEQ}}(\Sigma)$  preserve the satisfaction relation. That is, for any  $\psi \in \mathbf{Sen}_{\mathcal{FOPEQ}}(\Sigma)$  and  $M' \in |\mathbf{Mod}_{\mathcal{EVT}}(\rho^{Sign}(\Sigma))|$*

$$\rho_{\Sigma}^{Mod}(M') \models_{\mathcal{FOPEQ}_{\Sigma}} \psi \iff M' \models_{\mathcal{EVT}_{\rho^{Sign}(\Sigma)}} \rho_{\Sigma}^{Sen}(\psi)$$

*Proof.* By Definition 4.14,  $M' = \langle A, L, \emptyset \rangle$ ,  $\rho_{\Sigma}^{Mod}(M') = A$  and  $\rho_{\Sigma}^{Sen}(\psi) = \{\langle e, \psi \rangle \mid e \in \text{dom}(\Sigma.E)\}$ . Therefore, the above equivalence becomes

$$A \models_{\mathcal{FOP}\mathcal{E}\mathcal{Q}_{\Sigma}} \psi \iff M' \models_{\mathcal{E}\mathcal{V}\mathcal{T}_{\rho^{Sign}(\Sigma)}} \{\langle e, \psi \rangle \mid e \in \text{dom}(\Sigma.E)\}$$

Then, by the definition of the satisfaction relation in  $\mathcal{E}\mathcal{V}\mathcal{T}$  (Definition 4.12) we get the following for each  $\psi$  in  $\{\langle e, \psi \rangle \mid e \in \text{dom}(\Sigma.E)\}$ .

$$A \models_{\mathcal{FOP}\mathcal{E}\mathcal{Q}_{\Sigma}} \psi \iff \forall s' \in L \cdot A^{(s')} \models_{(\rho^{Sign}(\Sigma))_{\mathcal{FOP}\mathcal{E}\mathcal{Q}}^{(V')}} \psi$$

We deduce that  $\Sigma = (\rho^{Sign}(\Sigma))_{\mathcal{FOP}\mathcal{E}\mathcal{Q}}^{V'}$ , since there are no variable names in  $V'$  and thus no new operator symbols are added to the signature. As there are no variable names in  $V'$ , we have  $L = \{\{\}\}$ , so we can conclude that  $A^{(s')} = A$ . Thus the satisfaction condition holds.  $\square$

**Using the comorphism.** The purpose of this comorphism is to allow us to use  $\mathcal{FOP}\mathcal{E}\mathcal{Q}$  specifications in  $\mathcal{E}\mathcal{V}\mathcal{T}$ . Thus, given the institution comorphism  $\rho : \mathcal{FOP}\mathcal{E}\mathcal{Q} \rightarrow \mathcal{E}\mathcal{V}\mathcal{T}$ , for any  $\Sigma$ -specification written over  $\mathcal{FOP}\mathcal{E}\mathcal{Q}$  we can use the specification-building operator

$$\text{-- with } \rho : \text{Spec}_{\mathcal{FOP}\mathcal{E}\mathcal{Q}}(\Sigma) \rightarrow \text{Spec}_{\mathcal{E}\mathcal{V}\mathcal{T}}(\rho^{Sign}(\Sigma))$$

to interpret this as a specification over  $\mathcal{E}\mathcal{V}\mathcal{T}$  [ST12, §10.4]. This results in a specification with just the **Init** event and no variables, containing  $\mathcal{FOP}\mathcal{E}\mathcal{Q}$ -sentences that hold in the initial state. This process is used to represent contexts, specifically their axioms, which are written over  $\mathcal{FOP}\mathcal{E}\mathcal{Q}$  as sentences over  $\mathcal{E}\mathcal{V}\mathcal{T}$ .

We note that there is a dual nature to the relationship between  $\mathcal{FOP}\mathcal{E}\mathcal{Q}$  and  $\mathcal{E}\mathcal{V}\mathcal{T}$ . The first being the use of  $\mathcal{FOP}\mathcal{E}\mathcal{Q}$  to evaluate the  $\mathcal{E}\mathcal{V}\mathcal{T}$ -satisfaction condition via an embedding as presented in Section 4.4. The second is the comorphism that we have described above which allows us to project Event-B context sentences into  $\mathcal{E}\mathcal{V}\mathcal{T}$  and we illustrate this in our semantics in Section 5. This double relationship offers a number of interesting consequences. In a way,  $\mathcal{E}\mathcal{V}\mathcal{T}$  is tied to  $\mathcal{FOP}\mathcal{E}\mathcal{Q}$  in terms of the evaluation of the satisfaction relation. However, the use of the comorphism to represent context sentences means that it is possible to replace this use of  $\mathcal{FOP}\mathcal{E}\mathcal{Q}$  with another logic so long as a comorphism can be defined to  $\mathcal{E}\mathcal{V}\mathcal{T}$ . This second relationship illustrates the plug ‘n’ play nature of the  $\mathcal{E}\mathcal{V}\mathcal{T}$  institution.

In cases where a specification is enriched with new events, then the axioms and invariants should also apply to these new events. Our use of  $\mathcal{E}\mathcal{V}\mathcal{T}$ -sentences that define invariants mediates this as they are applied to all events in the specification when evaluating the satisfaction condition. In order for an institution to correctly utilise the specification-building operators, we must prove properties with regard to pushouts and amalgamation. We prove that  $\mathcal{E}\mathcal{V}\mathcal{T}$  meets these requirements in Section 4.6.

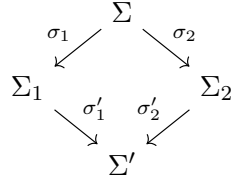
**4.6. Enabling specification-building via pushouts and amalgamation.** We ensure that the  $\mathcal{E}\mathcal{V}\mathcal{T}$  institution has good modularity properties by proving that  $\mathcal{E}\mathcal{V}\mathcal{T}$  admits the amalgamation property: all pushouts in  $\mathbf{Sign}_{\mathcal{E}\mathcal{V}\mathcal{T}}$  exist and every pushout diagram in  $\mathbf{Sign}_{\mathcal{E}\mathcal{V}\mathcal{T}}$  admits *weak* model amalgamation [ST12, §4.4]. Note that the weak amalgamation property corresponds to the usual definition of amalgamation (Definition 4.17) with the uniqueness constraint dropped [ST12, §4.4]. This requirement was actually first referred to as “exactness” and defined by Diaconescu et al. [DGS91]. In [ST12], structured specifications are built by consecutive application of these specification-building operators. The meaning of

structured specifications is obtained by applying functions that correspond to the specification-building operators on the associated signatures and model classes. In particular, the signature (resp. model class) corresponding to a structured specification that uses the **and** specification-building operator is obtained using the pushout (resp. amalgamation) construction shown below [ST12, §5].

**Proposition 4.16.** *Pushouts exist in  $\mathbf{Sign}_{\mathcal{EVT}}$ .*

In Definition 4.17 we present the definition of amalgamation as outlined by Sannella and Tarlecki and use it to structure the proof of amalgamation in  $\mathcal{EVT}$  [ST12, §4.4].

**Definition 4.17 (Amalgamation).** Let  $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_{\Sigma} \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$  be an institution. The following diagram in  $\mathbf{Sign}$



admits amalgamation if

- for any two models  $M_1 \in |\mathbf{Mod}(\Sigma_1)|$  and  $M_2 \in |\mathbf{Mod}(\Sigma_2)|$ , there exists a unique model  $M' \in |\mathbf{Mod}(\Sigma')|$  (amalgamation of  $M_1$  and  $M_2$ ) such that  $M'|_{\sigma'_1} = M_1$  and  $M'|_{\sigma'_2} = M_2$ .
- for any two model morphisms  $f_1 : M_{11} \rightarrow M_{12}$  in  $\mathbf{Mod}(\Sigma_1)$  and  $f_2 : M_{21} \rightarrow M_{22}$  in  $\mathbf{Mod}(\Sigma_2)$  such that  $f_1|_{\sigma_1} = f_2|_{\sigma_2}$ , there exists a unique model morphism  $f' : M'_1 \rightarrow M'_2$  in  $\mathbf{Mod}(\Sigma')$  (amalgamation of  $f_1$  and  $f_2$ ) such that  $f'|_{\sigma'_1} = f_1$  and  $f'|_{\sigma'_2} = f_2$ .

The institution  $\mathbf{INS}$  has the *amalgamation* property if all pushouts exist in  $\mathbf{Sign}$  and every pushout diagram in  $\mathbf{Sign}$  admits amalgamation [ST12, §4.4].

**Proposition 4.18.** *Every pushout diagram in  $\mathbf{Sign}_{\mathcal{EVT}}$  admits weak model amalgamation.*

So far, we have defined our institution for Event-B,  $\mathcal{EVT}$ , outlined the comorphism from  $\mathit{FOPEQ}$  to  $\mathcal{EVT}$ , and shown that  $\mathcal{EVT}$  meets the requirements needed for correct use of the specification-building operators. Next, we discuss the pragmatics of specification-building in  $\mathcal{EVT}$  (Section 4.7).

**4.7. Some pragmatics of Specification Building in  $\mathcal{EVT}$ .** We represent an Event-B specification, that is composed of machines and contexts, as a structured specification that is built from flat specifications (presentations in Definition 2.3) over  $\mathcal{EVT}$  using the specification-building operators. Technically,  $\mathcal{EVT}$  allows for loose specifications which are not possible in Event-B, however, we do not exploit this in our examples that follow.

Recall that, for any signature  $\Sigma$ , a  $\Sigma$ -presentation is a set of  $\Sigma$ -sentences. A model of a  $\Sigma$ -presentation is a  $\Sigma$ -model that satisfies all of the sentences in the presentation [GB92]. Thus, for a presentation in  $\mathcal{EVT}$ , model components corresponding to an event must satisfy all of the sentences specifying that event. This incorporates the standard semantics of the *extends* operator for events in Event-B where the extending event implicitly contains all the parameters, guards and actions of the extended event but can include additional parameters, guards and actions [AH07].

An interesting aspect is that if a variable is not assigned a value, within an action, then a model for the event may associate a new value with this variable. Some languages deal

with this using a *frame condition*, asserting implicitly that values for unmodified variables do not change. In Event-B such a condition would cause complications when combining presentations, since variables unreferenced in one event will be constrained not to change, and this may contradict an action for them in the other event. As far as we can tell, the informal semantics for the Event-B language does not require a frame condition, and we have not included one in our definition.

## 5. A FORMAL SEMANTICS FOR EVENT-B

In the previous sections, we described the Event-B formal specification language and the  $\mathcal{EVT}$  institution which provides the target for our translational semantics. This section is the centrepiece of this paper and shows how we map from the constructs of the Event-B language into the semantic domains given by  $\mathcal{EVT}$ . Our objective is to define a formal semantics for Event-B by representing Event-B specifications as structured specifications over  $\mathcal{EVT}$ , our institution for Event-B [FMP17b]. Our approach is structured around the three constituent sub-languages of Event-B that we identified in Section 3.

**Exploiting the three-layer model:** We utilise our three-layer model of the Event-B language that was presented in Figure 2 in order to translate Event-B into the institutional framework as follows.

- At the base of Figure 2 is the Event-B mathematical language. The institution of first-order predicate logic with equality,  $\mathcal{FOPEQ}$ , is embedded via an institution comorphism (Definition 4.14) into the institution for Event-B,  $\mathcal{EVT}$  [FMP17b]. The semantics that we define translates the constructs of this mathematical language into corresponding constructs over  $\mathcal{FOPEQ}$ .
- At the next level is the Event-B infrastructure, which consists of those language elements used to define variables, invariants, variants and events. These are translated into sentences over  $\mathcal{EVT}$ .
- At the topmost level is the Event-B superstructure which deals with the definition of Event-B machines and contexts, as well as their relationships (**refines**, **sees**, **extends**). These are translated into structured specifications over  $\mathcal{EVT}$  using the specification-building operators.

Closely following the syntax defined in Figure 1, we define the semantics of each of the Event-B language elements by describing a mechanism to translate them into  $\mathcal{EVT}$ -sentences. In order to carry out such a translation we first extract the corresponding  $\mathcal{EVT}$ -signature  $\Sigma = \langle S, \Omega, \Pi, E, V \rangle$  from any given Event-B specification. Contexts can be represented entirely by the underlying mathematical language and thus translated into specifications over  $\mathcal{FOPEQ}$ , and embedded using the comorphism.

In Section 5.1, we define an interface (Figure 8) in order to facilitate the use of some  $\mathcal{FOPEQ}$  operations and functions. We provide functions for extracting the signature of an Event-B specification in Section 5.2. In a bottom-up fashion, Sections 5.3 and 5.4 define the semantics of the Event-B infrastructure and superstructure languages, respectively.



<p><b><i>FOPÉQ</i> Operations</b></p> <ul style="list-style-type: none"> <li>• <b>F.and</b> : <math>\Sigma\text{-formula}^* \rightarrow \Sigma\text{-formula}</math> This corresponds to the logical conjunction (<math>\wedge</math>) of a set of formulae in <i>FOPÉQ</i>.</li> <li>• <b>F.lt</b> : <math>\Sigma\text{-term} \times \Sigma\text{-term} \rightarrow \Sigma\text{-formula}</math> This operation takes two terms and returns a formula corresponding to arithmetic less than (<math>&lt;</math>).</li> <li>• <b>F.leq</b> : <math>\Sigma\text{-term} \times \Sigma\text{-term} \rightarrow \Sigma\text{-formula}</math> This operation takes two terms and returns a formula corresponding to arithmetic less than or equal to (<math>\leq</math>).</li> <li>• <b>F.exists</b> : <math>\text{VarName}^* \times \Sigma\text{-formula} \rightarrow \Sigma\text{-formula}</math> This operation takes a sequence of <i>VarNames</i> and a formula and returns a formula corresponding to the existential quantification of the <i>VarNames</i> over the input formula.</li> <li>• <b>F.ι</b> : <math>\text{VarName}^* \rightarrow \Sigma\text{-formula} \rightarrow \Sigma\text{-formula}</math> This operation takes a list of <i>VarNames</i> and a formula and returns the input formula with the names of all of the free variables (as given by the list of <i>VarNames</i>) primed.</li> </ul> <p><b><i>FOPÉQ</i> Functions</b></p> <ul style="list-style-type: none"> <li>• <math>\mathbb{P}_\Sigma</math> : <math>\text{LabelledPred} \rightarrow \Sigma\text{-formula}</math></li> <li>• <math>\mathbb{T}_\Sigma</math> : <math>\text{Expression} \rightarrow \Sigma\text{-term}</math></li> <li>• <math>\mathbb{M}</math> : <math>\text{SetName}^* \times \text{ConstName}^* \times \text{LabelledPred}^* \rightarrow  \mathbf{Sign}_{\mathcal{FOPÉQ}} </math></li> </ul>
--

Figure 8: The *FOPÉQ* interface provides access to a range of operations and functions which we assume to exist. These are used throughout our definitions in Figures 9, 11 and 12.

**5.1. The *FOPÉQ* Interface.** Just as the Event-B formalism is built on an underlying mathematical language, our institution for Event-B is founded on *FOPÉQ*, the institution for first-order predicate logic with equality. In Figure 8, we define a *FOPÉQ* interface in order to facilitate the use of its operations and functions in our definitions. The description of these operations is outlined in Figure 8 and not specified further.

The function  $\mathbb{P}_\Sigma$  described in Figure 8 takes a labelled predicate and outputs a *FOPÉQ*-sentence ( $\Sigma\text{-formula}$ ). The function  $\mathbb{T}_\Sigma$  takes an expression and returns a  $\Sigma\text{-term}$ . These functions are used to translate Event-B predicates and expressions into  $\Sigma\text{-formulae}$  and  $\Sigma\text{-terms}$  respectively.

The purpose of the function  $\mathbb{M}$  is to take two lists of identifiers and a list of labelled predicates, and use these to form the *FOPÉQ* signature  $\langle S, \Omega, \Pi \rangle$ . The reason for this is that when extracting a signature from a context (described in Figure 9) carrier sets are interpreted as sorts and used to form  $S$ . The constants and axioms are used to form  $\Omega$  and  $\Pi$ . We assume that  $\mathbb{M}$  provides this translation.

For simplicity, we assume that it is possible to use Event-B identifiers in *FOPÉQ* and  $\mathcal{EVT}$ . Also, when we reference a  $\Sigma\text{-formula}$  in Figure 8 we mean a possibly open *FOPÉQ*-formula over the signature given by  $\Sigma$ . We only return a closed *FOPÉQ*-formula when applying  $\mathbb{P}_\Sigma$  to axiom sentences since they form closed predicates in Figure 11.

**5.2. Extracting the Signature.** Before we form sentences in any institution, we must first specify the corresponding signature. To aid this process, we define *Env* to be the type of all environments, which are just mappings from machine/context names to their corresponding

For an Event-B specification  $SP$ , we form an environment  $\xi = \mathbb{D}\llbracket SP \rrbracket \xi_0$  where  $\xi_0$  is the empty environment.

- $Env = (MachineName \cup ContextName) \rightarrow |\mathbf{Sign}|$  # An environment maps names to signatures
- $\mathbb{D} : Specification \rightarrow Env \rightarrow Env$  # Process a list of machine/context definitions
  - $\mathbb{D} \llbracket \langle \rangle \rrbracket \xi = \xi$
  - $\mathbb{D} \llbracket hd :: tl \rrbracket \xi = \mathbb{D} (\llbracket tl \rrbracket) (\mathbb{D} \llbracket hd \rrbracket \xi)$
- $\mathbb{D} : Machine \rightarrow Env \rightarrow Env$  # Extract and store the signature for one machine
  - $\mathbb{D} \left[ \begin{array}{l} \mathbf{machine} \ m \\ \mathbf{refines} \ a \\ \mathbf{sees} \ ctx_1, \dots, ctx_n \\ \mathbf{mbody} \\ \mathbf{end} \end{array} \right] \xi = \xi \cup \{ \llbracket m \rrbracket \mapsto (\langle S, \Omega, \Pi, E, V \rangle \cup r(\xi \llbracket a \rrbracket)) \}$
  - where
    - $\langle S, \Omega, \Pi \rangle = \{ (\xi \llbracket ctx_1 \rrbracket) \cup \dots \cup (\xi \llbracket ctx_n \rrbracket) \}$  # Include signatures from ‘seen’ contexts
    - $\langle E, V, RA \rangle = \mathbb{D} \llbracket mbody \rrbracket$  # Collect names from machine body
  - and
    - $r : |\mathbf{Sign}_{\mathcal{EVT}}| \rightarrow |\mathbf{Sign}_{\mathcal{EVT}}|$  # Signature for the abstract machine (less the refined events)
    - $r(\xi \llbracket a \rrbracket) = \mathbf{let} \ \Sigma_a = \xi \llbracket a \rrbracket \ \mathbf{in} \ \langle \Sigma_a.S, \Sigma_a.\Omega, \Sigma_a.\Pi, RA \triangleleft \Sigma_a.E, \Sigma_a.V \rangle$
- $\mathbb{D} : MachineBody \rightarrow (\mathcal{P}(EventName \times Stat) \times \mathcal{P}(VarName) \times \mathcal{P}(EventName))$  # Extract signature elements from machine-body
  - $\mathbb{D} \left[ \begin{array}{l} \mathbf{variables} \ v_1, \dots, v_n \\ \mathbf{invariants} \ i_1, \dots, i_n \\ \mathbf{theorems} \ t_1, \dots, t_n \\ \mathbf{variant} \ n \\ \mathbf{events} \ e_{init} \ e_1, \dots, e_n \end{array} \right] = \langle E, V, RA \rangle$
  - where
    - $E = \{ def \llbracket e_{init} \rrbracket, def \llbracket e_1 \rrbracket, \dots, def \llbracket e_n \rrbracket \}$  # Names of events defined here
    - $V = \langle \llbracket v_1 \rrbracket, \dots, \llbracket v_n \rrbracket \rangle$  # Names of variables declared here
    - $RA = ref \llbracket e_{init} \rrbracket \cup ref \llbracket e_1 \rrbracket \cup \dots \cup ref \llbracket e_n \rrbracket$  # Names of (abstract) events refined here
- $def : Event \rightarrow (EventName \times Stat)$  # Extract event name & status from an event definition
  - $def \llbracket \mathbf{event} \ e, \mathbf{status} \ s, \mathbf{refines} \ e_1, \dots, e_n, \dots \mathbf{end} \rrbracket = (e \mapsto s)$
  - $def \llbracket e_{init} \rrbracket = \langle \mathbf{Init}, \mathbf{ordinary} \rangle$
- $ref : Event \rightarrow \mathcal{P}(EventName)$  # Extract names of refined events from an event definition
  - $ref \llbracket \mathbf{event} \ e, \mathbf{status} \ s, \mathbf{refines} \ e_1, \dots, e_n, \dots \mathbf{end} \rrbracket = \{ \llbracket e_1 \rrbracket, \dots, \llbracket e_n \rrbracket \}$
  - $ref \llbracket e_{init} \rrbracket = \{ \llbracket e_{init} \rrbracket \}$
- $\mathbb{D} : Context \rightarrow Env \rightarrow Env$  # Extract and store the signature for one context
  - $\mathbb{D} \left[ \begin{array}{l} \mathbf{context} \ ctx \\ \mathbf{extends} \ ctx_1, \dots, ctx_n \\ \mathbf{cbody} \\ \mathbf{end} \end{array} \right] \xi = \xi \cup \{ \llbracket ctx \rrbracket \mapsto (\mathbb{D} \llbracket cbody \rrbracket \cup \xi \llbracket ctx_1 \rrbracket \cup \dots \cup \xi \llbracket ctx_n \rrbracket) \}$
- $\mathbb{D} : ContextBody \rightarrow |\mathbf{Sign}_{\mathcal{FOPEQ}}|$  # Extract the  $\mathcal{FOPEQ}$  signature from a context body
  - $\mathbb{D} \left[ \begin{array}{l} \mathbf{sets} \ s_1, \dots, s_n \\ \mathbf{constants} \ c_1, \dots, c_n \\ \mathbf{axioms} \ a_1, \dots, a_n \\ \mathbf{theorems} \ t_1, \dots, t_n \end{array} \right] = \langle S, \Omega, \Pi \rangle$  # Sorts, operations, predicates
  - where
    - $\langle S, \Omega, \Pi \rangle = \mathbb{M}(\llbracket s_1 \rrbracket, \dots, \llbracket s_n \rrbracket), (\llbracket c_1 \rrbracket, \dots, \llbracket c_n \rrbracket), (\llbracket a_1 \rrbracket, \dots, \llbracket a_m \rrbracket))$

Figure 9: These auxiliary functions are used to extract a signature from an Event-B specification, which is a list of context or machine definitions. The extraction of  $\mathcal{FOPEQ}$ -signatures uses the interface functions described in Figure 8.

```

1  $\Sigma_{m1} = \langle S, \Omega, \Pi, E, V \rangle$ 
2   where
3    $S = \{\mathbb{N}\}$ ,
4    $\Omega = \{0 : \mathbb{N}, d : \mathbb{N}\}$ ,
5    $\Pi = \{> : \mathbb{N} \times \mathbb{N}\}$ ,
6    $E = \{(\text{Init} \mapsto \text{ordinary}), (\text{ML\_in} \mapsto \text{ordinary}), (\text{ML\_out} \mapsto \text{ordinary}),$ 
7          $(\text{IL\_in} \mapsto \text{convergent}), (\text{IL\_out} \mapsto \text{convergent})\}$ ,
8    $V = \{\mathbf{n} : \mathbb{N}, \mathbf{a} : \mathbb{N}, \mathbf{b} : \mathbb{N}, \mathbf{c} : \mathbb{N}\}$ 

```

Figure 10: An example of the signature extraction: this is the result of applying the functions from Figure 9 to the Event-B machine specification of `m1` in Figure 4.

signatures. In all subsequent definitions we use  $\xi$  to denote an environment of type *Env*. We define the overloaded function  $\mathbb{D}$  in Figure 9 to build the environment from a given Event-B specification, which is a list of machine and context definitions.

The function *def* (Figure 9) extracts the function (event name  $\mapsto$  status) for each event in the machine and these functions form the *E* component of the signature. Each event name is tagged with a status in order to correctly form variant sentences which will be discussed in Section 5.3. The function *ref* (Figure 9) forms the set of events that a particular concrete event refines. We use this function to remove the names of the refined abstract events, and the status that each is paired with, from the abstract machine's signature, before we combine it with the concrete signature. We use the domain anti-restriction operator  $\triangleleft$  to express this, where for any set *S* and function *T*, we have

$$S \triangleleft T = \{x \mapsto y \mid x \mapsto y \in T \wedge x \notin S\}$$

In order to illustrate this signature extraction in practice, Figure 10 contains the signature that was extracted using these functions from `m1` in Figure 4. Notice how the variables and events in the signature include those from the abstract machine, `m0` in Figure 3 (lines 8–29), and the constant from the context (lines 1–7) has been included as a 0-ary operator using the *FOPÉQ* interface. We note that, internally, Rodin distinguishes typing and non-typing invariants and we have used the typing invariants on lines 7–9 of Figure 4 and line 13 of Figure 3 to infer the types of the variables, and we assume an appropriate mapping to types in *FOPÉQ*.

Once the environment has been formed, we can then define a systematic translation from specifications in Event-B to structured specifications over  $\mathcal{EVT}$ . We take a bottom-up approach to this translation which comprises two parts.

- The first mapping (in Figure 11) that we define is from the Event-B infrastructure sentences (invariants, variants, events and axioms) to sentences over  $\mathcal{EVT}$  (for invariants, variants and events) and sentences over *FOPÉQ* (for axioms) in Section 5.3.
- The second mapping (in Figure 12) that we provide is from the superstructure components of an Event-B specification to structured specifications over  $\mathcal{EVT}$  (for machines) and structured specifications over *FOPÉQ* (for contexts) in Section 5.4.

**5.3. Defining the Semantics of Event-B Infrastructure Sentences.** In this section, we define a translation from Event-B infrastructure sentences to sentences over  $\mathcal{EVT}$ . We

- $\mathbb{S}_\Sigma : \text{MachineBody} \rightarrow \text{Sen}_{\mathcal{EVT}}(\Sigma)$  # Build sentences from a machine body

$$\mathbb{S}_\Sigma \left[ \begin{array}{l} \text{variables } v_1, \dots, v_n \\ \text{invariants } i_1, \dots, i_n \\ \text{theorems } t_1, \dots, t_n \\ \text{variant } n \\ \text{events } e_{\text{init}}, e_1, \dots, e_n \end{array} \right] = \left( \begin{array}{l} \mathbb{I}_\Sigma[i_1] \cup \dots \cup \mathbb{I}_\Sigma[i_n] \\ \cup \mathbb{V}_\Sigma[n] \\ \cup \mathbb{E}_\Sigma[e_{\text{init}}] \\ \cup \mathbb{E}_\Sigma[e_1] \cup \dots \cup \mathbb{E}_\Sigma[e_n] \end{array} \right)$$
- $\mathbb{I}_\Sigma : \text{LabelledPred} \rightarrow \text{Sen}_{\mathcal{EVT}}(\Sigma)$  # Invariant sentences

$$\mathbb{I}_\Sigma \llbracket i \rrbracket = \{ \langle \llbracket e \rrbracket, \mathbf{F.and}(\mathbb{P}_\Sigma \llbracket i \rrbracket, \mathbf{F.l}(\Sigma.V)(\mathbb{P}_\Sigma \llbracket i \rrbracket)) \rangle \mid e \in \text{dom}(\Sigma.E) \}$$
- $\mathbb{V}_\Sigma : \text{Expression} \rightarrow \text{Sen}_{\mathcal{EVT}}(\Sigma)$  # Variant can't increase for non-ord. events

$$\mathbb{V}_\Sigma \llbracket n \rrbracket = \begin{array}{l} \{ \langle \llbracket e \rrbracket, \mathbf{F.lt}(\mathbf{F.l}(\Sigma.V)(\mathbb{T}_\Sigma \llbracket n \rrbracket), \mathbb{T}_\Sigma \llbracket n \rrbracket) \rangle \mid (e \mapsto \text{convergent}) \in \Sigma.E \} \\ \cup \{ \langle \llbracket e \rrbracket, \mathbf{F.leq}(\mathbf{F.l}(\Sigma.V)(\mathbb{T}_\Sigma \llbracket n \rrbracket), \mathbb{T}_\Sigma \llbracket n \rrbracket) \rangle \mid (e \mapsto \text{anticipated}) \in \Sigma.E \} \end{array}$$
- $\mathbb{E}_\Sigma : \text{InitEvent} \rightarrow \text{Sen}_{\mathcal{EVT}}(\Sigma)$  # Initial event: get sentences from actions

$$\mathbb{E}_\Sigma \left[ \begin{array}{l} \text{event Initialisation} \\ \text{status ordinary} \\ \text{then } act_1, \dots, act_n \\ \text{end} \end{array} \right] = \{ \langle \text{Init}, BA \rangle \}$$

where

$$BA = \mathbf{F.and}(\mathbb{P}_\Sigma \llbracket act_1 \rrbracket, \dots, \mathbb{P}_\Sigma \llbracket act_n \rrbracket)$$
- $\mathbb{E}_\Sigma : \text{Event} \rightarrow \text{Sen}_{\mathcal{EVT}}(\Sigma)$  # Non-initial event: get sentences from event body

$$\mathbb{E}_\Sigma \left[ \begin{array}{l} \text{event } e \\ \text{status } s \\ \text{refines } e_1, \dots, e_n \\ \text{ebody} \\ \text{end} \end{array} \right] = \{ \langle \llbracket e \rrbracket, \mathbb{F}_\Sigma \llbracket ebody \rrbracket \rangle \}$$
- $\mathbb{F}_\Sigma : \text{EventBody} \rightarrow \Sigma\text{-formula}$  # Build a  $\mathcal{FOPEQ}$  formula for an event definition

$$\mathbb{F}_\Sigma \left[ \begin{array}{l} \text{any } p_1, \dots, p_n \\ \text{where } grd_1, \dots, grd_n \\ \text{with } w_1, \dots, w_n \\ \text{then } act_1, \dots, act_n \end{array} \right] = \mathbf{F.exists}(p, \mathbf{F.and}(G, W, BA))$$

# Formula is existentially quantified over event parameters  $p$

where

$$\begin{array}{ll} p = \langle \llbracket p_1 \rrbracket, \dots, \llbracket p_n \rrbracket \rangle & \# \text{ List of parameters} \\ G = \mathbf{F.and}(\mathbb{P}_\Sigma \llbracket grd_1 \rrbracket, \dots, \mathbb{P}_\Sigma \llbracket grd_n \rrbracket) & \# \text{ Guards} \\ W = \mathbf{F.and}(\mathbb{P}_\Sigma \llbracket w_1 \rrbracket, \dots, \mathbb{P}_\Sigma \llbracket w_n \rrbracket) & \# \text{ Witnesses} \\ BA = \mathbf{F.and}(\mathbb{P}_\Sigma \llbracket act_1 \rrbracket, \dots, \mathbb{P}_\Sigma \llbracket act_n \rrbracket) & \# \text{ Actions} \end{array}$$
- $\mathbb{S}_\Sigma : \text{ContextBody} \rightarrow \text{Sen}_{\mathcal{FOPEQ}}(\Sigma)$  # Context: get  $\mathcal{FOPEQ}$  sentences from axioms

$$\mathbb{S}_\Sigma \left[ \begin{array}{l} \text{sets } s_1, \dots, s_n \\ \text{constants } c_1, \dots, c_n \\ \text{axioms } a_1, \dots, a_n \\ \text{theorems } t_1, \dots, t_n \end{array} \right] = \{ \mathbb{P}_\Sigma \llbracket a_1 \rrbracket, \dots, \mathbb{P}_\Sigma \llbracket a_n \rrbracket \}$$

Figure 11: The semantics for the Event-B infrastructure sub-language is provided by translating Event-B machines and contexts into sets of sentences. For machines we generate sentences over  $\mathcal{EVT}$ , denoted  $\text{Sen}_{\mathcal{EVT}}(\Sigma)$ , while for contexts we generate sentences over  $\mathcal{FOPEQ}$ , denoted  $\text{Sen}_{\mathcal{FOPEQ}}(\Sigma)$ . We use the interface operations and functions described in Figure 8 throughout this translation.

translate the axiom sentences that are found in Event-B contexts to sentences over  $\mathcal{FOPEQ}$  as they form part of the underlying Event-B mathematical language as shown in Figure 2. We refer the reader to Section 4.2, where we described how to translate Event-B variants, invariants and events into  $\mathcal{EVT}$ -sentences, and the equations described here mirrors this translation.

We define an overloaded meaning function,  $\mathbb{S}_\Sigma$ , for specifications in Figure 11.  $\mathbb{S}_\Sigma$  takes as input a specification and returns a set of sentences over  $\mathcal{EVT}$  ( $Sen_{\mathcal{EVT}}(\Sigma)$ ) for machines and a set of sentences over  $\mathcal{FOPEQ}$  ( $Sen_{\mathcal{FOPEQ}}(\Sigma)$ ) for contexts. When applying  $\mathbb{S}_\Sigma$  to a machine (resp. context) we also define functions for processing invariants, variants and events (resp. axioms). These are given by  $\mathbb{I}_\Sigma$ ,  $\mathbb{V}_\Sigma$  and  $\mathbb{E}_\Sigma$ . Axioms are predicates that can be translated into closed  $\mathcal{FOPEQ}$ -formulae using the function  $\mathbb{P}_\Sigma$  which is defined in the interface in Figure 8.

Given a list of invariants  $i_1, \dots, i_n$  we define the function  $\mathbb{I}_\Sigma$  in Figure 11. Each invariant,  $i$ , is a *LabelledPred* from which we form the following open  $\mathcal{FOPEQ}$ -sentence:  $\mathbf{F.and}(\mathbb{P}_\Sigma[i], \mathbf{F.i}(\Sigma.V)(\mathbb{P}_\Sigma[i]))$ . Each invariant sentence is paired with each event name in the signature to form the set  $\{(e, \mathbf{F.and}(\mathbb{P}_\Sigma[i], \mathbf{F.i}(\Sigma.V)(\mathbb{P}_\Sigma[i]))) \mid e \in \text{dom}(\Sigma.E)\}$  and this ensures that the invariant is applied to all events when evaluating the satisfaction condition.

The use of conjunction here for invariants ( $\mathbf{F.and}$ ) rather than implication may seem an unusual choice. Typically (e.g. Abrial's INV proof obligation, [Abr10, §5.2.1]), one demands that if the invariant holds in the before-state then it should also hold in the after-state. However, since our sentences are effectively indexed by events, and our semantics are stratified over events, we are always working in a context where a given event has occurred. Thus, in this (localised) context, we specify that the invariant must hold in *both* the before and after states. Since separate sentences are conjoined when grouped, using an implication here would also allow undesirable models when such sentences are combined. Specifically, if implication were used between the invariants in the before and after states then models that may not obey the invariant in the before state would be valid (i.e.  $\text{false} \Rightarrow \text{true}$ ). In our context, this is incorrect because we are assuming that a given event has happened which is only possible if the invariant is true in the before state. Further, we use logical conjunction to combine sentences so that the combination imposes restrictions on the set of allowable models. Including implication causes issues here because it would essentially create a sentence with multiple options (if-then branches) that would be carried through to the models.

Given a variant expression  $n$ , we define the function  $\mathbb{V}_\Sigma$  in Figure 11. The variant is only relevant for specific events so we pair it with an event name in order to meaningfully evaluate the variant expression. An event whose status is **convergent** must strictly decrease the variant expression [Abr10]. An event whose status is **anticipated** must not increase the variant expression. This expression can be translated into an open  $\mathcal{FOPEQ}$ -term using the function  $\mathbb{T}_\Sigma$  as described in Figure 8. From this we form a formula based on the status of the event(s) in the signature  $\Sigma$  as described in Section 5.3. Event-B machines are only permitted to have one variant [ABH<sup>+</sup>10].

In Figure 11 we define the function  $\mathbb{E}_\Sigma$  to process a given event definition. Event guard(s) and witnesses are predicates that can be translated via  $\mathbb{P}_\Sigma$  into open  $\mathcal{FOPEQ}$ -formulae denoted by  $G$  and  $W$  respectively in Figure 11. In Event-B, actions are interpreted as before-after predicates e.g.  $x := x + 1$  is interpreted as  $x' = x + 1$ . Therefore, actions can also be translated via  $\mathbb{P}_\Sigma$  into open  $\mathcal{FOPEQ}$ -formulae denoted by  $BA$  in Figure 11. Thus

the semantics of an *EventBody* definition is given by the function  $\mathbb{F}_\Sigma$  which returns the formula  $\mathbf{F.exists}(p, \mathbf{F.and}(G, W, BA))$  where  $p$  is a list of the event parameters.

A context can exist independently of a machine and is written as a specification over *FOPEQ*. Thus, we translate an axiom sentence directly as a *FOPEQ*-sentence which is a closed  $\Sigma$ -formula using the function  $\mathbb{P}_\Sigma$  given in Figure 8. Axiom sentences are closed *FOPEQ*-formulae (elements of  $\text{Sen}_{\text{FOPEQ}}(\Sigma)$ ) which are interpreted as valid sentences in *EV $\mathcal{T}$*  using the comorphism  $\rho$ .

Figure 11 describes our equations for the Event-B infrastructure constructs. Next we show the relevant equations for the Event-B superstructure constructs.

**5.4. Defining the Semantics of Event-B Superstructure Sentences.** Based on the syntax defined in Figure 1 we have identified the core constructs that form the Event-B superstructure language:

- A context **extends** other contexts
- A machine **sees** contexts and **refines** another (more abstract) machine
- An individual event **refines** other events from the abstract machine

In this section, we define a semantics for the Event-B superstructure language using specification-building operators. In Figure 12 we define the function  $\mathbb{B}$  to translate Event-B specifications written using the superstructure language to structured specifications over *EV $\mathcal{T}$*  that use the specification-building operators defined in the theory of institutions [ST12, §5.1]. We translate a specification as described by Figure 1 into a structured specification over the institution *EV $\mathcal{T}$* .

The construct that enables a context to extend others is used in Event-B to add more details to a context. Since a context in Event-B only refers to elements of the *FOPEQ* component of an *EV $\mathcal{T}$*  signature it is straightforward to translate this using the specification-building operator **then**. As outlined in Table 1, **then** is used to enrich the signature with new sorts/operations etc. [ST12, §5.2].

A context can extend and be extended by more than one context. A context that extends multiple others contains all constants and axioms of the contexts that it is extending and the additional specification that it adds to extend these contexts [ABH<sup>+</sup>10]. To give a semantics for context extension using the specification-building operators we use the **and** operator to merge all extended contexts and the **then** operator to incorporate the extending context itself. The specification-building operator **and** gives the sum of two specifications that can be written over different signatures (Table 1). It is the most straightforward way to combine specifications over different signatures [ST12, §5.2].

In Event-B, a machine **sees** one or more contexts. This construct is used to add a context(s) to a machine so that the machine can refer to elements of the context(s). We know that the relationship between *FOPEQ* and *EV $\mathcal{T}$*  is that of an institution comorphism (Definition 4.14) [FMP17b]. This enables us to directly use *FOPEQ*-sentences, as given by the context in this case, in an *EV $\mathcal{T}$*  specification. We use the specification-building operation **with**  $\rho$  which indicates translation by an institution comorphism  $\rho$  [ST12, §10.4]. The resulting machine specification is heterogeneous as it links two institutions, *EV $\mathcal{T}$*  and *FOPEQ*, by the institution comorphism.

An Event-B machine can refine at most one other machine and there are two types of machine refinement: superposition and data refinement [ABH<sup>+</sup>10]. The specification-building operator, **then**, can account for both of these types of refinement because either

The semantics of an Event-B specification  $SP$  are given by  $\mathbb{B}[\llbracket SP \rrbracket] \xi$ , where  $\xi = \mathbb{D}[\llbracket SP \rrbracket] \xi_0$  is the environment defined by Figure 9.

- $\mathbb{B} : \text{Specification} \rightarrow \text{Env} \rightarrow |\mathbf{Spec}|^*$  # Process specifications in an environment,  
 $\mathbb{B} \llbracket \langle \rangle \rrbracket \xi = \langle \rangle$  build a list of structured specifications  
 $\mathbb{B} \llbracket hd :: tl \rrbracket \xi = (\mathbb{B} \llbracket hd \rrbracket \xi) :: (\mathbb{B} \llbracket tl \rrbracket \xi)$
  
- $\mathbb{B} : \text{Machine} \rightarrow \text{Env} \rightarrow |\mathbf{Spec}_{\mathcal{EVT}}|$  # Build an  $\mathcal{EVT}$  structured specification for one machine  

$$\mathbb{B} \left[ \begin{array}{l} \text{machine } m \\ \text{refines } a \\ \text{sees } ctx_1, \dots, ctx_n \\ \text{mbody} \\ \text{end} \end{array} \right] \xi = \left\langle \Sigma, \left[ \begin{array}{l} \text{spec } \llbracket m \rrbracket \text{ over } \mathcal{EVT} = \\ \# \text{ Include contexts using the comorphism } \rho: \\ (\llbracket ctx_1 \rrbracket \text{ and } \dots \text{ and } \llbracket ctx_n \rrbracket) \text{ with } \rho \\ \# \text{ Sentences from the refined machine (if any):} \\ (\text{and } \mathbb{A}_\Sigma \llbracket mbody \rrbracket \llbracket a \rrbracket \xi) \\ \text{then} \\ \mathbb{S}_\Sigma \llbracket mbody \rrbracket \end{array} \right] \right\rangle$$

where  $\Sigma = \xi \llbracket m \rrbracket$ .
  
- $\mathbb{A}_\Sigma : \text{MachineBody} \rightarrow \text{EventName} \rightarrow \text{Env} \rightarrow |\mathbf{Spec}_{\mathcal{EVT}}|$  # Extract any relevant specification from the refined (abstract) machine  

$$\mathbb{A}_\Sigma \left[ \begin{array}{l} \text{variables } v_1, \dots, v_n \\ \text{invariants } i_1, \dots, i_n \\ \text{theorems } t_1, \dots, t_n \\ \text{variant } n \\ \text{events } e_{init}, e_1, \dots, e_n \end{array} \right] \llbracket a \rrbracket \xi = \mathbb{I}_\Sigma \llbracket i_1 \rrbracket \text{ and } \dots \text{ and } \mathbb{I}_\Sigma \llbracket i_n \rrbracket \\ \text{and } \mathbb{R}_\Sigma \llbracket e_1 \rrbracket \llbracket a \rrbracket \xi \text{ and } \dots \text{ and } \mathbb{R}_\Sigma \llbracket e_n \rrbracket \llbracket a \rrbracket \xi$$

# Conjoin sentences from each event definition
  
- $\mathbb{R}_\Sigma : \text{Event} \rightarrow \text{EventName} \rightarrow \text{Env} \rightarrow |\mathbf{Spec}_{\mathcal{EVT}}|$  # Extract specification from one refined event  

$$\mathbb{R}_\Sigma \left[ \begin{array}{l} \text{event } e_c \\ \text{status } s \\ \text{refines } e_1, \dots, e_n \\ \text{ebody} \\ \text{end} \end{array} \right] \llbracket a \rrbracket \xi =$$

let  
 $\Sigma_a = \xi \llbracket a \rrbracket$ , # Signature of abstract machine  
# Use  $\Sigma_h, \sigma_h$  to select only refined events:  
 $\Sigma_h = \langle \Sigma_a.S, \Sigma_a.\Omega, \Sigma_a.\Pi, \{ \llbracket e_1 \rrbracket, \dots, \llbracket e_n \rrbracket \} \triangleleft \Sigma_a.E, \Sigma_a.V \rangle$ ,  
 $\sigma_h : \Sigma_h \hookrightarrow \Sigma_a$ ,  
# Use  $\sigma_m$  to reassign refined event sentences to  $e_c$ :  
 $\sigma_m : \Sigma_h \rightarrow \Sigma$   
 $\sigma_m = \left\langle \begin{array}{l} \Sigma_h.S \hookrightarrow \Sigma.S, \Sigma_h.\Omega \hookrightarrow \Sigma.\Omega, \Sigma_h.\Pi \hookrightarrow \Sigma.\Pi, \\ \Sigma_h.E \mapsto \{ \llbracket e_c \rrbracket \} \triangleleft \Sigma.E, \\ \Sigma_h.V \hookrightarrow \Sigma.V \end{array} \right\rangle$

in  
 $(\llbracket a \rrbracket \text{ hide via } \sigma_h) \text{ with } \sigma_m$ .
  
- $\mathbb{B} : \text{Context} \rightarrow \text{Env} \rightarrow |\mathbf{Spec}_{\mathcal{FOPEQ}}|$  # Build a  $\mathcal{FOPEQ}$  structured specification for one context  

$$\mathbb{B} \left[ \begin{array}{l} \text{context } ctx \\ \text{extends } ctx_1, \dots, ctx_n \\ \text{cbody} \\ \text{end} \end{array} \right] \xi = \left\langle \Sigma, \left[ \begin{array}{l} \text{spec } \llbracket ctx \rrbracket \text{ over } \mathcal{FOPEQ} = \\ \llbracket ctx_1 \rrbracket \text{ and } \dots \text{ and } \llbracket ctx_n \rrbracket \\ \text{then} \\ \mathbb{S}_\Sigma \llbracket cbody \rrbracket \end{array} \right] \right\rangle$$

where  $\Sigma = \xi \llbracket ctx \rrbracket$ .

Figure 12: The semantics for the Event-B superstructure sub-language is defined by translating Event-B specifications into structured specifications over  $\mathcal{EVT}$  using the function  $\mathbb{B}$  and the specification-building operators defined in the theory of institutions (Table 1). Note that  $\mathbf{Spec}$  denotes the category of specifications as per [ST12, §5.5].

new signature components or constraints on the data (gluing invariants) are added to the specification. In Figure 12 the function  $\mathbb{A}_\Sigma$  is used to process the events in the concrete machine which refine those in the abstract machine.

Event refinement in Event-B is superposition refinement [ABH<sup>+</sup>10]. By superposition refinement all of the components of the corresponding abstract event are implicitly included in the refined version. This approach is useful for gradually adding more detail to the event. In  $\mathcal{EVT}$ , we have not prohibited multiple definitions of the same event name. When there are multiple definitions we combine them by taking the conjunction of their respective formulae which will constrain the model. As mentioned above, when refining an abstract machine we use the function  $\mathbb{A}_\Sigma$  in Figure 12 to process the events in the concrete machine which refine those in the abstract machine.  $\mathbb{A}_\Sigma$  in turn calls the function  $\mathbb{R}_\Sigma$  (Figure 12).

$\mathbb{R}_\Sigma$  restricts the event component of the abstract machine signature to those events contained in the `refines` clause of the event definition using domain restriction ( $\langle \cdot \rangle$ ).  $\mathbb{A}_\Sigma$  also extracts the invariant sentences from the abstract machine. This new signature is included in the abstract machine via the signature morphism  $\sigma_h$ . We then form the signature morphism  $\sigma_m$  by letting it be the identity on the sort, operation, predicate and variable components of  $\Sigma_h$ . For events,  $\sigma_m$  maps each of the abstract event signature components that are being refined by the concrete event  $e_c$  to the signature component corresponding to  $e_c$ . The resulting specification uses `hide via` and `with` to apply these signature morphisms ( $\sigma_h$  and  $\sigma_m$ ) correctly. Note that we could have used the `reveal` specification-building operator rather than `hide via` to achieve the same result [ST12, §5.2].

Thus, we have formalised an institution-based semantics for the three layers of the Event-B language. In previous empirical work, we discovered that the Event-B language is lacking a well-defined set of modularisation constructs [FMP17c]. It is clear that the specification-building operators available in the theory of institutions can provide a solution to this. We validate our semantics and show how these specifications can be modularised using the specification-building operators in the next sections.

## 6. THE SEMANTICS IN ACTION: A WORKED EXAMPLE

In order to validate our semantics we developed a tool to take Event-B specifications and generate the corresponding  $\mathcal{EVT}$ -specifications, following the definitions given in Section 5. The tool is called `EB2EVT` and takes as input an Event-B specification in the XML format used by Rodin, and generates as output the  $\mathcal{EVT}$ -specifications, syntactically sugared in a HETS-like notation. `EB2EVT` is written in Haskell [Tho99], a choice that allows our code to closely mirror the function definitions, and is compatible with the HETS code-base. Specifically, HETS facilitates heterogeneous specification and proof using formalisms that have been encoded as institutions. To illustrate our approach, Figure 13 shows the associated Haskell code snippet of the  $\mathbb{F}$  function from Figure 11. The reader need only refer back to the definition of  $\mathbb{F}_\Sigma$  in Figure 11 to see their correspondence. Since refinement is central to systems developed using Event-B, we demonstrate how a chain of Event-B refinement steps is captured using our translational semantics. We illustrate this via the application of our `EB2EVT` tool to the running example in this paper.

**6.1. The Abstract Model.** We illustrate the use of `EB2EVT` using the cars on a bridge example as introduced in Section 3. This Event-B specification has three refinement steps, resulting in four machines and three contexts. The final refinement step results in quite



```

1 bbEventDefines :: [(Ident, Ident)] -> [EventDef] -> [Sentence]
2 bbEventDefines xx' evtList =
3   let pp eBody = (eany eBody)
4       gg eBody = F.and (ewhere eBody)
5       ww eBody = F.and (ewith eBody)
6       ba eBody = F.and (ethen eBody)
7       bbEvent eBody = (F.exists (xx' ++(pp eBody)) (F.and [gg eBody, ww eBody, ba eBody]))
8       bbEventDef evt = (Sentence (ename evt) (bbEvent (ebody evt)))
9   in (map bbEventDef evtList)

```

Figure 13: The Haskell implementation of the  $\mathbb{F}$  function from Figure 11 as applied to a list of Event-B event definitions.

a large specification and so we omit it here but more details can be found in [Far17]. We present the resultant  $\mathcal{EVT}$ -specification corresponding to each of the first two refinement steps.

Figure 3 introduced an abstract Event-B specification and using our EB2EVT translator tool we obtain the corresponding  $\mathcal{EVT}$ -specification that is presented in Figure 14.

**Lines 1–5:** This is a  $CASL$ -specification that describes the context from Figure 3. Note that  $FOPEQ$  is a sub-logic of  $CASL$  [Mos04] and we use  $CASL$  in its place here because it is available in HETS [MML07]. The constant  $d$  is represented as an operation of the appropriate type and the non-typing axioms are included as predicates. This is achieved via the function  $\mathbb{P}$  that was defined in the  $FOPEQ$  interface (Figure 8).

**Lines 6–20:** This is the  $\mathcal{EVT}$ -specification that is generated corresponding to the abstract machine in Figure 3 (lines 8–29). The machine variable,  $n$ , is represented as an operation and the context specification is included using the **then** specification-building operator on lines 7 and 8. This corresponds to the application of the function  $\mathbb{B}$  (Figure 12) to the superstructure components of the machine.

With regard to the infrastructure translation described in Figure 11, note that the invariant and event sentences have been syntactically sugared in order to simplify the notation. In this and future examples, we use **thenAct** in place of the **then** Event-B keyword to distinguish it from the **then** specification-building operator.

**6.2. The First Refinement.** In Figure 4 we presented a refinement of the abstract specification that was introduced in Figure 3. Using our EB2EVT translator tool, we obtain the  $\mathcal{EVT}$ -specification corresponding to this machine as shown in Figure 15.

**Lines 1–3:** The application of  $\mathbb{B}$  to the **refines** and **sees** clauses of Figure 4 (lines 2–3) results in the application of the specification-building operators (**and**, **then**) to include the abstract machine and context specifications into the generated  $\mathcal{EVT}$ -specification.

**Lines 4–9:** The new variables and (non-theorem) invariants are included as operations and predicates respectively. For readability, we use a simple notation for the variant expression on line 9. The reader should understand that the semantics of this is evaluated as described in Figure 11.

**Lines 10–31:**  $\mathbb{R}_\Sigma$  returns (**mo hide via**  $\sigma_h$ ) **with**  $\sigma_m$  for each refined event. The event names have remained the same during event refinement in this example, and since events with the same name are implicitly merged we can omit this notation here.

```

1 spec CD =
2   sort  $\mathbb{N}$ 
3   ops d: $\mathbb{N}$ 
4   . d > 0
5 end

6 spec M0 =
7   CD
8   then
9     ops n: $\mathbb{N}$ 
10    . n ≤ d
11    EVENTS
12    Initialisation
13    thenAct n := 0
14    Event ML_out  $\hat{=}$  ordinary
15    when n < d
16    thenAct n := n+1
17    Event ML_in  $\hat{=}$  ordinary
18    when n > 0
19    thenAct n := n-1
20 end

```

Figure 14: A structured specification in the  $\mathcal{EVT}$  institution (syntactically sugared) as generated by EB2EVT. This specification corresponds to the Event-B specification of the abstract machine given in Figure 3.

```

1 spec M1 =
2   M0 and CD
3   then
4     ops a: $\mathbb{N}$ 
5         b: $\mathbb{N}$ 
6         c: $\mathbb{N}$ 
7     . n = a + b + c
8       a = 0  $\vee$  c = 0
9     variant 2 * a + b
10    EVENTS
11    Initialisation
12    thenAct a := 0
13           b := 0
14           c := 0
15    Event ML_out  $\hat{=}$  ordinary
16    when a + b < d
17           c = 0
18    thenAct a := a+1
19    Event IL_in  $\hat{=}$  convergent
20    when a > 0
21    thenAct a := a-1
22           b := b+1
23    Event IL_out  $\hat{=}$  convergent
24    when 0 < b
25           a = 0
26    thenAct b := b-1
27           c := c+1
28    Event ML_in  $\hat{=}$  ordinary
29    when c > 0
30    thenAct c := c-1
31 end

```

Figure 15: An  $\mathcal{EVT}$  specification corresponding to the results of the first refinement step, as represented by the Event-B specification in Figure 4.

**6.3. The Second Refinement.** Figure 16 contains an Event-B specification that shows the second refinement step, which adds a substantial amount of new behaviour to the specification. The first addition is a context (lines 1–8) containing colours which provide values for the new variables (`ml.t1` and `il.t1`). These are used to control the behaviour of a pair of traffic lights (line 12). The new variables `ml_pass` and `il_pass` on line 13 are used to record whether or not a car has passed through the mainland or island traffic light respectively. The objective of these variables is to ensure that the traffic lights do not change so rapidly that no car can pass through them [Abr10]. Events for these lights were added to the machine and the current events were modified to account for this behaviour. In

```

1  CONTEXT Color
2  SETS Color
3  CONSTANTS
4    red, green
5  AXIOMS
6    axm4: Color = {green, red}
7    axm3: green ≠ red
8  END

9  MACHINE m2
10 REFINES m1
11 SEES cd, Color
12 VARIABLES a, b, c, ml_tl, il_tl,
13   il_pass, ml_pass
14 INVARIANTS
15   inv1: ml_tl ∈ {red, green}
16   inv2: il_tl ∈ {red, green}
17   inv3: ml_tl = green ⇒ c = 0
18   inv12: ml_tl = green ⇒ a + b + c < d
19   inv4: il_tl = green ⇒ a = 0
20   inv11: il_tl = green ⇒ b > 0
21   inv6: il_pass ∈ {0, 1}
22   inv7: ml_pass ∈ {0, 1}
23   inv8: ml_tl = red ⇒ ml_pass = 1
24   inv9: il_tl = red ⇒ il_pass = 1
25   inv5: il_tl = red ∨ ml_tl = red
26   thm2: 0 ≥ a ⇒ a = 0 theorem
27   thm3: 0 ≥ b ⇒ b = 0 theorem
28   thm4: 0 ≥ c ⇒ c = 0 theorem
29   thm5: ¬(d ≤ 0) theorem
30   thm6: b + 1 ≥ d ∧ ¬(b + 1 = d)
31         ⇒ ¬(b < d) theorem
32   thm7: b ≤ 1 ∧ ¬(b = 1)
33         ⇒ ¬(b > 0) theorem
34   thm1: (ml_tl = green ∧ a + b + 1 < d)
35         ∨ (ml_tl = green ∧ a + b + 1 = d)
36         ∨ (il_tl = green ∧ b > 1)
37         ∨ (il_tl = green ∧ b = 1)
38         ∨ (ml_tl = red ∧ a + b < d
39           ∧ c = 0 ∧ il_pass = 1)
40         ∨ (il_tl = red ∧ 0 < b ∧ a = 0
41           ∧ ml_pass = 1)
42         ∨ 0 < a ∨ 0 < c theorem
43 VARIANT ml_pass + il_pass
44 EVENTS
45   Initialisation
46     then act2: a := 0
47           act3: b := 0
48           act4: c := 0
49           act1: ml_tl := red
50           act5: il_tl := red
51           act6: ml_pass := 1
52           act7: il_pass := 1
53
54   Event ML_out1 ≐ ordinary
55     REFINES ML_out
56     when grd1: ml_tl = green
57           grd2: a + b + 1 < d
58     then act1: a := a + 1
59           act2: ml_pass := 1
60
61   Event ML_out2 ≐ ordinary
62     REFINES ML_out
63     when grd1: ml_tl = green
64           grd2: a + b + 1 = d
65     then act1: a := a + 1
66           act2: ml_tl := red
67           act3: ml_pass := 1
68
69   Event IL_out1 ≐ ordinary
70     REFINES IL_out
71     when grd1: il_tl = green
72           grd2: b > 1
73     then act1: b := b - 1
74           act2: c := c + 1
75           act3: il_pass := 1
76
77   Event IL_out2 ≐ ordinary
78     REFINES IL_out
79     when grd1: il_tl = green
80           grd2: b = 1
81     then act1: b := b - 1
82           act2: il_tl := red
83           act3: c := c + 1
84           act4: il_pass := 1
85
86   Event ML_tl_green ≐ convergent
87     when grd1: ml_tl = red
88           grd2: a + b < d
89           grd3: c = 0
90           grd4: il_pass = 1
91     then act1: ml_tl := green
92           act2: il_tl := red
93           act3: ml_pass := 0
94
95   Event IL_tl_green ≐ convergent
96     when grd1: il_tl = red
97           grd2: 0 < b
98           grd3: a = 0
99           grd4: ml_pass = 1
100    then act1: il_tl := green
101          act2: ml_tl := red
102          act3: il_pass := 0
103
104   Event IL_in ≐ ordinary
105     REFINES IL_in
106     when grd11: 0 < a
107     then act11: a := a - 1
108           act12: b := b + 1
109
110   Event ML_in ≐ ordinary
111     REFINES ML_in
112     when grd1: 0 < c
113     then act1: c := c + 1
114
115 END

```

Figure 16: An Event-B specification of a machine `m2` that refines the Event-B machine in Figure 4 by adding new events `ML_tl_green` and `IL_tl_green`. The context `Color` on lines 1–8 adds a new data type which is used by the `ml_tl` and `il_tl` traffic light variables.

```

1 spec COLOR =
2   sorts Color
3   ops red: Color
4     green: Color
5   . Color = {green,red}
6     green ≠ red
7 end

8 spec M2 =
9   M1 and cd and COLOR and
10  (M1 hide via  $\sigma_h$  ML.out) with  $\sigma_m$  ML.out $\rightarrow$ ML.out1 and
11  (M1 hide via  $\sigma_h$  ML.out) with  $\sigma_m$  ML.out $\rightarrow$ ML.out2 and
12  (M1 hide via  $\sigma_h$  IL.out) with  $\sigma_m$  IL.out $\rightarrow$ IL.out1 and
13  (M1 hide via  $\sigma_h$  IL.out) with  $\sigma_m$  IL.out $\rightarrow$ IL.out2
14  then
15    ops ml.tl : {red,green}
16      il.tl : {red,green}
17      il.pass : {0,1}
18      ml.pass : {0,1}
19    . ml.tl=green  $\Rightarrow$  c=0
20      ml.tl=green  $\Rightarrow$  a+b+c<d
21      il.tl=green  $\Rightarrow$  a=0
22      il.tl=green  $\Rightarrow$  b>0
23      ml.tl=red  $\Rightarrow$  ml.pass=1
24      il.tl=red  $\Rightarrow$  il.pass=1
25      il.tl=red  $\vee$  ml.tl=red
26  variant ml.pass+il.pass
27  EVENTS
28    Initialisation
29      thenAct a := 0
30              b := 0
31              c := 0
32              ml.tl := red
33              il.tl := red
34              ml.pass := 1
35              il.pass := 1
36  Event ML.out1  $\hat{=}$  ordinary
37    when ml.tl=green
38      a+b+1<d
39    thenAct a := a+1
40           ml.pass := 1

41  Event ML.out2  $\hat{=}$  ordinary
42    when ml.tl=green
43      a+b+1=d
44    thenAct a := a+1
45           ml.tl := red
46           ml.pass := 1
47  Event IL.out1  $\hat{=}$  ordinary
48    when il.tl=green
49      b>1
50    thenAct b := b-1
51           c := c+1
52           il.pass := 1
53  Event IL.out2  $\hat{=}$  ordinary
54    when il.tl=green
55      b=1
56    thenAct b := b-1
57           il.tl := red
58           c := c+1
59           il.pass := 1
60  Event ML.tl.green  $\hat{=}$  anticipated
61    when ml.tl=red
62      a+b<d
63      c=0
64      il.pass=1
65    thenAct ml.tl := green
66           il.tl := red
67           ml.pass := 0
68  Event IL.tl.green  $\hat{=}$  anticipated
69    when il.tl=red
70      0<b
71      a=0
72      ml.pass=1
73    thenAct il.tl := green
74           ml.tl := red
75           il.pass := 0
76  Event IL.in  $\hat{=}$  ordinary
77    when 0<a
78    thenAct a := a-1
79           b := b+1
80  Event ML.in  $\hat{=}$  ordinary
81    when 0<c
82    thenAct c := c-1
83 end

```

Figure 17: A structured specification in the  $\mathcal{EVT}$  institution corresponding to the Event-B specification of Figure 16, representing the results of the second refinement step.

particular, each of the abstract events (ML.out and IL.out in Figure 4) are refined by two events (ML.out1, ML.out2, IL.out1 and IL.out2 in Figure 16).

As can be seen from Figure 16, there is a good degree of repetition in the Event-B specification resulting from common cases for the numeric variables. These can be represented much more succinctly as a structured specification in our institution as shown in Figure 17.

**Lines 1–7:** This *CASL* specification specifies the new data type *Color*. This corresponds to the context described in the Event-B model in Figure 16.

**Lines 8–14:** The superstructure component of the translation includes the abstract machine (m0), and the contexts (cd and COLOR). In this case the names of the events are changed

in the refinement step. The event `ML_out` is decomposed into the events `ML_out1` and `ML_out2`. The event `IL_out` follows a similar decomposition into `IL_out1` and `IL_out2`. We account for this refinement using the signature morphisms and specification-building operators on lines 10–13. We have not explicitly described these signature morphisms but their functionality should be obvious from their respective subscripts.

**Line 26:** As before, we use a simple variant notation for the variant expression.

**Lines 27–83:** The events are translated as before for the remainder of the specification.

This example illustrates the use of `EB2EVT` as a means for bridging the gap between the Rodin and HETS software eco-systems. The source code for `EB2EVT` along with more examples of its use can be found on our git repository<sup>1</sup>.

## 7. ENHANCED REFINEMENT AND MODULARISATION IN $\mathcal{EVT}$

Our formalisation of Event-B in terms of the theory of institutions offers a number of benefits. In this section, we provide a discussion of refinement in this setting and how we can exploit the use of specification-building operators for modularisation of Event-B specifications.

**7.1. Representing Refinement Explicitly.** As is evidenced by our running example and Section 3.1, refinement is a central aspect of the Event-B methodology, therefore any formalisation of Event-B must be capable of capturing refinement. The theory of institutions equips us with a basic notion of refinement as *model-class inclusion* where the class of models of a specification comprises the models that satisfy the specification. We formally define what is meant by *model-class* in Definition 7.1.

**Definition 7.1** (Model-Class). For any  $\Phi \subseteq \mathbf{Sen}(\Sigma)$  of  $\Sigma$ -sentences, the class  $Mod_{\Sigma}(\Phi) \subseteq |\mathbf{Mod}(\Sigma)|$  of models of  $\Phi$  is defined as the class of all  $\Sigma$ -models that satisfy all the sentences in  $\Phi$ .

Note that we omit the  $\Sigma$  subscript where the signature is clear from the context. With regard to institution-theoretic refinement, the class of models of the concrete specification is a subset of the class of models of the abstract specification [ST12, §7.1].

We consider two cases: (1) when the signatures are the same and (2) when the signatures are different. In the case where the signatures are the same, refinement is denoted as:

$$SP_A \sqsubseteq SP_C \iff Mod(SP_C) \subseteq Mod(SP_A)$$

where  $SP_A$  is an abstract specification that refines ( $\sqsubseteq$ ) to a concrete specification  $SP_C$ ,  $Mod(SP_A)$  and  $Mod(SP_C)$  denote the class of models of the abstract and concrete specifications, respectively. This means that, given specifications with the same signature, the concrete specification should not exhibit any model that was not possible in the abstract specification. This equates to the definition of general refinement outlined in [RS08a, RS08b].

In terms of Event-B, data refinement is not supported in this way because it would involve a change of signature. Instead, refinement occurs between machines by strengthening the invariants and the guards but does not provide any new variables or events. In this case, the specification is made more deterministic by further constraining the invariants and guards. This increase in determinism is supported by the refinement calculi [Bac90, BvW98, MRG88, Mor87].

<sup>1</sup><https://github.com/mariefarrell/EB2EVT.git>

```

1 refinement REF0 : M0 to M1 =
2   ML_in ↦ ML_in, ML_out ↦ ML_out
3 end

4 refinement REF1A : M1 to M2 =
5   ML_in ↦ ML_in, ML_out ↦ ML_out1,
6   IL_in ↦ IL_in, IL_out ↦ IL_out1
7 end

8 refinement REF1B : M1 to M2 =
9   ML_in ↦ ML_in, ML_out ↦ ML_out2,
10  IL_in ↦ IL_in, IL_out ↦ IL_out2,
11 end

```

Figure 18: A specification in the  $\mathcal{EVT}$  institution that explicitly defines the refinement relationships between the abstract machine and its two concrete refinements. Note that, for readability, we omit the status mappings since all statuses are ordinary in this case and remain as so after signature morphism.

When the signatures are different and related by a signature morphism  $\sigma : \text{Sig}[SP_A] \rightarrow \text{Sig}[SP_C]$  then we can use the corresponding model morphism in order to express refinement. This model morphism is used to interpret the concrete specification as containing only the signature items from the abstract specification. Here, refinement is the model-class inclusion of the models of the concrete specification restricted into the class of models of the abstract specification using the model morphism. In this case we write:

$$SP_A \sqsubseteq SP_C \iff \text{Mod}(\sigma)(SP_C) \subseteq \text{Mod}(SP_A)$$

where  $\text{Mod}(\sigma)(SP_C)$  is the model morphism applied to the model-class of the concrete specification  $SP_C$ . This interprets each of the models of  $SP_C$  as models of  $SP_A$  before a refinement relationship is determined. Note that we could have alternatively represented this kind of refinement using the **hide via** specification-building operator to transform  $SP_C$  into a specification over the signature  $\text{Sig}[SP_A]$  using the appropriate morphism.

In this scenario, both data refinement and superposition refinement are supported directly in  $\mathcal{EVT}$ , since the refined or added variables and events can be removed, using a suitable model morphism as outlined above, before refinement is completed. Schneider et al. used a similar notion of hiding when they provided a CSP semantics for Event-B refinement [STW14].

Figure 18 uses the refinement syntax available in HETS to specify each of the refinement steps in the cars on a bridge example. Note that this allows us to express the refinement as a separate entity, whereas in Event-B the refinement step is woven into the definition of the refined machine. In Figure 18, lines 1–3 define the refinement step between the abstract specification  $m0$  and the concrete specification  $m1$ . In this step the events are refined but no renaming or statuses are changed so it is very straightforward. Lines 4–11 define the refinement step between the specification  $m1$  and the concrete specification  $m2$ . There are two separate refinements (REF1A and REF1B) because the events  $ML\_out$  and  $IL\_out$  are split into two further events each as presented in Figure 17.

The example that we presented in this paper does not make use of data refinement in Event-B but if it did, it would also be possible to capture this kind of refinement in the HETS-style notation above. An example of this was presented in our previous work [FMP17b].

**7.2. Modularisation using Specification-Building Operators.** During the evolution of the Event-B formalism from Classical-B, certain facilities for the reuse of machine specifications disappeared. These include the modularisation properties supplied by the keywords `INCLUDES` and `USES` which facilitated the use of an existing machine in other developments [SB09]. Since then there have been numerous attempts at regaining such modularisation features [SB12, GPS09, GPB11, HISW11, But09, Pop08]. Modularisation in Event-B does not require re-engineering the language in the same way as Object-Z or VVSL, but rather, modifications are made by building plugins for Rodin.

Modularisation, in terms of decomposition, in Event-B was first described by Abrial as the act of cutting a large system of events into smaller pieces that can each be refined independently of the others [AH07]. This offers the advantage of separation of concerns even though there still needs to be a link between the parts. The central idea is that the specifications constructed via decomposition and further refinement may be easily recombined resulting in a specification that could have been obtained without using any decomposition techniques in the first place. Three primary types of Event-B decomposition have been identified [HISW11]. These are *shared variable* [Abr09], *shared event* [But09] and *modularisation* [ITL<sup>+</sup>10]. Another related approach is *generic instantiation* [AH07].

It is clear, from observations of industrial projects and the sheer volume of Rodin plugins developed for modularisation in Event-B, that there is an underlying demand for modularisation [ITL<sup>+</sup>10, HISW11]. In previous work, we assembled and analysed a corpus of Event-B specifications in light of their metrics and the incidence of specification clones throughout the corpus [FMP17c]. This work supported the claim that the Event-B formal specification language is in need of a set of well-defined modularisation constructs and we postulated that the specification-building operators of the theory of institutions could fill this void.

We have used the specification-building operators throughout our definitions in Section 5 as a way of naturally adding modularisation to these specifications and this shines through in the example that we have shown above. Notably, we can use these specification-building operators to further modularise the specifications in our illustrative example. In particular, the  $\mathcal{EVT}$ -specification in Figure 4 could be further modularised as illustrated in Figure 19.

**Lines 1–20:** The specifications `DATAM0` (lines 1–8) and `DATAM1` (lines 9–20) capture the data parts of the `m0` and `m1` specifications and their `Initialisation` events (see Figures 14 and 15 for correspondence).

**Lines 21–30:** Since the `ML_in` and `ML_out` events are quite similar, modulo renaming of variables, to the `IL_in` and `IL_out` events we construct the `INOUT` specification. This defines an `out` (lines 24–26) and an `in` event (lines 27–29) using natural number variables `v1` and `v2` (introduced on line 22).

**Lines 31–48:** This is a modular specification of `m1` that instantiates two copies of the `INOUT` specification with the appropriate signature morphisms mapping the `out` event to `ML_out` in the first instance and to `IL_out` in the second. Correspondingly, `in` maps to `ML_in` and then to `IL_in`. Further to this, the sentences after the `then` keyword

```

1 spec DATAM0 =
2   cd then
3     ops n:ℕ
4     . n ≤ d
5     EVENTS
6     Initialisation
7     thenAct n := 0
8 end

9 spec DATAM1 =
10  DATAM0 then
11    ops a, b, c : ℕ
12    . n = a + b + c
13    a = 0 ∨ c = 0
14    variant 2*a+b
15    EVENTS
16    Initialisation
17    thenAct a := 0
18    b := 0
19    c := 0
20 end

21 spec INOUT =
22   ops v1, v2 : ℕ
23   EVENTS
24   Event out ≐ ordinary
25   when v1 = 0
26   thenAct v2 := v2 + 1
27   Event in ≐ ordinary
28   when v1 > 0
29   thenAct v1 := v1 + 1
30 end

31 spec M1 =
32  DATAM1 and M0 and
33  INOUT with {⟨out, ordinary⟩ ↦ ⟨ML.out, ordinary⟩,
34             ⟨in, ordinary⟩ ↦ ⟨ML.in, ordinary⟩,
35             v1 ↦ c, v2 ↦ a} and
36  INOUT with {⟨out, ordinary⟩ ↦ ⟨IL.out, convergent⟩,
37             ⟨in, ordinary⟩ ↦ ⟨IL.in, convergent⟩,
38             v1 ↦ a, v2 ↦ c}
39  then
40  EVENTS
41  Event ML.out ≐ ordinary
42  when a + b < d
43  Event IL.in ≐ convergent
44  thenAct b := b + 1
45  Event IL.out ≐ convergent
46  when 0 < b
47  thenAct b := b + 1
48 end

```

Figure 19: A structured specification in the  $\mathcal{EVT}$  institution that is an even more modular version of the structured specification in Figure 15.

(lines 40–48) add extra behaviour to these events that was not made explicit in the `INOUT` specification.

This specification is evidently more modular than the original version by modelling the simple behaviour of coming in and going out in a separate specification and then instantiating it twice with the appropriate events and variables renamed. This kind of modularisation of specifications is currently not available in the Event-B formal specification language, but our  $\mathcal{EVT}$  institution allows us to enhance Event-B with these powerful modularisation constructs.

As already mentioned, a number of plugins exist with the objective of providing modularisation features for Event-B. However, although all of these plugins offer some form of modularisation for Event-B specifications, it is not clear how specifications developed utilising more than one of these plugins could be combined in practice, or, if this is even possible.

**7.3. Capturing Current Event-B Modularisation.** In this subsection, we illustrate how the functionality of a selection of the current Rodin modularisation plugins can be captured using the specification-building operators and parametrisation made available in the theory of institutions using  $\mathcal{EVT}$ . Most of these plugins embody a shared variable or shared event style of modularisation (e.g [SB12, But09, ITL<sup>+</sup>10]), and this section demonstrates



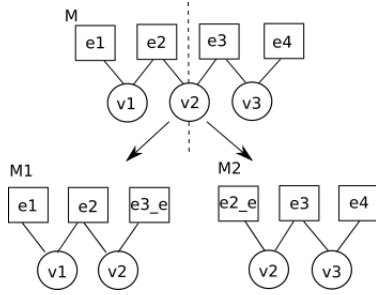


Figure 20: The *shared variable* decomposition of machine M into sub-machines M1 and M2. Here we wish to partition machine M into two sets of events, as shown by the dotted line.

```

1 spec M1 =
2   (M hide via  $\sigma_1$ )
3   with {e3  $\mapsto$  e3.e}
4 end
5 where  $\sigma_1 = \{v1 \mapsto v1, v2 \mapsto v2, e1 \mapsto e1,$ 
6          $e2 \mapsto e2, e3 \mapsto e3\}$ 
7 spec M2 =
8   (M hide via  $\sigma_2$ )
9   with {e2  $\mapsto$  e2.e}
10 end
11 where  $\sigma_2 = \{v2 \mapsto v2, v3 \mapsto v3, e2 \mapsto e2,$ 
12               $e3 \mapsto e3, e4 \mapsto e4\}$ 

```

Figure 21: Representing the *shared variable* modularisation used in Figure 20 using specification-building operators.

how our semantics provides a theoretical foundation for current modularisation approaches in Event-B. A more thorough analysis of the current modularisation approaches available in the Rodin Platform can be found in [Far17].

**7.3.1. Shared Variable and Shared Event Decomposition.** Modularisation is straightforward when the events in a machine each reference their own, separate set of variables. However, the more common situation is for events to share variables, as is illustrated by the machine M at the top of Figures 20 and 22. In machine M, each of the three variables  $\{v1, v2, v3\}$  is referenced by two different events (we have omitted the event status and variable sorts from this example to increase clarity). In each case we wish to split machine M into two equivalent sub-machines M1 and M2, and Figures 20 and 22 show two different approaches to achieving this in Event B.

Figure 20 illustrates the *shared variable* approach, where an Event-B machine is decomposed into sub-machines by grouping events which share the same variables [AH07]. Since this may not produce a partition, the user must choose which events to allocate to a sub-machine, e.g., in Figure 20 we allocate events e1 and e2 to sub-machine M1 and events e3 and e4 to sub-machine M2. Since e2 and e3 both refer to v2 in Figure 20, ‘external’ events e3\_e and e2\_e are added to abstractly describe the behaviour of the shared variable in the other sub-machine.

Figure 21 recasts the shared variable example of Figure 20 using specification-building operators. The specification-building operator **hide via**, used on lines 2 and 8 of Figure 21, is used to hide auxiliary signature items (variables and events in this case) by means of omitting them from the signature morphisms  $\sigma_1$  and  $\sigma_2$ . The two external events are created using the signature morphisms defined in lines 3 and 9 of Figure 21.

In contrast, Figure 22 illustrates *shared event* decomposition, which partitions an Event-B machine based on variables which participate in the same events [AH07]. In this example, a clean partition is not possible, and the user has chosen to allocate variable v1 to sub-machine M1 and variables v2 and v3 to sub-machine M2. The problem now is caused by event e2 which uses variables v1 and v2 and thus cannot be cleanly allocated to either machine. The

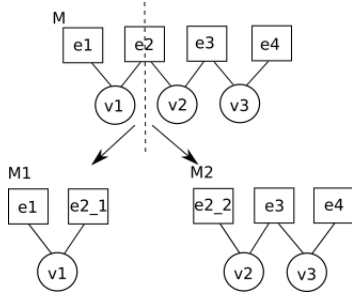


Figure 22: The *shared event* decomposition of machine  $M$  into sub-machines  $M1$  and  $M2$ . Here we wish to partition machine  $M$  into two sets of variables, as shown by the dotted line.

```

1 spec M1 =
2   (M hide via  $\sigma_1$ )
3   with {e2  $\mapsto$  e2.1}
4 end
5 where
6    $\sigma_1 = \{v1 \mapsto v1, e1 \mapsto e1, e2 \mapsto e2\}$ 

7 spec M2 =
8   (M hide via  $\sigma_2$ )
9   with {e2  $\mapsto$  e2.2}
10 end
11 where
12    $\sigma_2 = \{v2 \mapsto v2, v3 \mapsto v3, e2 \mapsto e2,$ 
13              $e3 \mapsto e3, e4 \mapsto e4\}$ 

```

Figure 23: Representing the *shared event* modularisation used in Figure 22 using specification-building operators

solution is to decompose event  $e2$  into two events  $e2.1$  and  $e2.2$ . These new events each correspond to a restricted version of the original event  $e2$ , each confined to the guards and actions that refer to each of the variable sets (i.e.  $e2.1$  contains only the guards and actions of  $e2$  that referenced  $v1$ ).

Figure 23 represents this decomposition using specification-building operators. In each case we use hiding with a signature morphism to specify the signature components that we wish to keep, and then the shared event is renamed. The identity mappings for the variables on lines 6, 12 and 13 have the effect of hiding those variables that belong to the other machine. Thus these variables can still be used in constraints, but are no longer externally visible. Since the machines  $M1$  and  $M2$  do not share any common variables they can now be refined independently.

**7.3.2. Generic Instantiation.** The *generic instantiation* approach to re-use in Event B is intended to facilitate the parametrisation of machines, so that a refinement chain can be re-used [AH07]. The idea is that a chain of existing machine refinements is made generic with respect to all of the carrier sets, constants and axioms that have been collected in the contexts used throughout. It is then possible for a user to instantiate these sets and constants differently in a new model. In order to reuse the proofs of the old model, its axioms must then be proven as theorems after instantiation in the new model. This process facilitates the reuse of existing Event-B models thus providing a more modular approach to specification in Event-B.

Figure 24 is an example of using the generic instantiation plugin for Rodin [SB09]. Here, a machine `mac` has been specified (in outline) on lines 6–10, and it depends on a context `ctx1` specified on lines 1–5. This is standard Event B syntax, and the machine is not marked as genericised in any way. The instantiation is shown on lines 11–20 of Figure 24, where a machine `maci` is specified (again, in outline), and declared to instantiate `mac`, by providing a context `ctx2` as an implementation of `ctx1` (lines 12–13). The details of the instantiation are made explicit in the following lines, where the elements of the context `ctx1` are replaced

```

1 CONTEXT ctx1
2 SETS S1, ..., Sm
3 CONSTANTS C1, ..., Cn
4 AXIOMS Ax1, ..., Axp
5 END

6 MACHINE mac
7 SEES ctx1
8 VARIABLES v1, ..., vq
9 EVENTS ev1, ..., evr
10 END

11 INSTANTIATED MACHINE maci
12 INSTANTIATES mac VIA ctx1
13 SEES ctx2 //contains instance properties
14 REPLACE //replace parameters defined in ctx1
15   SETS S1 := ctx2.S1, ..., Sm := ctx2.Sm
16   CONSTANTS C1 := ctx2.C2, ..., Cn := ctx2.Cn
17   RENAME // rename elements in mac
18   VARIABLES v1 := ctx2.v1, ..., vq := ctx2.vq
19   EVENTS ev1 := ctx2.ev1, ..., evr := ctx2.evr
20 END

```

Figure 24: The machine `maci` is defined as an instantiation of the machine `mac` using the generic instantiation plugin.

```

1 spec ctx1 =
2   sorts S1, ..., Sm
3   ops C1, ..., Cn
4   . Ax1, ..., Axp
5 end

6 spec MAC =
7   ctx1 then
8     ops v1, ..., vq
9     EVENTS ev1, ..., evr
10 end

11 spec MACI =
12   (ctx2 then MAC)
13   with
14     {S1 ↦ ctx2.S1, ..., Sm ↦ ctx2.Sm,
15      C1 ↦ ctx2.C2, ..., Cn ↦ ctx2.Cn,
16      v1 ↦ ctx2.v1, ..., vq ↦ ctx2.vq,
17      ev1 ↦ ctx2.ev1, ..., evr ↦ ctx2.evr}
18 end

```

Figure 25: A representation of generic instantiation using specification-building operators in  $\mathcal{EVT}$ .

by corresponding elements from `ctx2` (lines 14–16), and the variables and events of `mac` are renamed (lines 17–19).

The generic instantiation plugin generates extra proof obligations to ensure that the instantiation of the original machine is a valid one and that if the instantiated machine refines an existing model that this is a valid refinement. The machine and its instance are linked via refinement and once the proof obligations have been carried out successfully then the rest of the pattern refinement chain can be instantiated.

There are many ways of implementing this using the specification-building operators, and we provide one example in Figure 25. In fact, the theory of institutions is equipped with a concept of parameterisation and it is easy to see how we could have used this to represent the generic instantiation in Figure 25 by giving the context, `ctx1` as a parameter to the specification for `mac` with the appropriate signature morphisms in the  $\mathcal{EVT}$  institution. However, this would involve changing the original specification of `mac`, and so we have chosen a specification in Figure 25 that more closely resembles the approach shown in Figure 24. The main elements of the instantiation are the inclusion of `ctx2` and `mac` in `maci` (line 12), along with a series of appropriate renamings (lines 13–17) to match those in the original.

In fact, the possibilities provided by the specification-building operators are much more general than those provided by the generic instantiation plugin. Using the generic instantiation plugin, only user defined sets can be replaced so it is not possible, for example, to replace  $\mathbb{Z}$ ,  $\mathbb{N}$  or `BOOL`. Moreover, it is a requirement that *all* sets and contexts be replaced ensuring that there are no uninstantiated parameters. One of the main benefits of using specification-building operators over  $\mathcal{EVT}$  is that it is now possible to replace any sets

that a user wishes, once the appropriate signature morphisms have been defined by the user. It is also possible to generalise machines in  $\mathcal{EVT}$  using any signature elements, including other machines, as parameterisation is not limited to just sorts and constants.

## 8. CONCLUSIONS AND FUTURE WORK

In this paper, we provide a detailed description of the Event-B institution,  $\mathcal{EVT}$ . Our definition of  $\mathcal{EVT}$  allows the restructuring of Event-B specifications using the standard specification-building operators for institutions [ST12, §5.1]. Thus  $\mathcal{EVT}$  provides a means for writing down and splitting up the components of an Event-B system, facilitating increased modularity for Event-B specifications. We have also shown how the institutional framework subsumes and can explicitly represent Event-B refinement.

Moreover, we contribute a formal semantics for the Event-B formal specification language. We achieved this by constructing a three-layer model for the Event-B language that decomposed it into its mathematical, infrastructure and superstructure sub-languages. The semantics that we have presented for each of these three constituent languages and thus for Event-B itself is grounded in the theory of institutions using our institution for Event-B ( $\mathcal{EVT}$ ) and the institution for first-order predicate logic with equality ( $\mathcal{FOPEQ}$ ). We have described a Haskell implementation of this semantics in our `EB2EVT` translator tool and illustrated how it is used to generate syntactically sugared  $\mathcal{EVT}$ -specifications. We demonstrate our approach using the cars on a bridge example taken from Abrial's book [Abr10].

Further to this, we have discussed refinement and illustrated the use of the specification-building operators as modularisation constructs in the  $\mathcal{EVT}$  institution. Institutions are designed with modularity and interoperability in mind and so our approach does not inhibit using multiple modelling domains to evaluate the semantics of an Event-B model. In fact, we have provided scope for the interoperability of Event-B with other formalisms that have been described in this institutional framework. Future work includes defining further institution-theoretic relationships between  $\mathcal{EVT}$  and other formalisms that have been defined in the theory of institutions. This will provide a solid, mathematical foundation for their combination.

This work examined how the theory of institutions can provide a gateway to formalism-independent modularisation constructs for Event-B. The Event-B specification language does not natively embody strong modularisation features and one might wonder whether this work could motivate a more drastic modification to the Event-B formalism. It is our view that this work provides access to stronger modularisation for Event-B without the need to modify the formalism itself. This is particularly important since Event-B is industrial-strength and widely used in safety-critical systems already so dramatic modifications to the original language at this stage might cause issues in uptake. Although this paper is focused on Event-B, we have demonstrated how such modularisation capabilities can be added to a formal specification language using the theory of institutions and future work seeks to examine how this approach can be applied to other similar formal languages. We have also illustrated the kinds of modularisation capabilities that are useful in a formal specification language and this work should inspire the development of modularisation features for these formalisms.

## REFERENCES

- [ABH<sup>+</sup>10] Jean-Raymond Abrial, Michael Butler, Stefan Hallerstede, Thai Son Hoang, Farhad Mehta, and Laurent Voisin. Rodin: an open toolset for modelling and reasoning in Event-B. *International Journal on Software Tools for Technology Transfer*, 12(6):447–466, 2010.
- [Abr09] Jean-Raymond Abrial. Event model decomposition. Technical report, Department of Computer Science, ETH Zurich, 2009.
- [Abr10] Jean-Raymond Abrial. *Modeling in Event-B: System and Software Engineering*. Cambridge University Press, 1st edition, 2010.
- [AH07] Jean-Raymond Abrial and Stefan Hallerstede. Refinement, decomposition, and instantiation of discrete models: Application to Event-B. *Fundamenta Informaticae*, 77(1-2):1–28, 2007.
- [AJ16] Troy K. Astarte and Cliff B. Jones. An exegesis of four formal descriptions of ALGOL 60. Technical Report CS-TR-1498, Computing Science, Newcastle University, September 2016.
- [Bac90] Ralph-Johan Back. Refinement calculus, part II: Parallel and reactive programs. In *Stepwise Refinement of Distributed Systems Models, Formalisms, Correctness*, volume 430 of *Lecture Notes in Computer Science*, pages 67–93, 1990.
- [BG77] Rod M. Burstall and Joseph A. Goguen. Putting theories together to make specifications. In *5th International Joint Conference on Artificial intelligence, IJCAI*, volume 2, pages 1045–1058. Morgan Kaufmann, 1977.
- [BG80] Rod M. Burstall and Joseph A. Goguen. The semantics of Clear, a specification language. In *Abstract Software Specifications*, volume 86 of *Lecture Notes in Computer Science*, pages 292–332, 1980.
- [BHK89] J. A. Bergstra, J. Heering, and P. Klint, editors. *Algebraic Specification*. Addison Wesley, 1989.
- [Bj06] Dines Bjørner. *Software Engineering 1: Abstraction and Modelling*. Springer, 2006. doi:10.1007/3-540-31288-9\_12.
- [But09] Michael Butler. Decomposition structures for Event-B. In *7th International Conference on Integrated Formal Methods*, volume 5423 of *Lecture Notes in Computer Science*, pages 20–38, 2009.
- [BvW98] Ralph-Johan Back and Joakim von Wright. *Refinement Calculus: A systematic introduction*. Springer, 1998.
- [CAPT15] Pablo F. Castro, Nazareno Aguirre, Carlos L. Pombo, and Maibaum TSE. Categorical foundations for structured specifications in Z. *Formal Aspects of Computing*, 27(5–6):831–865, 2015.
- [DGS91] Razvan Diaconescu, Joseph Goguen, and Petros Stefaneas. Logical support for modularisation. In *Workshop on Logical Frameworks*, pages 83–130, 1991.
- [EMCO92] Hartmut Ehrig, Bernd Mahr, Ingo Classen, and Fernando Orejas. Introduction to algebraic specification. part 1: Formal methods for software development. *The Computer Journal*, 35(5):460–467, 1992.
- [Far17] Marie Farrell. *Event-B in the Institutional Framework: Defining a Semantics, Modularisation Constructs and Interoperability for a Specification Language*. PhD thesis, Department of Computer Science, Maynooth University, Ireland, 2017.
- [Fei94] L. M. G. Feijs. An overview of the development of COLD. In *Semantics of Specification Languages (SoSL)*, pages 15–22. Springer London, 1994.
- [FMP17a] Marie Farrell, Rosemary Monahan, and James F. Power. Combining Event-B and CSP: An Institution Theoretic Approach to Interoperability. In *19th International Conference on Formal Engineering Methods, ICFEM*, volume 10610 of *Lecture Notes in Computer Science*, pages 140–156, 2017.
- [FMP17b] Marie Farrell, Rosemary Monahan, and James F. Power. An institution for Event-B. In *Recent Trends in Algebraic Development Techniques*, volume 10644 of *Lecture Notes in Computer Science*, pages 104–119, 2017.
- [FMP17c] Marie Farrell, Rosemary Monahan, and James F. Power. Specification Clones: An empirical study of the structure of Event-B specifications. In *15th International Conference on Software Engineering and Formal Methods, SEFM*, volume 10469 of *Lecture Notes in Computer Science*, pages 152–167, 2017.
- [GB83] Joseph A. Goguen and Rod M. Burstall. Introducing institutions. In *Workshop on Logic of Programs*, volume 164 of *Lecture Notes in Computer Science*, pages 221–256, 1983.

- [GB92] Joseph A. Goguen and Rod M. Burstall. Institutions: abstract model theory for specification and programming. *Journal of the ACM*, 39(1):95–146, 1992.
- [GGM96] Joseph A Goguen, Joseph Goguen, and Grant Malcolm. *Algebraic semantics of imperative programs*. MIT Press, 1996.
- [GJM02] Carlo Ghezzi, Mehdi Jazayeri, and Dino Mandrioli. *Fundamentals of Software Engineering*. Prentice Hall, 2002.
- [GPB11] Ali Gondal, Michael Poppleton, and Michael Butler. Composing Event-B specifications – case-study experience. In *10th International Conference on Software Composition*, volume 6708 of *Lecture Notes in Computer Science*, pages 100–115, 2011.
- [GPS09] Ali Gondal, Michael Poppleton, and Colin Snook. Feature composition-towards product lines of Event-B models. In *3rd International Workshop on Model-Driven Product Line Engineering*, volume 6698 of *Lecture Notes in Computer Science*, pages 18–25, 2009.
- [Gun92] Carl A. Gunter. *Semantics of Programming Languages: Structures and Techniques*. MIT Press, Cambridge, Mass., November 1992.
- [Hal08] Stefan Hallerstede. On the purpose of Event-B proof obligations. In *1st International Conference on Abstract State Machines, B and Z, ABZ*, volume 5238 of *Lecture Notes in Computer Science*, pages 125–138, 2008.
- [Hen90] Matthew Hennessy. *The Semantics of Programming Languages*. Wiley, 1990.
- [HISW11] Thai Son Hoang, Alexei Iliasov, Renato Silva, and Wei Wei. A survey on Event-B decomposition. *Electronic Communications of the EASST*, 46:1–15, 2011.
- [HJ98] Charles Anthony Richard Hoare and He Jifeng. *Unifying Theories of Programming*. Prentice Hall, 1998.
- [HL18] Jifeng He and Qin Li. A new roadmap for linking theories of programming and its applications on GCL and CSP. *Science of Computer Programming*, 162:3–34, 2018.
- [Hoa69] Charles Antony Richard Hoare. An axiomatic basis for computer programming. *Communications of the ACM*, 12(10):576–580, 1969.
- [Hoa78] Charles Antony Richard Hoare. Communicating sequential processes. In *The Origin of Concurrent Programming*, pages 413–443. Springer, 1978.
- [ISO02] ISO/IEC. *Information Technology — Z Formal Specification Notation — Syntax, Type System and Semantics*. ISO/IEC 13568:2002. International Organization for Standardization, July 2002.
- [ITL<sup>+</sup>10] Alexei Iliasov, Elena Troubitsyna, Linas Laibinis, Alexander Romanovsky, Kimmo Varpaaniemi, Dubravka Ilic, and Timo Latvala. Supporting reuse in Event-B development: Modularisation approach. In *2nd International Conference on Abstract State Machines, Alloy, B and Z, ABZ*, volume 5977 of *Lecture Notes in Computer Science*, pages 174–188, 2010.
- [Jon01] Cliff B. Jones. The transition from VDL to VDM. *J. UCS*, 7(8):631–640, 2001. doi:10.3217/jucs-007-08-0631.
- [KP98] Yonit Kesten and Amir Pnueli. Modularization and abstraction: The keys to practical formal verification. In *40th International Symposium on Mathematical Foundations of Computer Science, MFCS*, volume 9234 of *Lecture Notes in Computer Science*, pages 54–71, 1998.
- [Lam02] Leslie Lamport. *Specifying Systems: The TLA+ Language and Tools for Hardware and Software Engineers*. Addison-Wesley, 2002.
- [Lei10] K. Rustan M. Leino. Dafny: An automatic program verifier for functional correctness. In *International Conference on Logic for Programming Artificial Intelligence and Reasoning*, volume 6355 of *Lecture Notes in Computer Science*, pages 348–370, 2010.
- [LP95] Peter Gorm Larsen and Wieslaw Pawlowski. The formal semantics of ISO VDM-SL. *Computer Standards and Interfaces*, 17(5):585–601, 1995. doi:[https://doi.org/10.1016/0920-5489\(95\)00026-Q](https://doi.org/10.1016/0920-5489(95)00026-Q).
- [Mer12] Stephan Merz. On the logic of TLA+. *Computing and Informatics*, 22(3-4):351–379, 2012.
- [Mid89] Cornelis Adam Middelburg. VVSL: A language for structured VDM specifications. *Formal Aspects of Computing*, 1:115–135, 1989.
- [MML07] Till Mossakowski, Christian Maeder, and Klaus Lüttich. The heterogeneous tool set, HETS. In *13th International Conference on Tools and Algorithms for the Construction and Analysis of Systems, TACAS*, volume 4424 of *Lecture Notes in Computer Science*, pages 519–522, 2007.
- [Mor87] Joseph Morris. A theoretical basis for stepwise refinement and the programming calculus. *Science of Computer Programming*, 9(3):287–306, 1987.

- [Mos04] Peter D. Mosses, editor. *CASL Reference Manual*, volume 2960 of *Lecture Notes in Computer Science*. Springer, 2004.
- [MRG88] Carroll Morgan, Ken Robinson, and Paul Gardiner. *On the Refinement Calculus*. Springer, 1988.
- [OCW09] Marcel Oliveira, Ana Cavalcanti, and Jim Woodcock. A UTP semantics for Circus. *Formal Aspects of Computing*, 21(1-2):3–32, 2009.
- [OMG11a] OMG. UML Infrastructure Specification, v2.4.1. Specification formal/2011-08-05, Object Management Group, August 2011.
- [OMG11b] OMG. UML Superstructure Specification, v2.4.1. Specification formal/2011-08-06, Object Management Group, August 2011.
- [Pop08] Michael Poppleton. The composition of Event-B models. In *1st International Conference on Abstract State Machines, B and Z, ABZ*, volume 5238 of *Lecture Notes in Computer Science*, pages 209–222, 2008.
- [Ros98] Bill Roscoe. *The Theory and Practice of Concurrency*. Prentice Hall, 1998.
- [RS08a] Steve Reeves and David Streader. General refinement, part one: Interfaces, determinism and special refinement. *Electronic Notes in Theoretical Computer Science*, 214:277–307, 2008.
- [RS08b] Steve Reeves and David Streader. General refinement, part two: Flexible refinement. *Electronic Notes in Theoretical Computer Science*, 214:309–329, 2008.
- [SB09] Renato Silva and Michael Butler. Supporting reuse of Event-B developments through generic instantiation. In *11th International Conference on Formal Engineering Methods, ICFEM*, volume 5885 of *Lecture Notes in Computer Science*, pages 466–484, 2009.
- [SB12] Renato Silva and Michael Butler. Shared event composition/decomposition in Event-B. In *9th International Symposium on Formal Methods for Components and Objects, FMCO*, volume 6957 of *Lecture Notes in Computer Science*, pages 122–141, 2012.
- [Sch86] David A. Schmidt. *Denotational Semantics: a methodology for language development*. Allyn and Bacon, 1986.
- [Sch01] Steve Schneider. *The B-method: An introduction*. Palgrave, 2001.
- [Sch10] Matthias Schmalz. The Logic of Event-B. Technical Report 698, Department of Computer Science, ETH Zurich, 2010.
- [Smi95] Graeme Smith. A fully abstract semantics of classes for Object-Z. *Formal Aspects of Computing*, 7(3):289–313, May 1995.
- [Smi12] Graeme Smith. *The Object-Z Specification Language*. Springer, 2012.
- [Spi88] J Michael Spivey. *Understanding Z: a specification language and its formal semantics*, volume 3 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1988.
- [SS71] Dana S. Scott and Christopher Strachey. Toward a mathematical semantics for computer languages. Technical Monograph PRG-6, Oxford University Computing Laboratory, Programming Research Group, 1971.
- [ST88] Donald Sannella and Andrzej Tarlecki. Specifications in an arbitrary institution. *Information and Computation*, 76(2-3):165–210, 1988.
- [ST12] Donald Sannella and Andrzej Tarlecki. *Foundations of Algebraic Specification and Formal Software Development*. Springer, 2012.
- [STW14] Steve Schneider, Helen Treharne, and Heike Wehrheim. The behavioural semantics of Event-B refinement. *Formal Aspects of Computing*, 26:251–280, 2014.
- [Tar44] Alfred Tarski. The semantic conception of truth: and the foundations of semantics. *Philosophy and Phenomenological Research*, 4(3):341–376, 1944.
- [Tho99] Simon Thompson. *Haskell: the craft of functional programming*. Addison-Wesley, 1999.
- [Wat91] David A. Watt. *Programming Language Syntax and Semantics*. Prentice Hall, 1991.
- [Win93] Glynn Winskel. *The Formal Semantics of Programming Languages*. MIT Press, February 1993.
- [Wor92] John B. Wordsworth. *Software Development with Z: a Practical Approach to Formal Methods in Software Engineering*. Addison-Wesley, 1992.

## APPENDIX

**Lemma 4.3.**  *$\mathcal{EVT}$ -signatures and signature morphisms define a category  $\mathbf{Sign}_{\mathcal{EVT}}$ . The objects are signatures and the arrows are signature morphisms. Composition of arrows is performed componentwise.*

*Proof.* Let  $\Sigma = \langle S, \Omega, \Pi, E, V \rangle$  be an  $\mathcal{EVT}$ -signature where  $\langle S, \Omega, \Pi \rangle$  is a signature over  $\mathcal{FOP\mathcal{E}Q}$ ,  $\Sigma_{\mathcal{FOP\mathcal{E}Q}}$ , the institution for first-order predicate logic with equality [ST12, §4.1].  $E$  is a set of tuples of the form (*event name*  $\mapsto$  *status*) and  $V$  is a set of sort-indexed variable names.

In a category, morphisms can be composed, their composition is associative and identity morphisms exist for every object in the category. We show that  $\mathbf{Sign}_{\mathcal{EVT}}$  preserves these three properties:

(a) Composition of  $\mathcal{EVT}$ -signature morphisms:

$\mathcal{EVT}$ -signature morphisms can be composed, the composition of  $\sigma_S, \sigma_\Omega$  and  $\sigma_\Pi$  is inherited from  $\mathcal{FOP\mathcal{E}Q}$ . Therefore, we only examine the composition of  $\sigma_E$  and  $\sigma_V$ .

$\sigma_E$ : Event names do not have a sort or arity, so the only restrictions are (1) on functions of the form (**Init**  $\mapsto$  **ordinary**) and (2) that the ordering in the status poset is preserved. In both of these cases the composition is well-defined for  $\sigma_E$ .

$\sigma_V$ : Variable names are sort-indexed so  $\sigma_V$  utilises  $\sigma_S$  on these sorts.

$$\begin{aligned} \sigma_2(\sigma_1(v : s)) &= \sigma_2((\sigma_{1_V}(v) : \sigma_{1_S}(s))) \\ &= \sigma_{2_V}(\sigma_{1_V}(v)) : \sigma_{2_S}(\sigma_{1_S}(s)) \end{aligned}$$

Let  $\sigma_1 : \Sigma_1 \rightarrow \Sigma_2$  and  $\sigma_2 : \Sigma_2 \rightarrow \Sigma_3$ , then we prove that  $\sigma_2 \circ \sigma_1$  is a morphism in the category of  $\mathcal{EVT}$ -signatures.

• For all  $(e_1 \mapsto st_1) \in \Sigma_1.E_1$ ,  $\sigma_1(e_1 \mapsto st_1) \in \Sigma_2.E_2$  and for all  $(e_2 \mapsto st_2) \in \Sigma_2.E_2$ ,  $\sigma_2(e_2 \mapsto st_2) \in \Sigma_3.E_3$ . Therefore  $\sigma_2(\sigma_1(e_1 \mapsto st_1)) \in \Sigma_3.E_3$  so

$$\forall (e_1 \mapsto st_1) \in \Sigma_1.E_1 \Rightarrow \sigma_2 \circ \sigma_1(e_1 \mapsto st_1) \in \Sigma_3.E_3$$

• For all  $(v_1 : s_1) \in \Sigma_1.V_1$ ,  $\sigma_1((v_1 : s_1)) \in \Sigma_2.V_2$  and for all  $(v_2 : s_2) \in \Sigma_2.V_2$ ,  $\sigma_2((v_2 : s_2)) \in \Sigma_3.V_3$ . Therefore  $\sigma_2(\sigma_1((v_1 : s_1))) \in \Sigma_3.V_3$  so

$$\forall (v_1 : s_1) \in \Sigma_1.V_1 \Rightarrow \sigma_2 \circ \sigma_1((v_1 : s_1)) \in \Sigma_3.V_3$$

Therefore,  $\mathcal{EVT}$ -signature morphisms can be composed.

(b) Composition of  $\mathcal{EVT}$ -signature morphisms is associative:

$$(\sigma_3 \circ \sigma_2) \circ \sigma_1 = \sigma_3 \circ (\sigma_2 \circ \sigma_1)$$

For  $(e \mapsto st) \in \Sigma.E$ :

$$\sigma_2 \circ \sigma_1(e \mapsto st) = \sigma_2(\sigma_1(e \mapsto st))$$

Then, by definition of composition

$$\begin{aligned} \sigma_3 \circ (\sigma_2 \circ \sigma_1)(e \mapsto st) &= \sigma_3(\sigma_2(\sigma_1(e \mapsto st))) \\ &= \sigma_3 \circ \sigma_2 \circ (\sigma_1(e \mapsto st)) \\ &= (\sigma_3 \circ \sigma_2) \circ \sigma_1(e \mapsto st) \end{aligned}$$

Similarly, for  $(v : s) \in \Sigma.V$ ,

$$\sigma_3 \circ (\sigma_2 \circ \sigma_1)((v : s)) = (\sigma_3 \circ \sigma_2) \circ \sigma_1((v : s))$$



Therefore, the composition of  $\mathcal{EVT}$ -signature morphisms is associative.

(c) Identity morphisms for  $\mathcal{EVT}$ -signatures:

For any  $\mathcal{EVT}$ -signature  $\Sigma$ , there exists an identity signature morphism  $id_\Sigma : \Sigma \rightarrow \Sigma$ .  $id_E$  and  $id_V$  are such that  $id_E(e \mapsto st) = (e \mapsto st)$  and  $id_V((v : s)) = (v : s)$ . This morphism satisfies the following signature morphism condition

$$(e \mapsto st) \in \Sigma.E \Rightarrow id_E(e \mapsto st) \in \Sigma.E \quad \wedge \quad (v : s) \in \Sigma.V \Rightarrow id_V(v : s) \in \Sigma.V$$

We have thus shown that  $\mathbf{Sign}_{\mathcal{EVT}}$  forms a category as instructed by the definition of an institution (Definition 2.1).  $\square$

**Lemma 4.5.** *There is a functor  $\mathbf{Sen}_{\mathcal{EVT}} : \mathbf{Sign}_{\mathcal{EVT}} \rightarrow \mathbf{Set}$  generating for each  $\mathcal{EVT}$ -signature  $\Sigma$  a set of  $\mathcal{EVT}$ -sentences which is an object in the category  $\mathbf{Set}$  and for each  $\mathcal{EVT}$ -signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  (morphisms in the category  $\mathbf{Sign}_{\mathcal{EVT}}$ ) a function  $Sen(\sigma) : Sen(\Sigma_1) \rightarrow Sen(\Sigma_2)$  (morphisms in the category  $\mathbf{Set}$ ) translating  $\mathcal{EVT}$ -sentences.*

*Proof.*  $\mathbf{Sen}_{\mathcal{EVT}}$  is a functor therefore it is necessary to map the  $\mathcal{EVT}$ -signature morphisms to corresponding functions over sentences. The functor maps morphisms to functions between sets of sentences respecting sort, arity and **Init** events. The image of a signature  $\Sigma_i$  ( $i \in \{1, 2, \dots\}$ ) in  $\mathbf{Sign}_{\mathcal{EVT}}$  is an object  $\mathbf{Sen}(\Sigma_i)$  in the category  $\mathbf{Set}$ . By the definition of the extension of the signature morphism to sentences [ST12, §4], the translation of the object  $\mathbf{Sen}(\Sigma_1)$  coincides with an object in  $\mathbf{Sen}(\Sigma_2)$  with sort, operation, predicate, event and variable names translated with respect to the signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$ . Thus, the image of a morphism in  $\mathbf{Sign}_{\mathcal{EVT}}$  is a function  $Sen(\sigma) : Sen(\Sigma_1) \rightarrow Sen(\Sigma_2)$  in the category  $\mathbf{Set}$ .

Then we prove that  $\mathbf{Sen}$  preserves the composition of  $\mathcal{EVT}$ -signature morphisms and identities as follows.

(a) Composition of  $\mathcal{EVT}$ -sentence morphisms:

$$Sen(\sigma_2 \circ \sigma_1) = Sen(\sigma_2) \circ Sen(\sigma_1)$$

Let  $\Sigma_i$  ( $i = 1..3$ ) be  $\mathcal{EVT}$ -signatures and let  $\sigma_1 : \Sigma_1 \rightarrow \Sigma_2$  and  $\sigma_2 : \Sigma_2 \rightarrow \Sigma_3$  be  $\mathcal{EVT}$ -signature morphisms. Given an  $\mathcal{EVT}$ -sentence of the form  $\langle e, \phi(\bar{x}, \bar{x}') \rangle$ , then by expanding each side of the following equivalence, since signature morphisms can be composed, we show that composition is preserved.

$$\begin{aligned} Sen(\sigma_2 \circ \sigma_1)(\langle e, \phi(\bar{x}, \bar{x}') \rangle) &= Sen(\sigma_2) \circ Sen(\sigma_1)(\langle e, \phi(\bar{x}, \bar{x}') \rangle) \\ \langle \sigma_2 \circ \sigma_1(e), \sigma_2 \circ \sigma_1(\phi(\bar{x}, \bar{x}')) \rangle &= Sen(\sigma_2)(\langle \sigma_1(e), \sigma_1(\phi(\bar{x}, \bar{x}')) \rangle) \\ \langle \sigma_2(\sigma_1(e)), \sigma_2(\sigma_1(\phi(\bar{x}, \bar{x}')) \rangle &= \langle \sigma_2(\sigma_1(e)), \sigma_2(\sigma_1(\phi(\bar{x}, \bar{x}')) \rangle \end{aligned}$$

Thus, composition is preserved.

(b) Preservation of identities:

Let  $id_{\Sigma_1} : \Sigma_1 \rightarrow \Sigma_1$  be an identity  $\mathcal{EVT}$ -signature morphism as in Lemma 4.3. Since  $\mathcal{EVT}$ -signature morphisms already preserve identity and  $Sen(id_{\Sigma_1})$  is the application of the identity signature morphisms to every element of the sentence, then the identities are preserved. We can illustrate this as follows. Given a  $\Sigma_1$ -sentence  $\langle e, \phi(\bar{x}, \bar{x}') \rangle$ ,

$$\begin{aligned} Sen(id_{\Sigma_1})(\langle e, \phi(\bar{x}, \bar{x}') \rangle) & \\ &= \langle id_{\Sigma_1}(e), id_{\Sigma_1}(\phi(\bar{x}, \bar{x}')) \rangle \\ &= \langle e, \phi(\bar{x}, \bar{x}') \rangle \end{aligned}$$

and so, identities are preserved.

Thus  $\mathbf{Sen}_{\mathcal{EVT}}$  is a functor.  $\square$

**Lemma 4.8.** *For any  $\mathcal{EVT}$ -signature  $\Sigma$  there is a category of  $\mathcal{EVT}$ -models  $\mathbf{Mod}_{\mathcal{EVT}}(\Sigma)$  where the objects in the category are  $\mathcal{EVT}$ -models and the arrows are  $\mathcal{EVT}$ -model morphisms.*

*Proof.* We begin by describing  $\mathcal{EVT}$ -model morphisms and then prove that composition and identities are preserved.

In  $\mathcal{FOPEQ}$  a model morphism  $h : A_1 \rightarrow A_2$  is a family of functions  $h = \langle h_s : |A_1|_s \rightarrow |A_2|_s \rangle_{s \in \mathcal{S}}$  which respects the sorts and arities of the operations and predicates. Recall from Definition 4.7 that  $\mathcal{EVT}$ -models have the form  $\langle A, L, R \rangle$ , therefore  $\mathcal{EVT}$ -model morphisms are given by extending the corresponding  $\mathcal{FOPEQ}$ -model morphisms for the  $A$  component of the model to the initialising set  $L$  and the relations in  $R$ .

Thus for each  $\mathcal{EVT}$ -model morphism  $\mu : \langle A_1, L_1, R_1 \rangle \rightarrow \langle A_2, L_2, R_2 \rangle$  there is an underlying  $\mathcal{FOPEQ}$ -model morphism  $h : A_1 \rightarrow A_2$ , and we extend this to the states in the set  $L_1$  and in the relation  $R_1$ . That is, for any element

$$\{x_1 \mapsto a_1, \dots, x_n \mapsto a_n, x_1' \mapsto a_1', \dots, x_n' \mapsto a_n'\} \in R_1.e$$

in  $R_1$  there is

$$\{x_1 \mapsto h(a_1), \dots, x_n \mapsto h(a_n), x_1' \mapsto h(a_1'), \dots, x_n' \mapsto h(a_n')\} \in R_2.e$$

in  $R_2$  where  $x_1, \dots, x_n, x_1', \dots, x_n'$  are variable names and their primed versions drawn from  $V$ . A similar construction follows for  $L_1$ . The composition of model morphisms, their associativity and identity derives from that of  $\mathcal{FOPEQ}$ .

(a) Composition of  $\mathcal{EVT}$ -model morphisms:

Let  $M_i = \langle A_i, L_i, R_i \rangle$  be a model and  $h_i : M_i \rightarrow M_{i+1}$  be an  $\mathcal{EVT}$ -model morphism where  $i \in \{1, 2, 3\}$ . We can now show that the composition of  $\mathcal{EVT}$ -model morphisms is associative as follows:

We know that

$$(h_3 \circ h_2) \circ h_1(A_1) = h_3 \circ (h_2 \circ h_1)(A_1)$$

since model morphisms and thus their associativity for  $A$  are inherited from  $\mathcal{FOPEQ}$ .

Next, we consider  $L_1$  and  $R_1$  whose fundamental components are  $\Sigma$ -states (variable to value mappings). Signature morphisms on  $\Sigma$ -states follow the associated sort mappings in  $\mathcal{FOPEQ}$  and thus model morphisms on  $\Sigma$ -states follow model morphisms on  $A$  as above. In this way, compositionality of model morphisms is associative for  $L_1$  and  $R_1$ .

(b) Identity morphism for  $\mathcal{EVT}$ -models:

For any  $\mathcal{EVT}$ -model  $M_i$  there exists an identity model morphism  $h_{id} : M_i \rightarrow M_i$ . If  $M_i = \langle A_i, L_i, R_i \rangle$  then  $h_{id}(M_i) = \langle A_i, L_i, R_i \rangle$ .

Thus  $\mathbf{Mod}_{\mathcal{EVT}}(\Sigma)$  forms a category.  $\square$

**Lemma 4.10.** *For each  $\mathcal{EVT}$ -signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  the  $\mathcal{EVT}$ -model reduct is a functor  $\mathbf{Mod}(\sigma)$  from  $\Sigma_2$ -models to  $\Sigma_1$ -models.*

Recall that each  $\Sigma$ -State $_A$  is a set of variable-to-value mappings of the form

$$\{x_1 \mapsto a_1, \dots, x_n \mapsto a_n\}$$

where  $x_1, \dots, x_n \in \Sigma.V$  (Definition 4.6).

*Proof.* Let  $M_2 = \langle A_2, L_2, R_2 \rangle$  be a  $\Sigma_2$ -model. Then the reduct  $M_2|_\sigma$  collapses the  $\mathcal{EVT}$ -model to only contain signature items supported by  $\Sigma_1$  and consists of the tuple  $M_2|_\sigma = \langle A_2|_\sigma, L_2|_\sigma, R_2|_\sigma \rangle$  such that

- $A_2|_\sigma$  is the reduct of the  $\mathcal{FOPEQ}$ -component of the  $\mathcal{EVT}$ -model along the  $\mathcal{FOPEQ}$ -components of  $\sigma : \Sigma \rightarrow \Sigma'$ .
- $L_2|_\sigma$  and  $R_2|_\sigma$  are based on the reduction of the states of  $A_2$  along  $\sigma$ . In particular, given  $e \in \text{dom}(E_1)$  and  $e \neq \text{Init}$  and  $R_2.\sigma(e) \in R_2$

$$R_2.\sigma(e) = \{s_1, \dots, s_m\}$$

where each  $s_i$  is a  $\Sigma_2$ -state $_{A_2}$  ( $1 \leq i \leq m$ ) is of the form

$$\{\sigma(x_1) \mapsto a_1, \dots, \sigma(x_n) \mapsto a_n, \sigma(x_{1'}) \mapsto a_{1'}, \dots, \sigma(x_{n'}) \mapsto a_{n'}\}$$

with  $x_1, \dots, x_n \in \Sigma.V$  and  $x_{1'}, \dots, x_{n'} \in \Sigma.V'$ .

Then for each  $e \in \text{dom}(E_1)$ ,  $e \neq \text{Init}$  and  $R_2|_\sigma.e \in R_2|_\sigma$  there is

$$R_2|_\sigma.e = \{s_1|_\sigma, \dots, s_m|_\sigma\}$$

where each  $s_i|_\sigma$  ( $1 \leq i \leq m$ ) is of the form

$$\{x_1 \mapsto a_1, \dots, x_n \mapsto a_n, x_{1'} \mapsto a_{1'}, \dots, x_{n'} \mapsto a_{n'}\}$$

In order to prove that the model reduct is a functor, we show that it preserves composition and identities as follows:

- (a) Preservation of composition for  $\mathcal{EVT}$ -model reducts:

Our objective here is to show that the reduct of a composition of two  $\mathcal{EVT}$ -model morphisms is equal to the composition of the reducts of those  $\mathcal{EVT}$ -model morphisms. Given  $\mathcal{EVT}$ -model morphisms  $h_1 : M_1 \rightarrow M_2$  and  $h_2 : M_2 \rightarrow M_3$ , then we show that

$$(h_2 \circ h_1)|_\sigma = h_2|_\sigma \circ h_1|_\sigma$$

for some  $\mathcal{EVT}$ -signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$ . Given an  $\mathcal{EVT}$ -model of the form  $M_1 = \langle A, L, R \rangle$  over  $\Sigma$ , then for any  $R.e \in R$ , as outlined above, of the form

$$\{x_1 \mapsto a_1, \dots, x'_n \mapsto a'_n\}_e$$

Then  $(h_2 \circ h_1)|_\sigma$  is defined as

$$(h_2 \circ h_1)\{\sigma(x_1) \mapsto a_1, \dots, \sigma(x'_n) \mapsto a'_n\}_\sigma(e)$$

This is equal to

$$h_2\{\sigma(x_1) \mapsto h_1(a_1), \dots, \sigma(x'_n) \mapsto h_1(a'_n)\}_\sigma(e)$$

Since  $\mathcal{EVT}$ -model morphisms can be composed, this is thus equal to  $h_2|_\sigma \circ h_1|_\sigma$ .

- (b) Preservation of identities for  $\mathcal{EVT}$ -model reducts:

The reduct of the identity is the identity. Let  $id_{M_2}$  be an identity  $\Sigma_2$ -morphism then  $id_{M_2}|_\sigma$  is an identity  $\Sigma_1$ -morphism  $h_1$  defined by  $h_1(R.e) = id_{M_2}|_\sigma(R.e) = R.e$  for any  $R.e \in R$  and  $e$  is an event other than  $\text{Init}$ .

For the components belonging to  $A$  these proofs follow the corresponding proofs in  $\mathcal{FOPEQ}$ .  $\square$

**Lemma 4.11.** *There is a functor  $\mathbf{Mod}_{\mathcal{EVT}}$  yielding a category  $\mathbf{Mod}(\Sigma)$  of  $\mathcal{EVT}$ -models for each  $\mathcal{EVT}$ -signature  $\Sigma$ , and for each  $\mathcal{EVT}$ -signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  a functor  $\mathbf{Mod}(\sigma)$  from  $\Sigma_2$ -models to  $\Sigma_1$ -models.*

*Proof.* For each  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  in  $\mathbf{Sign}_{\mathcal{EVT}}$  there is an arrow in  $\mathbf{Sign}_{\mathcal{EVT}}^{op}$  going in the opposite direction. By Lemma 4, the image of this arrow in  $\mathbf{Sign}_{\mathcal{EVT}}^{op}$  is  $\mathbf{Mod}(\sigma) : \mathbf{Mod}(\Sigma_2) \rightarrow \mathbf{Mod}(\Sigma_1)$  in  $\mathbf{Cat}$ . By Lemma 3, the image of a signature in  $\mathbf{Sign}_{\mathcal{EVT}}$  is an object  $\mathbf{Mod}(\Sigma)$  in  $\mathbf{Cat}$ . Therefore, domain and co-domain of the image of an arrow are the images of the domain and co-domain respectively.

(a) Preservation of composition:

$$\mathbf{Mod}(\sigma_2 \circ \sigma_1) = \mathbf{Mod}(\sigma_2) \circ \mathbf{Mod}(\sigma_1)$$

Let  $\sigma_1 : \Sigma_1 \rightarrow \Sigma_2$  and  $\sigma_2 : \Sigma_2 \rightarrow \Sigma_3$  be  $\mathcal{EVT}$ -signature morphisms and let  $M_i = \langle A_i, L_i, R_i \rangle$  be an  $\mathcal{EVT}$ -model over  $\Sigma_i$  and let  $h_i$  be a  $\Sigma_i$ -model morphism with  $i \in \{1, 2, 3\}$ .

- $M_3|_{\sigma_2 \circ \sigma_1} = (M_3|_{\sigma_2})|_{\sigma_1}$

By definition of reduct  $M_3|_{\sigma_2} = \langle A_3, L_3, R_3 \rangle|_{\sigma_2} = \langle A_2, L_2, R_2 \rangle = M_2$ .

Then  $(M_3|_{\sigma_2})|_{\sigma_1} = M_2|_{\sigma_1} = \langle A_2, L_2, R_2 \rangle|_{\sigma_1} = \langle A_1, L_1, R_1 \rangle = M_1$ .

By composition of signature morphisms  $\sigma_2 \circ \sigma_1 : \Sigma_1 \rightarrow \Sigma_3$ , then

$$M_3|_{\sigma_2 \circ \sigma_1} = \langle A_3, L_3, R_3 \rangle|_{\sigma_2 \circ \sigma_1} = \langle A_1, L_1, R_1 \rangle = M_1$$

Therefore  $M_3|_{\sigma_2 \circ \sigma_1} = (M_3|_{\sigma_2})|_{\sigma_1}$

- $h_3|_{\sigma_2 \circ \sigma_1} = (h_3|_{\sigma_2})|_{\sigma_1}$

Proof similar to above.

(b) Preservation of identities:

Let  $id_{\Sigma_1}$  be an identity signature morphism as defined in Lemma 4.3. Since  $\mathcal{EVT}$ -signature morphisms already preserve identity and  $Mod(id_{\Sigma_1})$  is the application of the identity signature morphisms to every part of the  $\mathcal{EVT}$ -model, the identities are preserved.  $\square$

**Proposition 4.16.** *Pushouts exist in  $\mathbf{Sign}_{\mathcal{EVT}}$ .*

*Proof.* Given two  $\mathcal{EVT}$ -signature morphisms  $\sigma_1 : \Sigma \rightarrow \Sigma_1$  and  $\sigma_2 : \Sigma \rightarrow \Sigma_2$  a pushout is a triple  $(\Sigma', \sigma'_1, \sigma'_2)$  that satisfies the universal property: for all triples  $(\Sigma'', \sigma''_1, \sigma''_2)$  there exists a unique morphism  $u : \Sigma' \rightarrow \Sigma''$  such that the diagram below commutes. Our pushout construction follows  $\mathcal{FOPEQ}$  for the elements that  $\mathcal{FOPEQ}$  has in common with  $\mathcal{EVT}$ . In  $\mathbf{Sign}_{\mathcal{EVT}}$  the additional elements are  $E$  and  $V$  as presented below.

$$\begin{array}{ccccc}
 & & \Sigma & & \\
 & \swarrow \sigma_1 & & \searrow \sigma_2 & \\
 \Sigma_1 & & & & \Sigma_2 \\
 & \searrow \sigma'_1 & & \swarrow \sigma'_2 & \\
 & & \Sigma' & & \\
 & \swarrow \sigma''_1 & & \searrow \sigma''_2 & \\
 & & \Sigma'' & & 
 \end{array}$$

We base our constructions of the pushout in  $E$  and  $V$  on the canonical pushout in  $\mathbf{Set}$ .

- *Set of functions of the form (event name  $\mapsto$  status) in  $E$ :* The set of all event names in the pushout is the pushout in  $\mathbf{Set}$  (disjoint union) on event names only. Then, the status of an event in the pushout is the supremum of all event statuses that are mapped to it, according to the ordering given in Definition 4.2. Since  $\mathcal{EVT}$ -signature morphisms map  $(\mathbf{Init} \mapsto \mathbf{ordinary})$  to  $(\mathbf{Init} \mapsto \mathbf{ordinary})$  the pushout does likewise. The universality property for  $E$  follows from that of  $\mathbf{Set}$ .

The canonical injections in **Set** for  $E$  are denoted by  $inl$  for inject left and  $inr$  for inject right. Then, let  $(e_1 \mapsto status_1) = \sigma_1(e \mapsto status)$ ,  $(e_2 \mapsto status_2) = \sigma_2(e \mapsto status)$  and  $status' = status_1 \sqcup status_2$  (where  $\sqcup$  denotes the supremum). Thus, the pushout in  $E$  is given by the formula  $E_1 \dot{\cup} E_2 / \sim$  where  $\sim$  is the least equivalence relation such that

$$(inl(e_1) \mapsto status') \sim (inr(e_2) \mapsto status')$$

- *Set of sort-indexed variable names  $V$* : The set of sort-indexed variable names in the pushout is the pushout in  $\mathcal{FOPEQ}$  for the sort components and the pushout in **Set** for the variable names. This is a similar construction to the pushout for operation names in  $\mathcal{FOPEQ}$  as these also have to follow the sort pushout. Thus, the universality property for  $V$  follows from that of **Set** and the  $\mathcal{FOPEQ}$  pushout for sorts. Thus the pushout in  $V$  is given by the formula  $V_1 \dot{\cup} V_2 / \sim$  where  $\sim$  is the least equivalence relation such that

$$\sigma'_1 \circ \sigma_1(v : s) \sim \sigma'_2 \circ \sigma_2(v : s)$$

for  $(v : s) \in \Sigma.V$ . □

**Proposition 4.18.** *Every pushout diagram in  $\mathbf{Sign}_{\mathcal{EVT}}$  admits weak model amalgamation.*

We decompose this proposition into two further sub-propositions:

**Sub-Proposition A.1.** *For  $M_1 \in |\mathbf{Mod}(\Sigma_1)|$  and  $M_2 \in |\mathbf{Mod}(\Sigma_2)|$  such that  $M_1|_{\sigma_1} = M_2|_{\sigma_2}$ , there exists an  $\mathcal{EVT}$ -model (the amalgamation of  $M_1$  and  $M_2$ )  $M' \in |\mathbf{Mod}(\Sigma')|$  such that  $M'|_{\sigma'_1} = M_1$  and  $M'|_{\sigma'_2} = M_2$ .*

*Proof.* Given the  $\mathcal{EVT}$ -models  $M \in |\mathbf{Mod}(\Sigma)|$ ,  $M_1 \in |\mathbf{Mod}(\Sigma_1)|$ ,  $M_2 \in |\mathbf{Mod}(\Sigma_2)|$  and  $\mathcal{EVT}$ -signature morphisms  $\sigma_1 : \Sigma \rightarrow \Sigma_1$ ,  $\sigma_2 : \Sigma \rightarrow \Sigma_2$  in the commutative diagram below.

$$\begin{array}{ccc}
 & M' = \langle A', L', R' \rangle & \\
 \text{Mod}(\sigma'_1) \swarrow & & \searrow \text{Mod}(\sigma'_2) \\
 M_1 = \langle A_1, L_1, R_1 \rangle & & M_2 = \langle A_2, L_2, R_2 \rangle \\
 \text{Mod}(\sigma_1) \searrow & & \swarrow \text{Mod}(\sigma_2) \\
 & M = \langle A, L, R \rangle & 
 \end{array}$$

We compute the model amalgamation (following the corresponding pushout diagram in **Sign** as illustrated in Proposition 4.16)  $M' = M_1 \otimes M_2$  which is of the form  $\langle A', L', R' \rangle$  where  $A' = A_1 \otimes A_2$  is the  $\mathcal{FOPEQ}$ -model amalgamation of  $A_1$  and  $A_2$ . We construct the initialising set  $L' = L_1 \otimes L_2$  by amalgamating the initialising sets  $L_1$  and  $L_2$  to get the set of all possible combinations of variable mappings, while respecting the amalgamations induced on variable names via the pushout  $\Sigma.V'$ .

For example, suppose that the sort-indexed variable  $x$  is in  $\Sigma.V$ . Then we can apply the  $\mathcal{EVT}$ -signature morphisms  $\sigma_1 : \Sigma \rightarrow \Sigma_1$  and  $\sigma_2 : \Sigma \rightarrow \Sigma_2$  to get  $\sigma_1(x) = x_1 \in \Sigma_1.V$  and  $\sigma_2(x) = x_2 \in \Sigma_2.V$ . From Proposition 4.16, we know that the pushout  $(\Sigma_1.V \cup \Sigma_2.V) / \sim$  contains the variable name  $x'$  such that  $\sigma'_1(x_1) = \sigma'_2(x_2) = x' \in \Sigma'.V$ . We follow this pushout construction in order to construct the corresponding model amalgamation. A model  $M \in |\mathbf{Mod}(\Sigma)|$  contains an initialising set  $L$  which, in turn, contains maplets of the form  $x \mapsto a$  where  $a$  is a sort-appropriate value for the variable  $x$ . In light of the  $\mathcal{EVT}$ -signature morphisms outlined above, we know that  $M_1 \in |\mathbf{Mod}(\Sigma_1)|$  contains maplets of the form

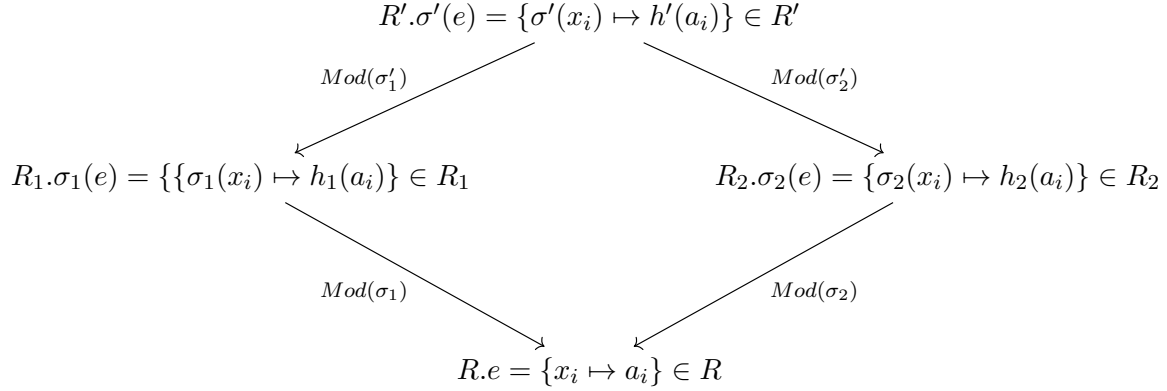


Figure 26: The construction of  $R' = R_1 \otimes R_2$ , the amalgamation of  $R_1$  and  $R_2$ .

$x_1 \mapsto a_1$  and  $M_2 \in |\mathbf{Mod}(\Sigma_2)|$  contains maplets of the form  $x_2 \mapsto a_2$  where  $a_1$  and  $a_2$  are sort-appropriate values for  $x_1$  and  $x_2$  respectively. Then the amalgamation  $M' \in |\mathbf{Mod}(\Sigma)|$  has the initialising set  $L' = L_1 \otimes L_2$  which contains all maplets of the form  $x' \mapsto a'$  where  $a'$  is a sort-appropriate value for  $x'$  drawn from  $A'$  (the  $\mathcal{FOP}\mathcal{EQ}$ -amalgamation of  $A_1$  and  $A_2$ ). Note that initialising sets contain sets of variable-to-value mappings for all of the variables in the  $V$  component of their signatures, as such, the amalgamation contains the above maplets in all possible combinations with those corresponding to any other variables that may be in the signature.

We construct the relation  $R' = R_1 \otimes R_2$ , which is the amalgamation of  $R_1$  and  $R_2$ , in a similar manner. Specifically, starting from any  $R.e = \{s_1, \dots, s_m\} \in R$  where  $s_1, \dots, s_m$  are states of the form

$$\{x_1 \mapsto a_1, \dots, x_n \mapsto a_n\}$$

where  $x_1, \dots, x_n$  are the variable names (and their primed versions drawn from  $\Sigma.V$ ). We construct the corresponding relation  $R'.\sigma'(e)$  in  $R'$  so that the diagram in Figure 26 commutes.

In Figure 26,  $h' = (h_1 + h_2)$  is the corresponding function over the carrier-sets in  $M'$  obtained from  $\mathcal{FOP}\mathcal{EQ}$ , and  $\sigma' = (\sigma'_1 \circ \sigma_1) + (\sigma'_2 \circ \sigma_2)$  is the mapping for variable and event names obtained from the corresponding construction in  $\mathbf{Sign}_{\mathcal{EVT}}$ .  $\square$

**Sub-Proposition A.2.** *For any two  $\mathcal{EVT}$ -model morphisms  $f_1 : M_{11} \rightarrow M_{12}$  in  $\mathbf{Mod}(\Sigma_1)$  and  $f_2 : M_{21} \rightarrow M_{22}$  in  $\mathbf{Mod}(\Sigma_2)$  such that  $f_1|_{\sigma_1} = f_2|_{\sigma_2}$ , there exists an  $\mathcal{EVT}$ -model morphism (the amalgamation of  $f_1$  and  $f_2$ ) called  $f' : M'_1 \rightarrow M'_2$  in  $\mathbf{Mod}(\Sigma')$ , such that  $f'|_{\sigma'_1} = f_1$  and  $f'|_{\sigma'_2} = f_2$ .*

*Proof.* Given the  $\mathcal{EVT}$ -model morphisms  $f_1$  and  $f_2$  and their common reduct  $f_0$ , we construct  $f'$  so that the following diagram commutes:

$$\begin{array}{ccc}
& f' : M'_1 \rightarrow M'_2 & \\
\text{Mod}(\sigma'_1) \swarrow & & \searrow \text{Mod}(\sigma'_2) \\
f_1 : M_{11} \rightarrow M_{12} & & f_2 : M_{21} \rightarrow M_{22} \\
\text{Mod}(\sigma_1) \searrow & & \swarrow \text{Mod}(\sigma_2) \\
& f_0 : M_{01} \rightarrow M_{02} &
\end{array}$$

Since each  $\mathcal{EVT}$ -model has a  $\mathcal{FOPEQ}$  model as its first component, each of the  $\mathcal{EVT}$ -model morphisms  $f_0$ ,  $f_1$ ,  $f_2$  and  $f'$  must have an underlying  $\mathcal{FOPEQ}$ -model morphism, which we denote  $f_0^-$ ,  $f_1^-$ ,  $f_2^-$  and  $f'^-$  respectively. To build the amalgamation for  $\mathcal{EVT}$ -models we must show how to extend these to cover the data states of the  $\mathcal{EVT}$ -models. This  $\mathcal{EVT}$ -model morphism follows the underlying  $\mathcal{FOPEQ}$ -model morphism on sort carrier sets for the values in the data states.

Given  $R.e \in R$ , suppose we start with any  $f_0$ -maplet of the form

$$\begin{aligned}
& \{\dots, \{x_1 \mapsto a_1, \dots, x_n \mapsto a_n\}, \dots\} \\
& \mapsto \{\dots, \{x_1 \mapsto f_0^-(a_1), \dots, x_n \mapsto f_0^-(a_n)\}, \dots\}_e \in f_0
\end{aligned}$$

where  $f_0^-$  is the underlying map on data types from the  $\mathcal{FOPEQ}$ -model morphism. Then the original two functions in  $f_1$  and  $f_2$  must have maplets of the form

$$\begin{aligned}
& \{\dots, \{\sigma_1(x_1) \mapsto h_1(a_1), \dots, \sigma_1(x_n) \mapsto h_1(a_n)\}, \dots\} \\
& \mapsto \{\dots, \{\sigma_1(x_1) \mapsto f_1^-(h_1(a_1)), \dots, \sigma_1(x_n) \mapsto f_1^-(h_1(a_n))\}, \dots\} \in f_1
\end{aligned}$$

and

$$\begin{aligned}
& \{\dots, \{\sigma_2(x_1) \mapsto h_2(a_1), \dots, \sigma_2(x_n) \mapsto h_2(a_n)\}, \dots\} \\
& \mapsto \{\dots, \{\sigma_2(x_1) \mapsto f_2^-(h_2(a_1)), \dots, \sigma_2(x_n) \mapsto f_2^-(h_2(a_n))\}, \dots\} \in f_2
\end{aligned}$$

where  $f_1^-$  and  $f_2^-$  are again the data type maps from the underlying  $\mathcal{FOPEQ}$ -model morphism, and  $h_1$  and  $h_2$  are obtained from  $\text{Mod}(\sigma_1)$  and  $\text{Mod}(\sigma_2)$ .

We then can construct the elements of the  $\mathcal{EVT}$ -model morphism  $f'$ , which is the amalgamation of  $f_1$  and  $f_2$ , as  $f'$ -maplets of the form:

$$\begin{aligned}
& \{\dots, \{\sigma'(x_1) \mapsto h'(a_1), \dots, \sigma'(x_n) \mapsto h'(a_n)\}, \dots\} \\
& \mapsto \{\dots, \{\sigma'(x_1) \mapsto f'^-(h'(a_1)), \dots, \sigma'(x_n) \mapsto f'^-(h'(a_n))\}, \dots\} \in f'
\end{aligned}$$

As before,  $h' = (h_1 + h_2)$  is the corresponding function over the carrier-sets in  $M'$  obtained from  $\mathcal{FOPEQ}$ , and  $\sigma' = (\sigma'_1 \circ \sigma_1) + (\sigma'_2 \circ \sigma_2)$  is the mapping for variable and event names obtained from the corresponding construction in **Sign**. Here  $f'^- = f_1^- + f_2^-$  is the model morphism amalgamation from the corresponding diagram for model morphisms in  $\mathcal{FOPEQ}$ , which ensures that the data states are mapped to corresponding states in the model  $M'_2$ .  $\square$