

DIRECT SPECTRA OF BISHOP SPACES AND THEIR LIMITS

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ABSTRACT. We apply fundamental notions of Bishop set theory (BST), an informal theory that complements Bishop’s theory of sets, to the theory of Bishop spaces, a function-theoretic approach to constructive topology. Within BST we develop the notions of a direct family of sets, of a direct spectrum of Bishop spaces, of the direct limit of a direct spectrum of Bishop spaces, and of the inverse limit of a contravariant direct spectrum of Bishop spaces. Within the extension of Bishop’s informal system of constructive mathematics BISH with inductive definitions with rules of countably many premises, we prove the fundamental theorems on the direct and inverse limits of spectra of Bishop spaces and the duality principle between them.

1. INTRODUCTION

1.1. Bishop’s set theory. The theory of sets underlying Bishop-style constructive mathematics (BISH) was only sketched in Chapter 3 of Bishop’s seminal book [4]. Since Bishop’s central aim in [4] was to show that a large part of advanced mathematics can be done within a constructive and computational framework that does not contradict the classical practice, the inclusion of a detailed account of the set-theoretic foundations of BISH could possibly be against the effective delivery of his message.

The Bishop-Cheng measure theory, developed in [7], was very different from the measure theory of [4], and the inclusion of an enriched version of the former into [8], the book on BISH that Bishop co-authored with Bridges later, affected the corresponding Chapter 3 in two main respects. First, the inductively defined notion of the set of Borel sets generated by a given family of complemented subsets of a set X , with respect to a set of real-valued functions on X , was excluded, as unnecessary, and, second, the operations on the complemented subsets of a set X were defined differently, and in accordance to the needs of the new measure theory.

Yet, in both books many issues were left untouched, a fact that often was a source of confusion. In many occasions, especially in the measure theory of [7] and [8], the powerset was treated as a set, while in the measure theory of [4], Bishop generally avoided the powerset by using appropriate families of subsets instead. In later works of Bridges and Richman, like [10] and [18], the powerset was clearly used as a set, in contrast though, to

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the predicative spirit of [4]. The concept of a family of sets indexed by a (discrete) set, was asked to be defined in [4] (Exercise 2, p. 72), and a definition, attributed to Richman, was given in [8] (Exercise 2, p. 78). An elaborate study of this concept within BISH is missing though, despite its central character in the measure theory of [4], its extensive use in the theory of Bishop spaces [25] and in abstract constructive algebra [18]. Actually, in [18] Richman introduced the more general notion of a family of objects of a category indexed by some set, but the categorical component in the resulting mixture of Bishop’s set theory and category theory was not explained in constructive terms¹.

Bishop briefly discussed some formal aspects of BISH in [6], where a variant of Gödel’s T was proposed as a formal system for BISH, while in his unpublished work [5] he elaborated a version of dependent type theory with one universe, in order to formalise BISH. The various set-theoretic formalisations of BISH, developed by Myhill [19], Friedman [13], Aczel [1], Feferman [12], Beeson [3], and Greenleaf [14], were, roughly speaking, quite influenced by Zermelo-Fraenkel set theory, and “top-down” approaches to BISH, with many “unexpected” features² with respect to its practice.

The type-theoretic interpretation of Bishop’s set theory into the theory of setoids (see especially the work of Palmgren [20]-[24]) has become nowadays the standard way to understand Bishop sets. The identity type of Martin-Löf’s type theory (MLTT) (see [17]), expresses, in a proof-relevant way, the existence of the least reflexive relation on a type, a fact with no counterpart in Bishop’s set theory. As a consequence, the free setoid on a type is definable (see [22], p. 90), and the presentation axiom in setoids is provable. Moreover, in MLTT the families of types over a type I form the type $I \rightarrow \mathcal{U}$, which belongs to the successor universe \mathcal{U}' of \mathcal{U} . In Bishop’s set theory though, where only one universe of sets is used, the set-character of the totality of all families of sets indexed by some set I is questionable from the predicative point of view (see our comment after Definition 3.2).

1.2. Bishop Set Theory. Bishop set theory (BST) is an informal, constructive theory of totalities and assignment routines that serves as a reconstruction of Bishop’s set theory. Its aim is, first, to fill in the “gaps” in Bishop’s account of the set theory underlying BISH, and, second, to serve as an intermediate step between Bishop’s informal set theory and a *suitable* i.e., an *adequate* and *faithful*, in the sense of Feferman [12], formalisation of BISH. To assure faithfulness, we need to use concepts or principles that appear, explicitly or implicitly, in BISH.

Next we describe briefly the features of BST that “complete” Bishop’s theory of sets in [35].

1. Explicit use of a universe of sets. Bishop used a universe of sets only implicitly. E.g., he “roughly” describes in [4], p. 72, a set-indexed family of sets as

... a rule which assigns to each t in a discrete set T a set $\lambda(t)$.

Every other rule, or assignment routine mentioned by Bishop is from one given totality, the domain of the rule, to some other totality, its codomain. The only way to make the rule of a family of sets compatible with this pattern is to employ a totality of sets. In the unpublished manuscript [5] Bishop explicitly used a universe in his formulation of dependent type theory as a formal system for BISH. Here we use an open-ended totality \forall_0 of sets, which contains

¹As it was done in the formulation of category theory in homotopy type theory (see Chapter 9 in [41]).

²A detailed presentation of the unexpected features of these formalisations is going to be found in [35].

the primitive set \mathbb{N} and all defined (predicative) sets. \mathbb{V}_0 itself is not a set, but a class. It is a notion instrumental to the definition of dependent operations, and of a set-indexed family of sets.

2. Clear distinction between sets and classes. A class is a totality defined through a membership condition in which a quantification over \mathbb{V}_0 occurs. The powerset $\mathcal{P}(X)$ of a set X , the totality $\mathcal{P}^{\llbracket}(X)$ of complemented subsets of a set X , and the totality $\mathcal{F}(X, Y)$ of partial functions from a set X to a set Y are characteristic examples of classes. A class is never used here as the domain of an assignment routine, but only as a codomain of an assignment routine.

3. Explicit use of dependent operations. The standard view, even among practitioners of Bishop-style constructive mathematics, is that dependency is not necessary to BISH. Dependent functions though, do appear explicitly in Bishop's definition of the intersection $\bigcap_{t \in T} \lambda(t)$ of a family λ of subsets of some set X indexed by an inhabited set T (see [4], p. 65, and [8], p. 70). As we try to show in [35], the elaboration of dependency within BISH is only fruitful to it. Dependent functions are not only necessary to the definition of products of families of sets indexed by an arbitrary set, but in many areas of constructive mathematics. As already mentioned, dependency is formulated in Bishop's type theory [5]. The somewhat "silent" role of dependency within Bishop's set theory is replaced by a central role within BST.

4. Elaboration of the theory of families of sets. With the use of the universe \mathbb{V}_0 , of the notion of a non-dependent assignment routine λ_0 from an index-set I to \mathbb{V}_0 , and of a certain dependent operation λ_1 , we define explicitly in Definition 3.1 the notion of a family of sets indexed by I . Although an I -family of sets is a certain function-like object, it can also be understood as an object with level the level of a set plus one. The corresponding notion of a "function" from an I -family Λ to an I -family M is that of a family-map. Operations between sets generate operations between families of sets and their family-maps. If the index-set I is a directed set, the corresponding notion of a family of sets over it is that of a direct family of sets. *Families of subsets* of a given set X over an index-set I are special I -families that deserve an independent treatment. Families of equivalence classes, families of partial functions, families of complemented subsets and direct families of subsets are some of the variations of set-indexed families of subsets that are studied in [35] with many applications in Bishop-style constructive mathematics.

Here we apply the general theory of families of sets, in order **to develop the theory of spectra of Bishop spaces**. A Bishop space is a constructive, function-theoretic alternative to the notion of a topological space. A Bishop topology F on a set X is a subset of the set $\mathbb{F}(X)$ of real-valued functions on X that includes the constant functions and it is closed under addition, composition with Bishop continuous functions from \mathbb{R} to \mathbb{R} , and uniform limits. Hence, in contrast to topological spaces, continuity of real-valued functions is a primitive notion and a concept of open set comes a posteriori. A Bishop topology on a set can be seen as an abstract and constructive approach to the ring of continuous functions $C(X)$ of a topological space X . Associating appropriately a Bishop topology to the set $\lambda_0(i)$ of a family of sets over a set I , for every $i \in I$, the notion of a spectrum of Bishop spaces is defined. If I is a directed set, we get a direct spectrum. The theory of direct spectra of Bishop spaces and their limits is developed here in complete analogy to the classical theory of spectra of topological spaces and their limits (see [11], Appendix Two). The constructive

theory of spectra of other structures, like groups, or rings, or modules, can be developed along the same lines.

1.3. Outline of this paper. We structure this paper³ as follows:

- (1) In Section 2 we present the fundamental notions of BST that are used in the subsequent sections.
- (2) In Section 3 we define the families of Bishop sets indexed by some set I and the family-maps between them. The corresponding \sum - and \prod -sets are introduced.
- (3) In Section 4 we introduce the notion of a directed set with a modulus of directedness and the notion of a cofinal subset of a directed set with a modulus of cofinality. These moduli help us avoid the use of some choice-principle in later proofs.
- (4) In Section 5 we define the families of Bishop sets indexed by some directed set (I, \preceq) and the family-maps between them. The corresponding \sum^{\preceq} - and \prod^{\preceq} -sets are introduced.
- (5) In Section 6 we include the basic notions and facts on Bishop spaces that are used in the subsequent sections.
- (6) In Section 7 we define the notion of a direct spectrum of Bishop spaces, the notion of a continuous, direct spectrum-map, and we define a canonical Bishop topology on the direct sum of Bishop spaces.
- (7) In Section 8 we define the families of subsets of a given set X indexed by some set I , and the family-maps between them. We also define sets of subsets of a set X indexed by some set I , and set-indexed families of equivalence classes of an equivalence structure. These notions are used in the definition of the direct limit of a (covariant) spectrum of Bishop spaces.
- (8) In Section 9 we define the direct limit $\varinjlim \mathcal{F}_i$ of a direct spectrum of Bishop spaces, we prove the universal property of the direct limit for $\varinjlim \mathcal{F}_i$ (Proposition 9.5), the existence of a unique map of the limit spaces from a spectrum-map (Theorem 9.8), the cofinality theorem for direct limits (Theorem 9.12), and the existence of a Bishop bijection for the product of spectra of Bishop spaces (Proposition 9.15).
- (9) In Section 10 we define the inverse limit $\varprojlim \mathcal{F}_i$ of a contravariant direct spectrum of Bishop spaces, we prove the universal property of the inverse limit for $\varprojlim \mathcal{F}_i$ (Proposition 10.2), the existence of a unique map of the limit spaces from a spectrum-map (Theorem 10.3), the cofinality theorem for inverse limits (Theorem 10.5), and the existence of a Bishop morphism for the product of inverse spectra of Bishop spaces (Proposition 10.6).
- (10) In Section 11 we prove the duality principle between the inverse and direct limits of Bishop spaces (Theorem 11.3).

The above results form a constructive counterpart to the theory of limits of topological spaces, as this is presented in [11], Appendix Two. As in the classic textbook of Dugundji, the choice of presentation is *non-categorical* and *purely topological*. By the first we mean that the language of category theory is avoided, and by the second, that the limit constructions are done directly for spaces. Our central aim is to present the basic theory of spectra of Bishop spaces **within** BST. The use of categorical arguments would be consistent with our aim, only if the corresponding category theory was formulated within BST. In [18]

³Here we continue our work in [32], where we showed the distributivity of dependent sums over dependent products within BST (there we used the acronym CSFT instead).

Richman used the notion of functor, to define the concept of a set-indexed family of sets. The problem with Richman’s approach was that the category theory involved was not explained in constructive terms, and its relation to BISH was left unclear. Here we avoid the use of categorical concepts in order to establish our results, as we want to work exclusively within BST. Actually, in [35], pp. 80–83 we use the notion of a set-indexed family of sets in order to *define* the notion of a functor between small categories within BST!

The theory of Bishop spaces, that was only sketched by Bishop in [4], and revived by Bridges in [9], and Ishihara in [15], was developed by the author in [25]–[31] and [33], [34]. Since it makes use of inductive definitions with rules of countably many premises, for their study we work within BST*, which is BST extended with such inductive definitions. Our concepts and results avoid the use even of countable choice (CC). Although practitioners of BISH usually embrace dependent choice, hence CC, using non-sequential or non-choice-based arguments instead, forces us to formulate “better” concepts and find “better” proofs. This standpoint was advocated first by Richman in [37]. There are results in [8] that require CC in their proof, but we make no use of them here. E.g., our proofs of the cofinality theorems (Theorem 9.12 and Theorem 10.5) are choice-free as we use a proof-relevant definition of a cofinal subset. Notice that the formulation of the universal properties of the limits of spectra of Bishop spaces is *impredicative*, as it requires quantification over the *class* of Bishop spaces⁴. This is one reason we avoided the use of the universal properties in our proofs.

A formal system for BISH extended with such definitions is Myhill’s formal system CST* where CST* is Myhill’s extension of his formal system of constructive set theory CST with inductive definitions (see [19]). A variation of CST* is Aczel’s system CZF together with a weak version of Aczel’s regular extension axiom (REA), to accommodate these inductive definitions (see [1]).

For all basic facts on constructive analysis we refer to [8], for all results on Bishop spaces that are used here without proof we refer to [25], and for all results on BST that are not shown here we refer to [35].

2. FUNDAMENTALS OF BISHOP SET THEORY

The logical framework of BST is first-order intuitionistic logic with equality (see [39], chapter 1). The primitive equality between terms is denoted by $s := t$, and it is understood as a *definitional*, or *logical*, equality. I.e., we read the equality $s := t$ as “the term s is by definition equal to the term t ”. If ϕ is an appropriate formula, for the standard axiom for equality $[a := b \ \& \ \phi(a)] \Rightarrow \phi(b)$ we use the notation $[a := b \ \& \ \phi(a)] :\Rightarrow \phi(b)$. The equivalence notation $:\Leftrightarrow$ is understood in the same way. The set $(\mathbb{N}, =_{\mathbb{N}}, \neq_{\mathbb{N}})$ of natural numbers, where its canonical equality is given by $m =_{\mathbb{N}} n :\Leftrightarrow m := n$, and its canonical inequality by $m \neq_{\mathbb{N}} n :\Leftrightarrow \neg(m =_{\mathbb{N}} n)$, is primitive. The standard Peano-axioms are associated to \mathbb{N} .

A global operation (\cdot, \cdot) of pairing is also considered primitive. I.e., if s, t are terms, their pair (s, t) is a new term. The corresponding equality axiom is $(s, t) := (s', t') :\Leftrightarrow s := s' \ \& \ t := t'$. The n -tuples of given terms, for every n larger than 2, are definable. The global projection routines $\mathbf{pr}_1(s, t) := s$ and $\mathbf{pr}_2(s, t) := t$ are also considered primitive. The corresponding global projection routines for any n -tuples are definable.

An undefined notion of mathematical construction, or algorithm, or of finite routine is considered as primitive. The main objects of BST are totalities and assignment routines.

⁴A predicative formulation of a universal property can be given, if one is restricted to a given set-indexed family of Bishop spaces.

Sets are special totalities and functions are special assignment routines, where an assignment routine is a special finite routine. All other equalities in BST are equalities on totalities defined though an equality condition. A predicate on a set X is a bounded formula $P(x)$ with x a free variable ranging through X , where a formula is bounded, if every quantifier occurring in it is over a given set.

Definition 2.1. (i) *A primitive set \mathbb{A} is a totality with a given membership $x \in \mathbb{A}$, and a given equality $x =_{\mathbb{A}} y$, that satisfies axiomatically the properties of an equivalence relation. The set \mathbb{N} of natural numbers is the only primitive set considered here.*

(ii) *A (non-inductive) defined totality X is defined by a membership condition $x \in X :\Leftrightarrow \mathcal{M}_X(x)$, where \mathcal{M}_X is a formula with x as a free variable.*

(iii) *There is a special “open-ended” defined totality \mathbb{V}_0 , which is called the universe of (predicative) sets. \mathbb{V}_0 is not defined through a membership-condition, but in an open-ended way. When we say that a defined totality X is considered to be a set we “introduce” X as an element of \mathbb{V}_0 . We do not add the corresponding induction, or elimination principle, as we want to leave open the possibility of adding new sets in \mathbb{V}_0 .*

(iv) *A defined preset X , or simply, a preset, is a defined totality X the membership condition \mathcal{M}_X of which expresses a construction.*

(v) *A defined totality X with equality, or simply, a totality X with equality is a defined totality X equipped with an equality condition $x =_X y :\Leftrightarrow \mathcal{E}_X(x, y)$, where $\mathcal{E}_X(x, y)$ is a formula with free variables x and y that satisfies the conditions of an equivalence relation.*

(vi) *A defined set is a preset with a given equality, specified by a bounded formula.*

(vii) *A set is either a primitive set, or a defined set.*

(viii) *A totality is a class, if it is the universe \mathbb{V}_0 , or if quantification over \mathbb{V}_0 occurs in its membership condition.*

Definition 2.2. A bounded formula on a set X is called an *extensional property* on X , if

$$\forall_{x, y \in X} ([x =_X y \ \& \ P(x)] \Rightarrow P(y)).$$

The totality X_P generated by $P(x)$ is defined by $x \in X_P :\Leftrightarrow x \in X \ \& \ P(x)$,

$$x \in X_P :\Leftrightarrow x \in X \ \& \ P(x),$$

and the equality of X_P is inherited from the equality of X . We also write $X_P := \{x \in X \mid P(x)\}$, X_P is considered to be a set, and it is called the *extensional subset* of X generated by P .

Using the properties of an equivalence relation, it is immediate to show that an equality condition $\mathcal{E}_X(x, y)$ on a totality X is an extensional property on the product $X \times X$ i.e., $[(x, y) =_{X \times X} (x', y') \ \& \ x =_X y] \Rightarrow x' =_X y'$. We consider the following extensional subsets of \mathbb{N} :

$$\mathbb{1} := \{x \in \mathbb{N} \mid x =_{\mathbb{N}} 0\} := \{0\},$$

$$\mathbb{2} := \{x \in \mathbb{N} \mid x =_{\mathbb{N}} 0 \ \vee \ x =_{\mathbb{N}} 1\} := \{0, 1\}.$$

Since $n =_{\mathbb{N}} m :\Leftrightarrow n := m$, the property $P(x) :\Leftrightarrow x =_{\mathbb{N}} 0 \ \vee \ x =_{\mathbb{N}} 1$ is extensional.

Definition 2.3. If $(X, =_X)$ is a set, its *diagonal* is the extensional subset of $X \times X$

$$D(X, =_X) := \{(x, y) \in X \times X \mid x =_X y\}.$$

If $=_X$ is clear from the context, we just write $D(X)$.

Definition 2.4. Let X, Y be totalities. A *non-dependent assignment routine* f from X to Y , in symbols $f: X \rightsquigarrow Y$, is a finite routine that assigns an element y of Y to each given element x of X . In this case we write $f(x) := y$. If $g: X \rightsquigarrow Y$, let

$$f := g \Leftrightarrow \forall_{x \in X} (f(x) := g(x)).$$

If $f := g$, we say that f and g are *definitionally equal*. If $(X, =_X)$ and $(Y, =_Y)$ are sets, an *operation* from X to Y is a non-dependent assignment routine from X to Y , while a *function* from $(X, =_X)$ to $(Y, =_Y)$, in symbols $f: X \rightarrow Y$, is an operation from $(X, =_X)$ to $(Y, =_Y)$ that respects equality i.e.,

$$\forall_{x, x' \in X} (x =_X x' \Rightarrow f(x) =_Y f(x')).$$

If $f: X \rightsquigarrow Y$ is a function from X to Y , we say that f is a function, without mentioning the expression “from X to Y ”. If X is a set, the *identity function* $\text{id}_X: X \rightarrow X$ is defined by the rule $x \mapsto x$, for every $x \in X$. A function $f: X \rightarrow Y$ is an *embedding*, in symbols $f: X \hookrightarrow Y$, if

$$\forall_{x, x' \in X} (f(x) =_Y f(x') \Rightarrow x =_X x').$$

Let X, Y be sets. The totality $\mathcal{O}(X, Y)$ of operations from X to Y is equipped with the following canonical equality:

$$f =_{\mathcal{O}(X, Y)} g \Leftrightarrow \forall_{x \in X} (f(x) =_Y g(x)).$$

The totality $\mathcal{O}(X, Y)$ is considered to be a set. The set $\mathbb{F}(X, Y)$ of functions from X to Y is defined by separation on $\mathcal{O}(X, Y)$ through the extensional property $P(f) \Leftrightarrow \forall_{x, x' \in X} (x =_X x' \Rightarrow f(x) =_Y f(x'))$. The equality $=_{\mathbb{F}(X, Y)}$ is inherited from $=_{\mathcal{O}(X, Y)}$.

The canonical equality on \mathbb{V}_0 is defined by

$$X =_{\mathbb{V}_0} Y \Leftrightarrow \exists_{f \in \mathbb{F}(X, Y)} \exists_{g \in \mathbb{F}(Y, X)} (g \circ f = \text{id}_X \ \& \ f \circ g = \text{id}_Y).$$

In this case we write $(f, g): X =_{\mathbb{V}_0} Y$.

Definition 2.5. Let I be a set and $\lambda_0: I \rightsquigarrow \mathbb{V}_0$ a non-dependent assignment routine from I to \mathbb{V}_0 . A *dependent operation* Φ over λ_0 , in symbols

$$\Phi: \bigwedge_{i \in I} \lambda_0(i),$$

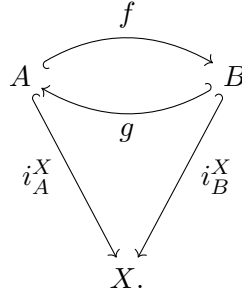
is an assignment routine that assigns to each element i in I an element $\Phi(i)$ in the set $\lambda_0(i)$. If $i \in I$, we call $\Phi(i)$ the *i -component* of Φ , and we also use the notation $\Phi_i := \Phi(i)$. An assignment routine is either a non-dependent assignment routine, or a dependent operation over some non-dependent assignment routine from a set to the universe. If $\Psi: \bigwedge_{i \in I} \lambda_0(i)$, $\Phi := \Psi \Leftrightarrow \forall_{i \in I} (\Phi_i := \Psi_i)$. If $\Phi := \Psi$, we say that Φ and Ψ are definitionally equal. Let $\mathbb{A}(I, \lambda_0)$ be the totality of dependent operations over λ_0 , equipped with the canonical equality $\Phi =_{\mathbb{A}(I, \lambda_0)} \Psi \Leftrightarrow \forall_{i \in I} (\Phi_i =_{\lambda_0(i)} \Psi_i)$. The totality $\mathbb{A}(I, \lambda_0)$ is considered to be a set.

Definition 2.6. If X is a set, a *subset* of X is a pair (A, i_A^X) , where A is a set and $i_A^X: A \hookrightarrow X$ is an embedding. If (A, i_A^X) and (B, i_B^X) are subsets of X , we say that A is a *subset* of B , and we write $A \subseteq B$, if there is $f: A \rightarrow B$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ i_A^X \searrow & & \swarrow i_B^X \\ & X & \end{array}$$

In this case we write $f : A \subseteq B$, and usually we write A instead of (A, i_A^X) , and i_A instead of i_A^X , if X is clear from the context. The totality of the subsets of X is the *powerset* $\mathcal{P}(X)$, and it is equipped with the equality $(A, i_A) =_{\mathcal{P}(X)} (B, i_B) :\Leftrightarrow A \subseteq B \ \& \ B \subseteq A$. If $f : A \subseteq B$ and $g : B \subseteq A$, we write $(f, g) : A =_{\mathcal{P}(X)} B$. If $f : A \subseteq B$, then f is an embedding: if $a, a' \in A$ are such that $f(a) =_B f(a')$, then $i_B(f(a)) =_X i_B(f(a')) \Leftrightarrow i_A(a) =_X i_A(a')$, which implies $a =_A a'$.

If $(f, g) : A =_{\mathcal{P}(X)} B$, all the following diagrams commute



Moreover, the internal equality of subsets implies their external equality as sets i.e.,

$$(f, g) : A =_{\mathcal{P}(X)} B \Rightarrow (f, g) : A =_{\mathbb{V}_0} B.$$

If $a \in A$, then $i_A(g(f(a))) =_X i_B(f(a)) =_X i_A(a)$, hence $g(f(a)) =_A a$, and $g \circ f =_{\mathbb{F}(A,A)} \text{id}_A$. Similarly, $f \circ g =_{\mathbb{F}(B,B)} \text{id}_B$. Since the membership condition of $\mathcal{P}(X)$ requires quantification over \mathbb{V}_0 , the totality $\mathcal{P}(X)$ is a class. If X_P is an extensional subset of X , then $(X_P, i_{X_P}) \subseteq X$, where $i_{X_P} : X_P \rightarrow X$ is defined by $x \mapsto x$, for every $x \in X_P$.

3. SET-INDEXED FAMILIES OF SETS

An I -family of sets is an assignment routine $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$ that behaves like a function i.e., if $i =_I j$, then $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$. The following definition is an exact formulation of this rough description⁵.

Definition 3.1. If I is a set, a *family of sets* indexed by I , or an I -family of sets, is a pair $\Lambda := (\lambda_0, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and⁶

$$\lambda_1 : \bigwedge_{(i,j) \in D(I)} \mathbb{F}(\lambda_0(i), \lambda_0(j)), \quad \lambda_1(i, j) =: \lambda_{ij}, \quad (i, j) \in D(I),$$

such that the following conditions hold:

- (a) For every $i \in I$, we have that $\lambda_{ii} := \text{id}_{\lambda_0(i)}$.
- (b) If $i =_I j$ and $j =_I k$, the following diagram commutes

⁵In [18] Richman saw a set I as a category with objects its elements and $\text{Hom}_{=I}(i, j) := \{x \in \{0\} \mid i =_I j\}$, for every $i, j \in I$. If we view \mathbb{V}_0 as a category with objects its elements and $\text{Hom}_{=\mathbb{V}_0}(X, Y) := \{(f, f') : \mathbb{F}(X, Y) \times \mathbb{F}(Y, X) \mid (f, f') : X =_{\mathbb{V}_0} Y\}$, for every $X, Y \in \mathbb{V}_0$, then an I -family of sets is a functor from the category I to the category \mathbb{V}_0 . Notice that in the definitions of $\text{Hom}_{=I}(i, j)$ and of $\text{Hom}_{=\mathbb{V}_0}(X, Y)$ the properties $P_I(x) := i =_I j$ and $Q_{X,Y}(f, f') := (f, f') : X =_{\mathbb{V}_0} Y$ are extensional.

⁶More accurately, we should write $\lambda_1 : \bigwedge_{z \in D(I)} \mathbb{F}(\lambda_0(\text{pr}_1(z)), \lambda_0(\text{pr}_2(z)))$.

$$\begin{array}{ccc}
\lambda_0(i) & & \\
\lambda_{ij} \downarrow & \searrow \lambda_{ik} & \\
\lambda_0(j) & \xrightarrow{\lambda_{jk}} & \lambda_0(k).
\end{array}$$

If $i =_I j$, we call the function λ_{ij} the *transport map*⁷ from $\lambda_0(i)$ to $\lambda_0(j)$. We call the assignment routine λ_1 the *modulus of function-likeness* of λ_0 ⁸. An I -family of sets is called an I -set of sets, if

$$\forall_{i,j \in I} (\lambda_0(i) =_{\nu_0} \lambda_0(j) \Rightarrow i =_I j).$$

If A is a set, the *constant I -family* A is the pair $\Lambda^A := (\lambda_0^A, \lambda_1^A)$, where $\lambda_0(i) := A$, for every $i \in I$, and $\lambda_1(i, j) := \text{id}_A$, for every $(i, j) \in D(I)$.

Definition 3.2. Let $\Lambda := (\lambda_0, \lambda_1)$ and $M := (\mu_0, \mu_1)$ be I -families of sets. A *family-map* from Λ to M is a dependent operation⁹ $\Psi : \lambda_{i \in I} \mathbb{F}(\lambda_0(i), \mu_0(i))$, such that for every $(i, j) \in D(I)$ the following diagram commutes

$$\begin{array}{ccc}
\lambda_0(i) & \xrightarrow{\lambda_{ij}} & \lambda_0(j) \\
\Psi_i \downarrow & & \downarrow \Psi_j \\
\mu_0(i) & \xrightarrow{\mu_{ij}} & \mu_0(j).
\end{array}$$

where $\Psi_i := \Psi(i)$ is the i -component of Ψ , for every $i \in I$. We denote by $\text{Map}_I(\Lambda, M)$ the totality of family-maps from Λ to M , which is equipped with the equality

$$\Psi =_{\text{Map}_I(\Lambda, M)} \Xi := \Leftrightarrow \forall_{i \in I} (\Psi_i =_{\mathbb{F}(\lambda_0(i), \mu_0(i))} \Xi_i).$$

We also write $\Phi : \Lambda \Rightarrow M$ to denote an element of $\text{Map}_I(\Lambda, M)$. If $\Psi : \Lambda \Rightarrow M$ and $\Xi : M \Rightarrow N$, the *composition family-map* $\Xi \circ \Psi : \Lambda \Rightarrow N$ is defined by $(\Xi \circ \Psi)(i) := \Xi_i \circ \Psi_i$

$$(\Xi \circ \Psi)_i \left(\begin{array}{ccc}
\lambda_0(i) & \xrightarrow{\lambda_{ij}} & \lambda_0(j) \\
\downarrow \Psi_i & & \downarrow \Psi_j \\
\mu_0(i) & \xrightarrow{\mu_{ij}} & \mu_0(j) \\
\downarrow \Xi_i & & \downarrow \Xi_j \\
\nu_0(i) & \xrightarrow{\nu_{ij}} & \nu_0(j),
\end{array} \right) (\Xi \circ \Psi)_j$$

⁷We draw this term from MLTT.

⁸Since $\lambda_{ii} = \lambda_{ji} \circ \lambda_{ij}$ and $\lambda_{jj} = \lambda_{ij} \circ \lambda_{ji}$, we have that $(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\nu_0} \lambda_0(j)$.

⁹In the categorical setting of Richman, a family map $\Psi \in \text{Map}_I(\Lambda, M)$ is a natural transformation from the functor Λ to the functor M . In our view, the fact that the most fundamental concepts of category theory, that of a functor and of a natural transformation, are formulated in a natural way in BST through the notion of a dependent operation explains why category theory is so closely connected to BST, or MLTT. Both, BST and MLTT, highlight the role of dependent operations.

for every $i \in I$. The dependent operation

$$\text{Id}_\Lambda : \prod_{i \in I} \mathbb{F}(\lambda_0(i), \lambda_0(i)), \quad \text{Id}_\Lambda(i) := \text{id}_{\lambda_0(i)}, \quad i \in I,$$

is the *identity family-map* from Λ to Λ . The totality of I -families is denoted by $\mathbf{Fam}(I)$, and

$$\Lambda =_{\mathbf{Fam}(I)} M :\Leftrightarrow \exists \Phi \in \mathbf{Map}_I(\Lambda, M) \exists \Xi \in \mathbf{Map}_I(M, \Lambda) (\Phi \circ \Xi = \text{id}_M \ \& \ \Xi \circ \Phi = \text{id}_\Lambda).$$

The equalities on $\mathbf{Map}_I(\Lambda, M)$ and $\mathbf{Fam}(I)$ are equivalence relations. It is natural to accept the totality $\mathbf{Map}(\Lambda, M)$ as a set. If $\mathbf{Fam}(I)$ was a set though, the constant I -family with value $\mathbf{Fam}(I)$ would be defined though a totality in which it belongs to. From a predicative point of view, this cannot be accepted. The membership condition of the totality $\mathbf{Fam}(I)$ though, does not involve quantification over the universe \mathbb{V}_0 , therefore it is also natural not to consider $\mathbf{Fam}(I)$ to be a class. Hence, $\mathbf{Fam}(I)$ is a totality “between” a (predicative) set and a class. For this reason, we say that $\mathbf{Fam}(I)$ is an *impredicative set*.

Definition 3.3. If K is a set, $\Sigma := (\sigma_0, \sigma_1)$ is a K -family of sets and $h : I \rightarrow K$, the h -*subfamily* of Σ is the pair $\Sigma \circ h := (\sigma_0 \circ h, \sigma_1 \circ h)$, where

$$\begin{aligned} (\sigma_0 \circ h)(i) &:= \sigma_0(h(i)); \quad i \in I, \\ (\sigma_1 \circ h)_{ij} &:= (\sigma_1 \circ h)(i, j) : \sigma_0(h(i)) \rightarrow \sigma_0(h(j)); \quad (i, j) \in D(I), \\ (\sigma_1 \circ h)_{ij} &:= \sigma_{h(i)h(j)}. \end{aligned}$$

Clearly, $\Sigma \circ h \in \mathbf{Fam}(I)$, and we write $(\Sigma \circ h)_I \leq \Sigma_K$.

Definition 3.4. Let $\Lambda := (\lambda_0, \lambda_1)$ be an I -family of sets. The¹⁰ *exterior union*, or *disjoint union*, $\sum_{i \in I} \lambda_0(i)$ of Λ is defined by

$$w \in \sum_{i \in I} \lambda_0(i) :\Leftrightarrow \exists i \in I \exists x \in \lambda_0(i) (w := (i, x)),$$

$$(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \ \& \ \lambda_{ij}(x) =_{\lambda_0(j)} y.$$

The totality $\prod_{i \in I} \lambda_0(i)$ of *dependent functions over* Λ is defined by

$$\Phi \in \prod_{i \in I} \lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{A}(I, \lambda_0) \ \& \ \forall (i, j) \in D(I) (\Phi_j =_{\lambda_0(j)} \lambda_{ij}(\Phi_i)),$$

and it is equipped with the equality of $\mathbb{A}(I, \lambda_0)$.

The equalities on $\sum_{i \in I} \lambda_0(i)$ and $\prod_{i \in I} \lambda_0(i)$ are equivalence relations, and both these totalities are sets. If X, Y are sets, let $\Lambda(X, Y) := (\lambda_0^{X, Y}, \lambda_1^{X, Y})$ be the 2-family of X and Y , where $\lambda_0^{X, Y} : \{0, 1\} \rightsquigarrow \mathbb{V}_0$ is defined by

$$\lambda_0^{X, Y}(0) := X \ \& \ \lambda_0^{X, Y}(1) := Y, \quad \& \quad \lambda_1^{X, Y}(0, 0) := \text{id}_X \ \& \ \lambda_1^{X, Y}(1, 1) := \text{id}_Y.$$

It is easy to show that the dependent functions over $\Lambda(X, Y)$ are equal in \mathbb{V}_0 to $X \times Y$. The *first projection* on $\sum_{i \in I} \lambda_0(i)$ is the assignment routine $\mathbf{pr}_1^\Lambda : \sum_{i \in I} \lambda_0(i) \rightsquigarrow I$, defined by

$$\mathbf{pr}_1^\Lambda(i, x) := \mathbf{pr}_1(i, x) := i,$$

for every $(i, x) \in \sum_{i \in I} \lambda_0(i)$. It is immediate to show that \mathbf{pr}_1^Λ is a function from $\sum_{i \in I} \lambda_0(i)$ to I . Moreover, it is easy to show that the pair $\Sigma^\Lambda := (\sigma_0^\Lambda, \sigma_1^\Lambda)$, where $\sigma_0^\Lambda : \sum_{i \in I} \lambda_0(i) \rightsquigarrow \mathbb{V}_0$

¹⁰For the sake of readability we denote the totalities of exterior union and dependent functions over Λ only with reference to the assignment routine λ_0 , while λ_1 is used in the equality formula of both totalities. In this way our notation is similar to the notation of the Σ -type and Π -type in MLTT.

is defined by $\sigma_0^\Lambda(i, x) := \lambda_0(i)$, and $\sigma_1^\Lambda((i, x), (j, y)) := \lambda_{ij}$, is a family of sets over $\sum_{i \in I} \lambda_0(i)$. The *second projection* on $\sum_{i \in I} \lambda_0(i)$ is the dependent operation

$$\begin{aligned} \text{pr}_2^\Lambda &: \bigwedge_{(i,x) \in \sum_{i \in I} \lambda_0(i)} \lambda_0(i), \\ \text{pr}_2^\Lambda(i, x) &:= \text{pr}_2(i, x) := x, \end{aligned}$$

and it is a dependent function over Σ^Λ . The following facts are easy to show.

Proposition 3.5. *Let $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathbf{Fam}(I)$, and $\Psi : \Lambda \Rightarrow M$.*

(i) *For every $i \in I$ the assignment routine $e_i^\Lambda : \lambda_0(i) \rightsquigarrow \sum_{i \in I} \lambda_0(i)$, defined by $x \mapsto (i, x)$, is an embedding of $\lambda_0(i)$ into $\sum_{i \in I} \lambda_0(i)$.*

(ii) *The assignment routine $\Sigma\Psi : \sum_{i \in I} \lambda_0(i) \rightsquigarrow \sum_{i \in I} \mu_0(i)$, defined by $\Sigma\Psi(i, x) := (i, \Psi_i(x))$, is a function from $\sum_{i \in I} \lambda_0(i)$ to $\sum_{i \in I} \mu_0(i)$, such that for every $i \in I$ the following diagram commutes*

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\Psi_i} & \mu_0(i) \\ e_i^\Lambda \downarrow & & \downarrow e_i^M \\ \sum_{i \in I} \lambda_0(i) & \xrightarrow{\Sigma\Psi} & \sum_{i \in I} \mu_0(i). \end{array}$$

(iii) *If Ψ_i is an embedding, for every $i \in I$, then $\Sigma\Psi$ is an embedding.*

(iv) *For every $i \in I$ the assignment routine $\pi_i^\Lambda : \prod_{i \in I} \lambda_0(i) \rightsquigarrow \lambda_0(i)$, defined by $\Theta \mapsto \Theta_i$, is a function from $\prod_{i \in I} \lambda_0(i)$ to $\lambda_0(i)$.*

(v) *The assignment routine $\Pi\Psi : \prod_{i \in I} \lambda_0(i) \rightsquigarrow \prod_{i \in I} \mu_0(i)$, defined by $[\Pi\Psi(\Theta)]_i := \Psi_i(\Theta_i)$, for every $i \in I$, is a function from $\prod_{i \in I} \lambda_0(i)$ to $\prod_{i \in I} \mu_0(i)$, such that for every $i \in I$ the following diagram commutes*

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\Psi_i} & \mu_0(i) \\ \pi_i^\Lambda \uparrow & & \uparrow \pi_i^M \\ \prod_{i \in I} \lambda_0(i) & \xrightarrow{\Pi\Psi} & \prod_{i \in I} \mu_0(i). \end{array}$$

(vi) *If Ψ_i is an embedding, for every $i \in I$, then $\Pi\Psi$ is an embedding.*

4. DIRECTED SETS

Definition 4.1. Let I be a set and $i \preceq_I j$ a binary extensional relation on I i.e., an extensional property on $I \times I$

$$\forall_{i,j,i',j' \in I} (i =_I i' \ \& \ j =_I j' \ \& \ i \preceq_I j \Rightarrow i' \preceq_I j').$$

If $i \preceq_I j$ is reflexive and transitive, then (I, \preceq_I) is called a *preorder*. We call a preorder (I, \preceq_I) a *directed set*, and *inverse-directed*, respectively, if

$$\forall_{i,j \in I} \exists_{k \in I} (i \preceq_I k \ \& \ j \preceq_I k),$$

$$\forall_{i,j \in I} \exists_{k \in I} (i \succ_I k \ \& \ j \succ_I k).$$

The *covariant diagonal* $D^{\preceq}(I)$ of \preceq_I , the *contravariant diagonal* $D^{\succ}(I)$ of \preceq_I , and the \preceq_I -upper set I_{ij}^{\preceq} of $i, j \in I$ are defined, respectively, by

$$D^{\preceq}(I) := \{(i, j) \in I \times I \mid i \preceq_I j\},$$

$$D^{\succ}(I) := \{(j, i) \in I \times I \mid j \succ_I i\},$$

$$I_{ij}^{\preceq} := \{k \in I \mid i \preceq_I k \ \& \ j \preceq_I k\}.$$

As $i \preceq_I j$ is extensional, $D^{\preceq}(I), D^{\succ}(I)$ are extensional subsets of $I \times I$, and I_{ij}^{\preceq} of I .

Definition 4.2. Let (I, \preceq_I) be a poset i.e., a preorder such that $[i \preceq_I j \ \& \ j \preceq_I i] \Rightarrow i =_I j$, for every $i, j \in I$. A *modulus of directedness* for I is a function $\delta: I \times I \rightarrow I$, such that for every $i, j, k \in I$ the following conditions are satisfied:

(δ_1) $i \preceq_I \delta(i, j)$ and $j \preceq_I \delta(i, j)$.

(δ_2) If $i \preceq_I j$, then $\delta(i, j) =_I \delta(j, i) =_I j$.

(δ_3) $\delta(\delta(i, j), k) =_I \delta(i, \delta(j, k))$.

In what follows we avoid for simplicity the use of subscripts on the relation symbols. If (I, \preceq) is a directed set and $(J, e) \subseteq I$, where $e: J \hookrightarrow I$, and using for simplicity the same symbol \preceq , if we define $j \preceq j' :\Leftrightarrow e(j) \preceq e(j')$, for every $j, j' \in J$, then (J, \preceq) is only a preordered set. If J is a cofinal subset of I , which classically it is defined by the condition $\forall_{i \in I} \exists_{j \in J} (i \preceq j)$, then (J, \preceq) becomes a directed set. To avoid the use of dependent choice, we add in the definition of a cofinal subset J of I a modulus of cofinality for J .

Definition 4.3. Let (I, \preceq) be a directed set and $(J, e) \subseteq I$, and let $j \preceq j' :\Leftrightarrow e(j) \preceq e(j')$, for every $j, j' \in J$. We say that J is *cofinal in I* , if there is a function¹¹ $\mathbf{cof}_J: I \rightarrow J$, which we call a *modulus of cofinality* of J in I , that satisfies the following conditions:

(\mathbf{Cof}_1) $\forall_{j \in J} (\mathbf{cof}_J(e(j)) =_J j)$.

$$\begin{array}{ccc} & \text{id}_J & \\ & \curvearrowright & \\ J & \xrightarrow{e} I \xrightarrow{\mathbf{cof}_J} & J. \end{array}$$

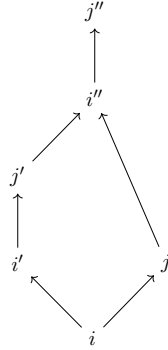
(\mathbf{Cof}_2) $\forall_{i, i' \in I} (i \preceq i' \Rightarrow \mathbf{cof}_J(i) \preceq \mathbf{cof}_J(i'))$.

(\mathbf{Cof}_3) $\forall_{i \in I} (i \preceq e(\mathbf{cof}_J(i)))$.

We denote the fact that J is cofinal in I by $(J, e, \mathbf{cof}_J) \subseteq^{\mathbf{cof}} I$, or, simpler, by $J \subseteq^{\mathbf{cof}} I$.

Taking into account the embedding e of J into I , condition (\mathbf{Cof}_3) is the exact writing of the classical defining condition $\forall_{i \in I} \exists_{j \in J} (i \preceq j)$. To add condition (\mathbf{Cof}_1) is natural, as \preceq is reflexive. If we consider condition (\mathbf{Cof}_3) on $e(j)$, for some $j \in J$, then by condition (\mathbf{Cof}_1) we get $e(j) \preceq e(\mathbf{cof}_J(e(j))) = e(j)$. Condition (\mathbf{Cof}_2) is also harmless to add. In the classical setting if $i \preceq i'$, and $j, j' \in J$ such that $i \preceq j$ and $i' \preceq j'$, then there is some $i'' \in I$ such that $j \preceq i''$ and $j' \preceq i''$. If $i'' \preceq j''$, for some $j'' \in J$,

¹¹Categorically speaking, e is a split monomorphism.



then $j \preceq j''$. Since $i' \preceq j''$ too, condition (\mathbf{Cof}_2) is justified. The added conditions (\mathbf{Cof}_1) and (\mathbf{Cof}_2) are used in the proofs of Theorem 9.12 and Lemma 9.11(ii), respectively. Moreover, they are used in the proof of Theorem 10.5. The extensionality of \preceq is also used in the proofs of Theorem 9.12 and Theorem 10.5.

E.g., if \mathbf{Even} and \mathbf{Odd} denote the sets of even and odd natural numbers, respectively, let $e: \mathbf{Even} \hookrightarrow \mathbb{N}$, defined by the identity map-rule, and $\mathbf{cof}_{\mathbf{Even}}: \mathbb{N} \rightarrow 2\mathbb{N}$, defined by the rule

$$\mathbf{cof}_{\mathbf{Even}}(n) := \begin{cases} n & , n \in \mathbf{Even} \\ n + 1 & , n \in \mathbf{Odd}. \end{cases}$$

Then $(\mathbf{Even}, e, \mathbf{cof}_{\mathbf{Even}}) \subseteq \mathbb{N}$.

Remark 4.4. If (I, \preceq) is a directed set and $(J, e, \mathbf{cof}_J) \subseteq^{\mathbf{cof}} I$, then (J, \preceq) is directed.

Proof. Let $j, j' \in J$ and let $i \in I$ such that $e(j) \preceq i$ and $e(j') \preceq i$. Since $i \preceq e(\mathbf{cof}_J(i))$, we get $e(j) \preceq e(\mathbf{cof}_J(i))$ and $e(j') \preceq e(\mathbf{cof}_J(i))$ i.e., $j \preceq \mathbf{cof}_J(i)$ and $j' \preceq \mathbf{cof}_J(i)$. \square

5. DIRECT FAMILIES OF SETS

The next concept is a variation of the notion of a set-indexed family of sets¹². A family of sets over a partial order is also used in the definition of a Kripke model for intuitionistic predicate logic, and the corresponding transport maps λ_{ij}^{\preceq} are called *transition functions* (see [40], p. 85).

Definition 5.1. Let (I, \preceq_I) be a directed set, and $D^{\preceq}(I) := \{(i, j) \in I \times I \mid i \preceq_I j\}$ the *diagonal* of \preceq_I . A *direct family of sets* (I, \preceq_I) , or an (I, \preceq_I) -*family of sets*, is a pair $\Lambda^{\preceq} := (\lambda_0, \lambda_1^{\preceq})$, where $\lambda_0: I \rightsquigarrow \mathbb{V}_0$, and λ_1^{\preceq} , a *modulus of transport maps* for λ_0 , where

$$\lambda_1^{\preceq}: \bigwedge_{(i,j) \in D^{\preceq}(I)} \mathbb{F}(\lambda_0(i), \lambda_0(j)), \quad \lambda_1^{\preceq}(i, j) := \lambda_{ij}^{\preceq}, \quad (i, j) \in D^{\preceq}(I),$$

such that the *transport maps* λ_{ij}^{\preceq} of Λ^{\preceq} satisfy the following conditions:

- (a) For every $i \in I$, we have that $\lambda_{ii}^{\preceq} := \text{id}_{\lambda_0(i)}$.
- (b) If $i \preceq_I j$ and $j \preceq_I k$, the following diagram commutes

¹²A directed set (I, \preceq_I) can also be seen as a category with objects the elements of I , and $\mathbf{Hom}_{\preceq_I}(i, j) := \{x \in \{0\} \mid i \preceq_I j\}$. If the universe \mathbb{V}_0 is seen as a category with objects its elements and $\mathbf{Hom}_{\preceq_{\mathbb{V}_0}}(X, Y) := \mathbb{F}(X, Y)$, an (I, \preceq_I) -family of sets is a functor from the category (I, \preceq_I) to this new category \mathbb{V}_0 .

hence by transitivity $i \preceq_I k''$ and $j' \preceq_I k''$. Moreover,

$$\begin{aligned}
\lambda_{ik''}^{\preceq}(x) &\stackrel{i \preceq_I k \preceq_I k''}{=} \lambda_{kk''}^{\preceq}(\lambda_{ik}^{\preceq}(x)) \\
&= \lambda_{kk''}^{\preceq}(\lambda_{jk}^{\preceq}(y)) \\
&\stackrel{j \preceq_I k \preceq_I k''}{=} \lambda_{jk''}^{\preceq}(y) \\
&\stackrel{j \preceq_I k' \preceq_I k''}{=} \lambda_{k'k''}^{\preceq}(\lambda_{jk'}^{\preceq}(y)) \\
&= \lambda_{k'k''}^{\preceq}(\lambda_{j'k'}^{\preceq}(z)) \\
&\stackrel{j' \preceq_I k' \preceq_I k''}{=} \lambda_{j'k''}^{\preceq}(z). \quad \square
\end{aligned}$$

Notice that the projection operation from $\sum_{i \in I}^{\preceq} \lambda_0(i)$ to I is not necessarily a function.

Proposition 5.4. *If (I, \preceq_I) is a directed set, $\Lambda^{\preceq} := (\lambda_0, \lambda_1^{\preceq})$, $M^{\preceq} := (\mu_0, \mu_1^{\preceq})$ are (I, \preceq_I) -families of sets, and $\Psi^{\preceq} : \Lambda^{\preceq} \Rightarrow M^{\preceq}$, the following hold.*

- (i) *For every $i \in I$ the operation $e_i^{\Lambda^{\preceq}} : \lambda_0(i) \rightsquigarrow \sum_{i \in I}^{\preceq} \lambda_0(i)$, defined by $x \mapsto (i, x)$, for every $x \in \lambda_0(i)$, is a function from $\lambda_0(i)$ to $\sum_{i \in I}^{\preceq} \lambda_0(i)$.*
- (ii) *The operation $\Sigma^{\preceq} \Psi : \sum_{i \in I}^{\preceq} \lambda_0(i) \rightsquigarrow \sum_{i \in I}^{\preceq} \mu_0(i)$, defined by $(\Sigma^{\preceq} \Psi)(i, x) := (i, \Psi_i(x))$, for every $(i, x) \in \sum_{i \in I}^{\preceq} \lambda_0(i)$, is a function from $\sum_{i \in I}^{\preceq} \lambda_0(i)$ to $\sum_{i \in I}^{\preceq} \mu_0(i)$ such that, for every $i \in I$, the following left diagram commutes*

$$\begin{array}{ccc}
\lambda_0(i) & \xrightarrow{\Psi_i} & \mu_0(i) \\
e_i^{\Lambda^{\preceq}} \downarrow & & \downarrow e_i^{M^{\preceq}} \\
\sum_{i \in I}^{\preceq} \lambda_0(i) & \xrightarrow{\Sigma^{\preceq} \Psi} & \sum_{i \in I}^{\preceq} \mu_0(i)
\end{array}
\qquad
\begin{array}{ccc}
\lambda_0(i) & \xrightarrow{\Psi_i} & \mu_0(i) \\
\pi_i^{\Lambda^{\preceq}} \uparrow & & \uparrow \pi_i^{M^{\preceq}} \\
\prod_{i \in I}^{\preceq} \lambda_0(i) & \xrightarrow{\prod^{\preceq} \Psi} & \prod_{i \in I}^{\preceq} \mu_0(i).
\end{array}$$

- (iii) *If Ψ_i is an embedding, for every $i \in I$, then $\Sigma^{\preceq} \Psi$ is an embedding.*
- (iv) *For every $i \in I$ the operation $\pi_i^{\Lambda^{\preceq}} : \prod_{i \in I}^{\preceq} \lambda_0(i) \rightsquigarrow \lambda_0(i)$, defined by $\Theta \mapsto \Theta_i$, for every $\Theta \in \prod_{i \in I}^{\preceq} \lambda_0(i)$, is a function from $\prod_{i \in I}^{\preceq} \lambda_0(i)$ to $\lambda_0(i)$.*
- (v) *The operation $\prod^{\preceq} \Psi : \prod_{i \in I}^{\preceq} \lambda_0(i) \rightsquigarrow \prod_{i \in I}^{\preceq} \mu_0(i)$, defined by $[\prod^{\preceq} \Psi(\Theta)]_i := \Psi_i(\Theta_i)$, for every $i \in I$ and $\Theta \in \prod_{i \in I}^{\preceq} \lambda_0(i)$, is a function from $\prod_{i \in I}^{\preceq} \lambda_0(i)$ to $\prod_{i \in I}^{\preceq} \mu_0(i)$, such that, for every $i \in I$, the above right diagram commutes.*
- (vi) *If Ψ_i is an embedding, for every $i \in I$, then $\prod^{\preceq} \Psi$ is an embedding.*

Proof. (i) Let $x, y \in \lambda_0(i)$ with $x =_{\lambda_0(i)} y$. As \preceq is reflexive, if we take $k := i$, we get $\lambda_{ii}^{\preceq}(x) := \text{id}_{\lambda_0(i)}(x) := x =_{\lambda_0(i)} y =: \text{id}_{\lambda_0(i)}(y) =: \lambda_{ii}^{\preceq}(y)$, hence $(i, x) =_{\sum_{i \in I}^{\preceq} \lambda_0(i)} (i, y)$.

(ii) If $(i, x) =_{\sum_{i \in I}^{\preceq} \lambda_0(i)} (j, y)$, there is $k \in I$ such that $i \preceq_I k$, $j \preceq_I k$ and $\lambda_{ik}^{\preceq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preceq}(y)$. We show the following equality:

$$\begin{aligned}
(\Sigma^{\preceq} \Psi)(i, x) &=_{\sum_{i \in I}^{\preceq} \mu_0(i)} (\Sigma^{\preceq} \Psi)(j, y) \Leftrightarrow (i, \Psi_i(x)) =_{\sum_{i \in I}^{\preceq} \mu_0(i)} (j, \Psi_j(y)) \\
&\Leftrightarrow \exists k' \in I (i, j \preceq_I k' \ \& \ \mu_{ik'}^{\preceq}(\Psi_i(x)) =_{\mu_0(k')} \mu_{jk'}^{\preceq}(\Psi_j(y))).
\end{aligned}$$

If we take $k' := k$, by the commutativity of the following diagrams, and since Ψ_k is a function,

$$\begin{array}{ccc}
\lambda_0(i) & \xrightarrow{\lambda_{ik}^{\lessdot}} & \lambda_0(k) & & \lambda_0(j) & \xrightarrow{\lambda_{jk}^{\lessdot}} & \lambda_0(k) \\
\Psi_i \downarrow & & \downarrow \Psi_k & & \Psi_j \downarrow & & \downarrow \Psi_k \\
\mu_0(i) & \xrightarrow{\mu_{ik}^{\lessdot}} & \mu_0(k) & & \mu_0(j) & \xrightarrow{\mu_{jk}^{\lessdot}} & \mu_0(k)
\end{array}$$

$$\mu_{ik}^{\lessdot}(\Psi_i(x)) =_{\mu_0(k)} \Psi_k(\lambda_{ik}^{\lessdot}(x)) =_{\mu_0(k)} \Psi_k(\lambda_{jk}^{\lessdot}(y)) =_{\mu_0(k)} \mu_{jk}^{\lessdot}(\Psi_j(y)).$$

(iii) If we suppose $(\Sigma^{\lessdot}\Psi)(i, x) =_{\Sigma_{i \in I} \mu_0(i)} (\Sigma^{\lessdot}\Psi)(j, y)$ i.e., $\mu_{ik}^{\lessdot}(\Psi_i(x)) =_{\mu_0(k)} \mu_{jk}^{\lessdot}(\Psi_j(y))$, for some $k \in I$ with $i, j \preceq_I k$, by the proof of case (ii) we get $\Psi_k(\lambda_{ik}^{\lessdot}(x)) =_{\mu_0(k)} \Psi_k(\lambda_{jk}^{\lessdot}(y))$, and since Ψ_k is an embedding, we get $\lambda_{ik}^{\lessdot}(x) =_{\lambda_0(k)} \lambda_{jk}^{\lessdot}(y)$ i.e., $(i, x) =_{\Sigma_{i \in I} \lambda_0(i)} (j, y)$.

(iv)-(vi) Their proof is omitted, since a proof of their contravariant version¹³ is given in the proof of Theorem 10.3. \square

Since the transport functions λ_{ik}^{\lessdot} are not in general embeddings, we cannot show in general that $e_i^{\Lambda^{\lessdot}}$ is an embedding, as it is the case for the map e_i^{Λ} in Proposition 3.5(i).

6. ON BISHOP SPACES

From now on we work within the extension BST^* of BST .

Definition 6.1. If X is a set and \mathbb{R} is the set of real numbers, we denote by $\mathbb{F}(X)$ the set of functions from X to \mathbb{R} , and by $\text{Const}(X)$ the subset of $\mathbb{F}(X)$ of all constant real functions on X . If $a \in \mathbb{R}$, we denote by \bar{a}^X the constant function on X with value a . We denote by \mathbb{N}^+ the set of non-zero natural numbers. A function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is called *Bishop continuous*, or simply continuous, if for every $n \in \mathbb{N}^+$ there is a function $\omega_{\phi, n} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\epsilon \mapsto \omega_{\phi, n}(\epsilon)$, which is called a *modulus of continuity* of ϕ on $[-n, n]$, such that the following condition is satisfied

$$\forall x, y \in [-n, n] (|x - y| < \omega_{\phi, n}(\epsilon) \Rightarrow |\phi(x) - \phi(y)| \leq \epsilon),$$

for every $\epsilon > 0$ and every $n \in \mathbb{N}^+$. We denote by $\text{Bic}(\mathbb{R})$ the set of continuous functions from \mathbb{R} to \mathbb{R} , which is equipped with the equality inherited from $\mathbb{F}(\mathbb{R})$.

We could have defined the modulus of continuity $\omega_{\phi, n}$ as a function from \mathbb{N}^+ to \mathbb{N}^+ . A continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous on every bounded subset of \mathbb{R} . The latter is an impredicative formulation of uniform continuity, as it requires quantification over the class $\mathcal{P}(\mathbb{R})$. The formulation of uniform continuity in the Definition 6.1 though, is predicative, since it requires quantification over the sets $\mathbb{N}^+, \mathbb{F}(\mathbb{R}^+, \mathbb{R}^+)$ and $[-n, n]$.

Definition 6.2. If X is a set, $f, g \in \mathbb{F}(X)$, $\epsilon > 0$, and $\Phi \subseteq \mathbb{F}(X)$, let

$$U(X; f, g, \epsilon) := \Leftrightarrow \forall x \in X (|g(x) - f(x)| \leq \epsilon),$$

$$U(X; \Phi, f) := \Leftrightarrow \forall \epsilon > 0 \exists g \in \Phi (U(f, g, \epsilon)).$$

If the set X is clear from the context, we write simply $U(f, g, \epsilon)$ and $U(\Phi, f)$, respectively. We denote by Φ^* the bounded elements of Φ , and its uniform closure $\bar{\Phi}$ is defined by

$$\bar{\Phi} := \{f \in \mathbb{F}(X) \mid U(\Phi, f)\}.$$

¹³In a contravariant family the transport maps are of type $\lambda_0(j) \rightarrow \lambda_0(i)$, if $i \preceq j$.

A Bishop topology on X is a certain subset of $\mathbb{F}(X)$. Since the Bishop topologies considered here are all extensional subsets of $\mathbb{F}(X)$, we do not mention the embedding $i_F^{\mathbb{F}(X)}: F \hookrightarrow \mathbb{F}(X)$, which is given in all cases by the identity map-rule.

Definition 6.3. A *Bishop space* is a pair $\mathcal{F} := (X, F)$, where F is an extensional subset of $\mathbb{F}(X)$, which is called a *Bishop topology*, or simply a *topology* of functions on X , that satisfies the following conditions:

(BS₁) If $a \in \mathbb{R}$, then $\bar{a}^X \in F$.

(BS₂) If $f, g \in F$, then $f + g \in F$.

(BS₃) If $f \in F$ and $\phi \in \text{Bic}(\mathbb{R})$, then $\phi \circ f \in F$

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{R} \\ & \searrow & \downarrow \phi \in \text{Bic}(\mathbb{R}) \\ F \ni \phi \circ f & & \mathbb{R}. \end{array}$$

(BS₄) $\bar{F} = F$.

If F is inhabited, then (BS₁) is provable by (BS₃). The set of constant functions $\text{Const}(X)$ is the *trivial* topology on X , while $\mathbb{F}(X)$ is the *discrete* topology on X . Clearly, if F is a topology on X , then $\text{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$, and the set of its bounded elements F^* is also a topology on X . We denote by $\mathcal{F}^* := (X, F^*)$ the Bishop space of bounded elements of a Bishop topology F . It is easy to see that the pair $\mathcal{R} := (\mathbb{R}, \text{Bic}(\mathbb{R}))$ is a Bishop space, which we call the *Bishop space of reals*. A Bishop topology F is a ring and a lattice; since $|\text{id}_{\mathbb{R}}| \in \text{Bic}(\mathbb{R})$, where $\text{id}_{\mathbb{R}}$ is the identity function on \mathbb{R} , by BS₃, if $f \in F$, then $|f| \in F$. By BS₂ and BS₃, and using the following equalities

$$f \cdot g = \frac{(f + g)^2 - f^2 - g^2}{2} \in F,$$

$$f \vee g = \max\{f, g\} = \frac{f + g + |f - g|}{2} \in F,$$

$$f \wedge g = \min\{f, g\} = \frac{f + g - |f - g|}{2} \in F,$$

we get similarly that if $f, g \in F$, then $f \cdot g, f \vee g, f \wedge g \in F$. Turning the definitional clauses of a Bishop topology into inductive rules, Bishop defined in [4], p. 72, the least topology including a given subbase F_0 . This inductive definition, which is also found in [8], p. 78, is crucial to the definition of new Bishop topologies from given ones.

Definition 6.4. The *Bishop closure* of F_0 , or the *least topology* $\bigvee F_0$ generated by some $F_0 \subseteq \mathbb{F}(X)$, is defined by the following inductive rules:

$$\frac{f_0 \in F_0}{f_0 \in \bigvee F_0}, \quad \frac{a \in \mathbb{R}}{\bar{a}^X \in \bigvee F_0}, \quad \frac{f, g \in \bigvee F_0}{f + g \in \bigvee F_0}, \quad \frac{f \in \bigvee F_0, g =_{\mathbb{F}(X)} f}{g \in \bigvee F_0},$$

$$\frac{f \in \bigvee F_0, \phi \in \text{Bic}(\mathbb{R})}{\phi \circ f \in \bigvee F_0}, \quad \frac{(g \in \bigvee F_0 \ \& \ U(f, g, \epsilon))_{\epsilon > 0}}{f \in \bigvee F_0}.$$

We call $\bigvee F_0$ the *Bishop closure* of F_0 , and F_0 a *subbase* of $\bigvee F_0$.

The last, most complex rule above can be replaced by the rule

$$\frac{g_1 \in \bigvee F_0 \ \& \ U(f, g_1, \frac{1}{2}), \quad g_2 \in \bigvee F_0 \ \& \ U(f, g_2, \frac{1}{2^2}), \quad \dots}{f \in \bigvee F_0},$$

a rule with countably many premisses. The corresponding induction principle $\text{Ind}_{\bigvee F_0}$ is

$$\left[\begin{array}{l} \forall_{f_0 \in F_0} (P(f_0)) \ \& \ \forall_{a \in \mathbb{R}} (P(\bar{a}^X)) \ \& \ \forall_{f, g \in \bigvee F_0} (P(f) \ \& \ P(g) \Rightarrow P(f + g)) \\ \& \ \forall_{f \in \bigvee F_0} \forall_{g \in \mathbb{F}(X)} (g =_{\mathbb{F}(X)} f \ \& \ P(f) \Rightarrow P(g)) \\ \& \ \forall_{f \in \bigvee F_0} \forall_{\phi \in \text{Bic}(\mathbb{R})} (P(f) \Rightarrow P(\phi \circ f)) \\ \& \ \forall_{f \in \bigvee F_0} (\forall_{\epsilon > 0} \exists_{g \in \bigvee F_0} (P(g) \ \& \ U(f, g, \epsilon)) \Rightarrow P(f)) \end{array} \right] \\ \Rightarrow \forall_{f \in \bigvee F_0} (P(f)),$$

where P is any bounded formula. Next we define the notion of a Bishop morphism between Bishop spaces. The Bishop morphisms are the arrows in the category of Bishop spaces **Bis**.

Definition 6.5. If $\mathcal{F} := (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, a function $h : X \rightarrow Y$ is called a *Bishop morphism*, if $\forall_{g \in G} (g \circ h \in F)$

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow & \downarrow \\ F \ni g \circ h & & g \in G \\ & & \mathbb{R}. \end{array}$$

We denote by $\text{Mor}(\mathcal{F}, \mathcal{G})$ the set of Bishop morphisms from \mathcal{F} to \mathcal{G} . As F is an extensional subset of $\mathbb{F}(X)$, $\text{Mor}(\mathcal{F}, \mathcal{G})$ is an extensional subset of $\mathbb{F}(X, Y)$. If $h \in \text{Mor}(\mathcal{F}, \mathcal{G})$, the *induced mapping* $h^* : G \rightarrow F$ from h is defined by the rule

$$h^*(g) := g \circ h; \quad g \in G.$$

If $\mathcal{F} := (X, F)$ is a Bishop space, then $F = \text{Mor}(\mathcal{F}, \mathbb{R})$, and one can show inductively that if $\mathcal{G} := (Y, \bigvee G_0)$, then $h : X \rightarrow Y \in \text{Mor}(\mathcal{F}, \mathcal{G})$ if and only if $\forall_{g_0 \in G_0} (g_0 \circ h \in F)$

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow & \downarrow \\ F \ni g_0 \circ h & & g_0 \in G_0 \\ & & \mathbb{R}. \end{array}$$

We call this fundamental fact the *\bigvee -lifting of morphisms*. A Bishop morphism is a *Bishop isomorphism*, if it is an isomorphism in the category **Bis**. We write $\mathcal{F} \simeq \mathcal{G}$ to denote that \mathcal{F} and \mathcal{G} are Bishop isomorphic. If $h \in \text{Mor}(\mathcal{F}, \mathcal{G})$ is a bijection, then h is a Bishop isomorphism if and only if it is *open* i.e., $\forall_{f \in F} \exists_{g \in G} (f = g \circ h)$.

Definition 6.6. Let $\mathcal{F} := (X, F)$, $\mathcal{G} := (Y, G)$ be Bishop spaces, $(A, i_A) \subseteq X$ inhabited, and $\phi : X \rightarrow Y$ a surjection. The *product* Bishop space $\mathcal{F} \times \mathcal{G} := (X \times Y, F \times G)$ of \mathcal{F} and \mathcal{G} ,

the *relative* Bishop space $\mathbb{F}|_A := (A, F|_A)$ on A , and the *pointwise exponential Bishop space* $\mathcal{F} \rightarrow \mathcal{G} = (\text{Mor}(\mathcal{F}, \mathcal{G}), F \rightarrow G)$ are defined, respectively, by

$$F \times G := \bigvee [\{f \circ \text{pr}_X, | f \in F\} \cup \{g \circ \text{pr}_Y | g \in G\}] =: \bigvee_{f \in F}^{g \in G} f \circ \text{pr}_X, g \circ \text{pr}_Y,$$

$$F|_A = \bigvee \{f|_A | f \in F\} =: \bigvee_{f \in F} f|_A$$

$$\begin{array}{ccc} A & \xrightarrow{i_A} & X & \xrightarrow{f} & \mathbb{R}, \\ & & \searrow & \nearrow & \\ & & & & \mathbb{R}, \\ & & & & \uparrow \\ & & & & f|_A \end{array}$$

$$F \rightarrow G := \bigvee \{\phi_{x,g} | x \in X, g \in G\} =: \bigvee_{x \in X}^{g \in G} \phi_{x,g},$$

$$\phi_{x,g} : \text{Mor}(\mathcal{F}, \mathcal{G}) \rightarrow \mathbb{R}, \quad \phi_{x,g}(h) = g(h(x)); \quad x \in X, g \in G.$$

One can show inductively the following \bigvee -liftings

$$\begin{aligned} \bigvee F_0 \times \bigvee G_0 &:= \bigvee [\{f_0 \circ \text{pr}_X, | f_0 \in F_0\} \cup \{g_0 \circ \text{pr}_Y | g_0 \in G_0\}] \\ &=: \bigvee_{f_0 \in F_0}^{g_0 \in G_0} f_0 \circ \text{pr}_X, g_0 \circ \text{pr}_Y, \end{aligned}$$

$$\left(\bigvee F_0\right)|_A = \bigvee \{f_0|_A | f_0 \in F_0\} =: \bigvee_{f_0 \in F_0} f_0|_A,$$

$$F \rightarrow \bigvee G_0 = \bigvee \{\phi_{x,g_0} | x \in X, g_0 \in G_0\} =: \bigvee_{x \in X}^{g_0 \in G_0} \phi_{x,g_0}.$$

$F|_A$ is the least topology on A that makes i_A a Bishop morphism, and the product topology $F \times G$ is the least topology on $X \times Y$ that makes the projections pr_X and pr_Y Bishop morphisms. The term pointwise exponential Bishop topology is due to the fact that $F \rightarrow G$ behaves like the classical topology of the pointwise convergence on $C(X, Y)$, the set of continuous functions from the topological space X to the topological space Y .

7. DIRECT SPECTRA OF BISHOP SPACES

Roughly speaking, if S is a structure on some set, an S -spectrum is an I -family of sets Λ such that each set $\lambda_0(i)$ is equipped with a structure S_i , which is compatible with the transport maps λ_{ij} of Λ . Accordingly, a spectrum of Bishop spaces is an I -family of sets Λ such that each set $\lambda_0(i)$ is equipped with a Bishop topology F_i , which is compatible with the transport maps of Λ . As expected, in the case of a spectrum of Bishop spaces this compatibility condition is that the transport maps λ_{ij} are Bishop morphisms i.e. $\lambda_{ij} \in \text{Mor}(\mathcal{F}_i, \mathcal{F}_j)$. It is natural to associate to Λ an I -family of sets $\Phi := (\phi_0^\Lambda, \phi_1^\Lambda)$ such that $\mathcal{F}_i := (\lambda_0(i), \phi_0^\Lambda(i))$ is the Bishop space corresponding to $i \in I$. If $i =_I j$, and if we put no restriction to the definition of $\phi_{ij}^\Lambda : F_i \rightarrow F_j$, we need to add extra data in the definition of a map between spectra of Bishop spaces. Since the map $\lambda_{ji}^* : F_i \rightarrow F_j$, where λ_{ji}^* is the element of $\mathbb{F}(F_i, F_j)$ induced by the Bishop morphism $\Lambda_{ji} \in \text{Mor}(\mathcal{F}_j, \mathcal{F}_i)$, is generated by the data of Λ , it is

natural to define $\phi_{ij} := \lambda_{ji}^*$. In this way proofs of properties of maps between spectra of Bishop spaces become easier. Every subset of $\mathbb{F}(X)$ considered in this section is an extensional subset of it.

Definition 7.1. Let $\Lambda := (\lambda_0, \lambda_1)$. A *family of Bishop topologies associated to Λ* is a pair $\Phi^\Lambda := (\phi_0^\Lambda, \phi_1^\Lambda)$, where $\phi_0^\Lambda: I \rightsquigarrow \mathbb{V}_0$ and $\phi_1^\Lambda: \mathcal{A}_{(i,j) \in D(I)} \mathbb{F}(\phi_0^\Lambda(i), \phi_0^\Lambda(j))$, such that the following conditions hold:

- (i) $\phi_0^\Lambda(i) := F_i \subseteq \mathbb{F}(\lambda_0(i))$, and $\mathcal{F}_i := (\lambda_0(i), F_i)$ is a Bishop space, for every $i \in I$.
- (ii) $\lambda_{ij} \in \text{Mor}(\mathcal{F}_i, \mathcal{F}_j)$, for every $(i, j) \in D(I)$.
- (iii) $\phi_1^\Lambda(i, j) := \lambda_{ji}^*$, for every $(i, j) \in D(I)$, where, if $f \in F_i$, the induced map $\lambda_{ji}^*: F_i \rightarrow F_j$ from λ_{ji} is defined by $\lambda_{ji}^*(f) := f \circ \lambda_{ji}$, for every $f \in F_i$.

The structure $S(\Lambda) := (\lambda_0, \lambda_1, \phi_0^\Lambda, \phi_1^\Lambda)$ is called a *spectrum of Bishop spaces over I* , or an *I -spectrum* with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop isomorphisms $(\lambda_{ij})_{(i,j) \in D(I)}$.

If $M := (\mu_0, \mu_1) \in \mathbf{Fam}(I)$ and $S(M) := (\mu_0, \mu_1, \phi_0^M, \phi_1^M)$ is an I -spectrum with Bishop spaces $(\mathcal{G}_i)_{i \in I}$ and Bishop isomorphisms $(\mu_{ij})_{(i,j) \in D(I)}$, a *spectrum-map Ψ* from $S(\Lambda)$ to $S(M)$, in symbols $\Psi: S(\Lambda) \Rightarrow S(M)$, is a family-map $\Psi: \Lambda \Rightarrow M$. A spectrum-map $\Phi: S(\Lambda) \Rightarrow S(M)$ is called *continuous*, if $\Psi_i \in \text{Mor}(\mathcal{F}_i, \mathcal{G}_i)$, for every $i \in I$.

As in the case of a family of Bishop spaces associated to an I -family of sets, the family of Bishop spaces associated to an (I, \preccurlyeq) -family of sets is defined in a minimal way from the data of Λ^\preccurlyeq . According to these data, the corresponding functions ϕ_{ij}^\preccurlyeq behave necessarily in a contravariant manner i.e., $\phi_{ij}^\preccurlyeq: F_j \rightarrow F_i$. Moreover, the transport maps $\lambda_{ij}^\preccurlyeq$ are Bishop morphisms, and not necessarily Bishop isomorphisms.

Definition 7.2. Let (I, \preccurlyeq) be a directed set, and let $\Lambda^\preccurlyeq := (\lambda_0, \lambda_1^\preccurlyeq) \in \mathbf{Fam}(I, \preccurlyeq)$. A *family of Bishop topologies associated to Λ^\preccurlyeq* is a pair $\Phi^{\Lambda^\preccurlyeq} := (\phi_0^{\Lambda^\preccurlyeq}, \phi_1^{\Lambda^\preccurlyeq})$, where $\phi_0^{\Lambda^\preccurlyeq}: I \rightsquigarrow \mathbb{V}_0$ and $\phi_1^{\Lambda^\preccurlyeq}: \mathcal{A}_{(i,j) \in \preccurlyeq(I)} \mathbb{F}(\phi_0^{\Lambda^\preccurlyeq}(j), \phi_0^{\Lambda^\preccurlyeq}(i))$, such that the following conditions hold:

- (i) $\phi_0^{\Lambda^\preccurlyeq}(i) := F_i \subseteq \mathbb{F}(\lambda_0(i))$, and $\mathcal{F}_i := (\lambda_0(i), F_i)$ is a Bishop space, for every $i \in I$.
- (ii) $\lambda_{ij}^\preccurlyeq \in \text{Mor}(\mathcal{F}_i, \mathcal{F}_j)$, for every $(i, j) \in D^\preccurlyeq(I)$.
- (iii) $\phi_1^{\Lambda^\preccurlyeq}(i, j) := (\lambda_{ij}^\preccurlyeq)^*$, for every $(i, j) \in D^\preccurlyeq(I)$, where, if $f \in F_j$, $(\lambda_{ij}^\preccurlyeq)^*(f) := f \circ \lambda_{ij}^\preccurlyeq$.

The structure $S(\Lambda^\preccurlyeq) := (\lambda_0, \lambda_1^\preccurlyeq, \phi_0^{\Lambda^\preccurlyeq}, \phi_1^{\Lambda^\preccurlyeq})$ is called a *direct spectrum over (I, \preccurlyeq)* , or an *(I, \preccurlyeq) -spectrum* with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^\preccurlyeq)_{(i,j) \in D^\preccurlyeq(I)}$.

If $M^\preccurlyeq := (\mu_0, \mu_1^\preccurlyeq) \in \mathbf{Fam}(I, \preccurlyeq)$ and $S(M^\preccurlyeq) := (\mu_0, \mu_1^\preccurlyeq, \phi_0^{M^\preccurlyeq}, \phi_1^{M^\preccurlyeq})$ is an (I, \preccurlyeq) -spectrum with Bishop spaces $(\mathcal{G}_i)_{i \in I}$ and Bishop morphisms $(\mu_{ij}^\preccurlyeq)_{(i,j) \in D^\preccurlyeq(I)}$, a *direct spectrum-map Ψ* from $S(\Lambda^\preccurlyeq)$ to $S(M^\preccurlyeq)$, in symbols $\Psi: S(\Lambda^\preccurlyeq) \Rightarrow S(M^\preccurlyeq)$, is a direct family-map $\Psi: \Lambda^\preccurlyeq \Rightarrow M^\preccurlyeq$. The totality of direct spectrum-maps from $S(\Lambda^\preccurlyeq)$ to $S(M^\preccurlyeq)$ is denoted by $\mathbf{Map}_{(I, \preccurlyeq)}(S(\Lambda^\preccurlyeq), S(M^\preccurlyeq))$ and it is equipped with the equality of $\mathbf{Map}_I(\Lambda^\preccurlyeq, M^\preccurlyeq)$. A direct spectrum-map $\Psi: S(\Lambda^\preccurlyeq) \Rightarrow S(M^\preccurlyeq)$ is called *continuous*, if $\forall i \in I (\Psi_i \in \text{Mor}(\mathcal{F}_i, \mathcal{G}_i))$, and let $\mathbf{Cont}_{(I, \preccurlyeq)}(S(\Lambda^\preccurlyeq), S(M^\preccurlyeq))$ be their totality, equipped with the equality of $\mathbf{Map}_I(\Lambda^\preccurlyeq, M^\preccurlyeq)$. The totality $\mathbf{Spec}(I, \preccurlyeq)$ of direct spectra over (I, \preccurlyeq) is equipped with an equality defined similarly to the equality on $\mathbf{Spec}(I)$. A contravariant direct spectrum $S(\Lambda^\succcurlyeq) := (\lambda_0, \lambda_1^\succcurlyeq; \phi_0^{\Lambda^\succcurlyeq}, \phi_1^{\Lambda^\succcurlyeq})$ over (I, \preccurlyeq) , a contravariant direct spectrum-map $\Psi: S(\Lambda^\succcurlyeq) \Rightarrow S(M^\succcurlyeq)$, and their totalities $\mathbf{Map}_{(I, \preccurlyeq)}(S(\Lambda^\succcurlyeq), S(M^\succcurlyeq))$, $\mathbf{Spec}(I, \succcurlyeq)$ are defined similarly.

The following is straightforward to show.

Remark 7.3. Let (I, \preccurlyeq) be a directed set, $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathbf{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in D^{\preccurlyeq}(I)}$, $S(M^{\preccurlyeq}) := (\mu_0, \mu_1, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}}) \in \mathbf{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(\mathcal{G}_i)_{i \in I}$ and Bishop morphisms $(\mu_{ij}^{\preccurlyeq})_{(i,j) \in D^{\preccurlyeq}(I)}$, and $\Psi: S(\Lambda^{\preccurlyeq}) \Rightarrow S(M^{\preccurlyeq})$. Then $\Phi^{\Lambda^{\preccurlyeq}} := (\phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ is an (I, \succcurlyeq) -family of sets, defined in the obvious dual way, and if Ψ is continuous, then, for every $(i, j) \in D^{\preccurlyeq}(I)$, the following diagram commutes

$$\begin{array}{ccc} G_j & \xrightarrow{(\mu_{ij}^{\preccurlyeq})^*} & G_i \\ (\Psi_j)^* \downarrow & & \downarrow (\Psi_i)^* \\ F_j & \xrightarrow{(\lambda_{ij}^{\preccurlyeq})^*} & F_i \end{array}$$

Remark 7.4. Let (I, \preccurlyeq) be a directed set and $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathbf{Spec}(I, D^{\preccurlyeq}(I))$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in D^{\preccurlyeq}(I)}$. If $\Theta \in \prod_{i \in I}^{\preccurlyeq} F_i$, the following operation is a function

$$f_{\Theta} : \left(\sum_{i \in I}^{\preccurlyeq} \lambda_0(i) \right) \rightsquigarrow \mathbb{R}, \quad f_{\Theta}(i, x) := \Theta_i(x), \quad (i, x) \in \sum_{i \in I}^{\preccurlyeq} \lambda_0(i).$$

Proof. Let $(i, x) = \sum_{i \in I}^{\preccurlyeq} \lambda_0(i) (j, y) \Leftrightarrow \exists k \succcurlyeq i, j (\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y))$. Since $\Theta_i = \phi_{ki}^{\preccurlyeq}(\Theta_k) := (\lambda_{ik}^{\preccurlyeq})^*(\Theta_k) := \Theta_k \circ \lambda_{ik}^{\preccurlyeq}$, and similarly $\Theta_j = \Theta_k \circ \lambda_{jk}^{\preccurlyeq}$, we have that $\Theta_i(x) = [\Theta_k \circ \lambda_{ik}^{\preccurlyeq}](x) := \Theta_k(\lambda_{ik}^{\preccurlyeq}(x)) = \Theta_k(\lambda_{jk}^{\preccurlyeq}(y)) := [\Theta_k \circ \lambda_{jk}^{\preccurlyeq}](y) = \Theta_j(y)$. \square

Definition 7.5. Let (I, \preccurlyeq) be a directed set and $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}) \in \mathbf{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in D^{\preccurlyeq}(I)}$. The Bishop space

$$\sum_{i \in I}^{\preccurlyeq} \mathcal{F}_i := \left(\sum_{i \in I}^{\preccurlyeq} \lambda_0(i), \int_{i \in I}^{\preccurlyeq} F_i \right) \quad \text{where} \quad \int_{i \in I}^{\preccurlyeq} F_i := \bigvee_{\Theta \in \prod_{i \in I}^{\preccurlyeq} F_i} f_{\Theta},$$

is the *sum Bishop space* of $S(\Lambda^{\preccurlyeq})$. If S^{\succcurlyeq} is a contravariant direct spectrum over (I, \preccurlyeq) , the sum Bishop space of $S(\Lambda^{\succcurlyeq})$ is defined dually.

Lemma 7.6. Let $S(\Lambda^{\preccurlyeq}) := (\lambda_0, \lambda_1^{\preccurlyeq}, \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}), S(M^{\preccurlyeq}) := (\mu_0, \mu_1^{\preccurlyeq}, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}}) \in \mathbf{Spec}(I, \preccurlyeq_I)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and with Bishop spaces $(\mathcal{G}_i)_{i \in I}$, respectively, and let $\Psi: S(\Lambda^{\preccurlyeq}) \rightarrow S(M^{\preccurlyeq})$ be continuous. If $H \in \prod_{i \in I}^{\preccurlyeq} G_i$, the dependent operation $H^* : \lambda_{i \in I} F_i$, defined by $H_i^* := \Psi_i^*(H_i) := H_i \circ \Psi_i$, for every $i \in I$, is in $\prod_{i \in I}^{\preccurlyeq} F_i$.

Proof. If $i \preccurlyeq j$, we need to show that $H_i^* = (\lambda_{ij}^{\preccurlyeq})^*(H_j^*) = H_j^* \circ \lambda_{ij}^{\preccurlyeq}$. Since $H \in \prod_{i \in I}^{\preccurlyeq} G_i$, we have that $H_i = H_j \circ \mu_{ij}^{\preccurlyeq}$, and by the continuity of Ψ and the commutativity of the diagram

$$\begin{array}{ccc}
\lambda_0(i) & \xrightarrow{\lambda_{ij}^\preceq} & \lambda_0(j) \\
\Psi_i \downarrow & & \downarrow \Psi_j \\
\mu_0(i) & \xrightarrow{\mu_{ij}^\preceq} & \mu_0(j),
\end{array}$$

$$\begin{aligned}
H_j^* \circ \lambda_{ij}^\preceq &:= \Psi_j^*(H_j) \circ \lambda_{ij}^\preceq := (H_j \circ \Psi_j) \circ \lambda_{ij}^\preceq := H_j(0) \circ (\Psi_j \circ \lambda_{ij}^\preceq) \\
&= H_j \circ (\mu_{ij}^\preceq \circ \Psi_i) = (H_j \circ \mu_{ij}^\preceq) \circ \Psi_i = H_i \circ \Psi_i := \Psi_i^*(H_i) := H_i^*. \quad \square
\end{aligned}$$

Proposition 7.7. Let $S(\Lambda^\preceq) := (\lambda_0, \lambda_1^\preceq, \phi_0^{\Lambda^\preceq}, \phi_1^{\Lambda^\preceq})$ and $S(M^\preceq) := (\mu_0, \mu_1^\preceq, \phi_0^{M^\preceq}, \phi_1^{M^\preceq})$ be spectra over (I, \preceq_I) , and let $\Psi : S(\Lambda^\preceq) \Rightarrow S(M^\preceq)$.

- (i) If $i \in I$, then $e_i^{\Lambda^\preceq} \in \text{Mor}(\mathcal{F}_i, \sum_{i \in I}^\preceq \mathcal{F}_i)$.
- (ii) If Ψ is continuous, then $\Sigma^\preceq \Psi \in \text{Mor}(\sum_{i \in I}^\preceq \mathcal{F}_i, \sum_{i \in I}^\preceq \mathcal{G}_i)$.

Proof. (i) By the \vee -lifting of morphisms it suffices to show that $\forall_{\Theta \in \prod_{i \in I}^\preceq F_i} (f_\Theta \circ e_i^{\Lambda^\preceq} \in F_i)$.

If $x \in \lambda_0(i)$, then $(f_\Theta \circ e_i^{\Lambda^\preceq})(x) := f_\Theta(i, x) := \Theta_i(x)$, hence $f_\Theta \circ e_i^{\Lambda^\preceq} := \Theta_i \in F_i$.

(ii) By the \vee -lifting of morphisms and Definition 7.5 it suffices to show that

$$\forall_{H \in \prod_{i \in I}^\preceq G_i} \left(g_H \circ \Sigma^\preceq \Psi \in \int_{i \in I}^\preceq F_i \right).$$

If $i \in I$ and $x \in \lambda_0(i)$, and if $H^* \in \prod_{i \in I}^\preceq F_i$, defined in Lemma 7.6, then $(g_H \circ \Sigma^\preceq \Psi)(i, x) := g_H(i, \Psi_i(x)) := H_i(\Psi_i(x)) =: (H_i \circ \Psi_i)(x) := f_{H^*}(i, x)$, and $g_H \circ \Sigma^\preceq \Psi := f_{H^*} \in \int_{i \in I}^\preceq F_i$. \square

8. ON FAMILIES OF EQUIVALENCE CLASSES

Roughly speaking, a family of subsets of a set X indexed by some set I is an assignment routine $\lambda_0 : I \rightsquigarrow \mathcal{P}(X)$ that behaves like a function i.e., if $i =_I j$, then $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$. The following definition is a formulation of this rough description that reveals the witnesses of the equality $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$. This is done “internally”, through the embeddings of the subsets into X . The equality $\lambda_0(i) =_{\forall_0} \lambda_0(j)$, which is defined “externally” through the transport maps, follows, and a family of subsets is also a family of sets. From the theory of families of subsets (see [35], chapter 4) we present here only what is relevant to the topological part of this paper.

Definition 8.1. Let X and I be sets. A *family of subsets* of X indexed by I , or an *I -family of subsets* of X , is a triplet $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \forall_0$,

$$\mathcal{E}^X : \bigwedge_{i \in I} \mathbb{F}(\lambda_0(i), X), \quad \mathcal{E}^X(i) =: \mathcal{E}_i^X; \quad i \in I,$$

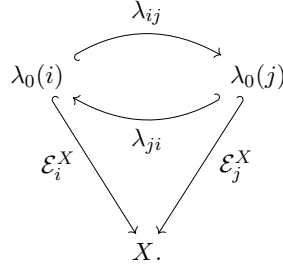
$$\lambda_1 : \bigwedge_{(i,j) \in D(I)} \mathbb{F}(\lambda_0(i), \lambda_0(j)), \quad \lambda_1(i, j) =: \lambda_{ij}; \quad (i, j) \in D(I),$$

such that the following conditions hold:

- (a) For every $i \in I$, the function $\mathcal{E}_i^X : \lambda_0(i) \rightarrow X$ is an embedding.

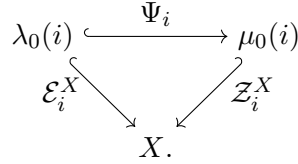
(b) For every $i \in I$, we have that $\lambda_{ii} := \text{id}_{\lambda_0(i)}$.

(c) For every $(i, j) \in D(I)$ we have that $\mathcal{E}_i^X = \mathcal{E}_j^X \circ \lambda_{ij}$ and $\mathcal{E}_j^X = \mathcal{E}_i^X \circ \lambda_{ji}$

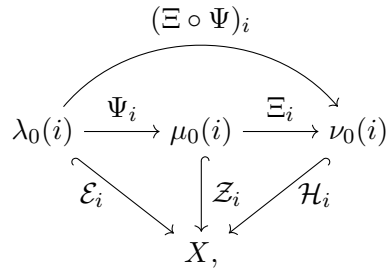


\mathcal{E}^X is a modulus of embeddings for λ_0 , and λ_1 a modulus of transport maps for λ_0 . Let $\Lambda := (\lambda_0, \lambda_1)$ be the I -family of sets that corresponds to $\Lambda(X)$. If $(A, i_A^X) \in \mathcal{P}(X)$, the constant I -family of subsets A is the pair $C^A(X) := (\lambda_0^A, \mathcal{E}^{X,A}, \lambda_1^A)$, where $\lambda_0(i) := A$, $\mathcal{E}_i^{X,A} := i_A^X$, and $\lambda_1(i, j) := \text{id}_A$, for every $i \in I$ and $(i, j) \in D(I)$

Definition 8.2. If $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1)$, $M(X) := (\mu_0, \mathcal{Z}^X, \mu_1)$ and $N(X) := (\nu_0, \mathcal{H}^X, \nu_1)$ are I -families of subsets of X , a family of subsets-map $\Psi: \Lambda(X) \Rightarrow M(X)$ from $\Lambda(X)$ to $M(X)$ is a dependent operation $\Psi: \bigwedge_{i \in I} \mathbb{F}(\lambda_0(i), \mu_0(i))$, where $\Psi(i) := \Psi_i$, for every $i \in I$, such that, for every $i \in I$, the following diagram commutes¹⁴



The totality $\text{Map}_I(\Lambda(X), M(X))$ of family of subsets-maps from $\Lambda(X)$ to $M(X)$ is equipped with the pointwise equality. If $\Psi: \Lambda(X) \Rightarrow M(X)$ and $\Xi: M(X) \Rightarrow N(X)$, the composition family of subsets-map $\Xi \circ \Psi: \Lambda(X) \Rightarrow N(X)$ is defined by $(\Xi \circ \Psi)(i) := \Xi_i \circ \Psi_i$,



for every $i \in I$. The identity family of subsets-map $\text{Id}_{\Lambda(X)}: \Lambda(X) \Rightarrow \Lambda(X)$ and the equality on the totality $\text{Fam}(I, X)$ of I -families of subsets of X are defined as in Definition 3.2.

We see no obvious reason, like the one for $\text{Fam}(I)$, not to consider $\text{Fam}(I, X)$ to be a set. In the case of $\text{Fam}(I)$ the constant I -family $\text{Fam}(I)$ would be in $\text{Fam}(I)$, while the constant I -family $\text{Fam}(I, X)$ is not clear how could be seen as a family of subsets of X . If $\nu_0(i) := \text{Fam}(I, X)$, for every $i \in I$, we need to define a modulus of embeddings $\mathcal{N}_i^X: \text{Fam}(I, X) \hookrightarrow X$, for every $i \in I$. From the given data one could define the assignment routine \mathcal{N}_i^X by the rule $\mathcal{N}_i^X(\Lambda(X)) := \mathcal{E}_i^X(u_i)$, if it is known that $u_i \in \lambda_0(i)$. Even in that

¹⁴Trivially, for every $i \in I$ the map $\Psi_i: \lambda_0(i) \rightarrow \mu_0(i)$ is an embedding.

case, the assignment routine \mathcal{N}_i^X cannot be shown to satisfy the expected properties. If \mathcal{N}_i^X was defined by the rule $\mathcal{N}_i^X(\Lambda(X)) := x_0$, for some $x_0 \in X$, then it cannot be an embedding.

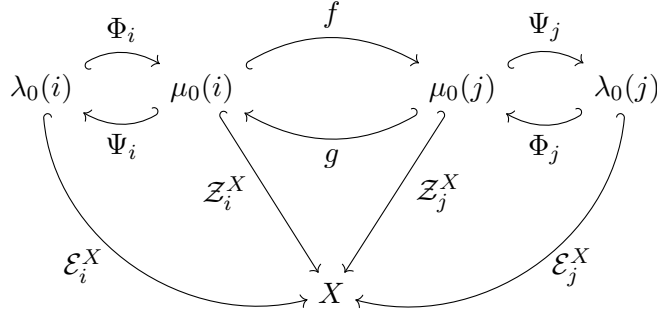
Definition 8.3. If $I, X \in \mathbb{V}_0$, a *set of subsets* of X indexed by I , or an *I -set of subsets* of X , is triplet $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathbf{Fam}(I, X)$ such that the following condition is satisfied:

$$Q(\Lambda(X)) := \Leftrightarrow \forall_{i,j \in I} (\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j) \Rightarrow i =_I j).$$

Let $\text{Set}(I, X)$ be their totality, equipped with the canonical equality on $\mathbf{Fam}(I, X)$.

Remark 8.4. If $\Lambda(X) \in \text{Set}(I, X)$ and $M(X) \in \mathbf{Fam}(I, X)$ such that $\Lambda(X) =_{\mathbf{Fam}(I, X)} M(X)$, then $M(X) \in \text{Set}(I, X)$.

Proof. Let $\Phi: \Lambda(X) \Rightarrow M(X)$ and $\Psi: M(X) \Rightarrow \Lambda(X)$ such that $(\Phi, \Psi): \Lambda(X) =_{\mathbf{Fam}(I, X)} M(X)$. Let also $(f, g): \mu_0(i) =_{\mathcal{P}(X)} \mu_0(j)$. It suffices to show that $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$.



If we define $f' := \Psi_j \circ f \circ \Phi_i$ and $g' := \Psi_i \circ g \circ \Phi_j$, it is straightforward to show that $(f', g'): \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$, hence $i =_I j$. \square

By the previous remark $Q(\Lambda(X))$ is an extensional property on $\mathbf{Fam}(I, X)$. Since $\text{Set}(I, X)$ is defined by separation on $\mathbf{Fam}(I, X)$, and since we see no objection to consider $\mathbf{Fam}(I, X)$ to be a set, we also see no objection to consider $\text{Set}(I, X)$ to be a set.

Definition 8.5. Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathbf{Fam}(I, X)$. Let the equality $=_I^{\Lambda(X)}$ on I given by $i =_I^{\Lambda(X)} j := \Leftrightarrow \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$, for every $i, j \in I$. The *set* $\lambda_0 I(X)$ of subsets of X generated by $\Lambda(X)$ is the totality I equipped with the equality $=_I^{\Lambda(X)}$. We write $\lambda_0(i) \in \lambda_0 I(X)$, instead of $i \in I$, when I is equipped with the equality $=_I^{\Lambda(X)}$. The new operation $\lambda_0^*: I \rightsquigarrow I$ from $(I, =_I)$ to $(I, =_I^{\Lambda(X)})$ is defined by the identity-rule.

Clearly, λ_0^* is a function. The following is easy to show (see also [35], section 3.7).

Proposition 8.6. Let $\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \text{Set}(I, X)$, and let Y be a set. If $f: I \rightarrow Y$, there is a unique function $\lambda_0 f: \lambda_0 I(X) \rightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} I & \xrightarrow{f} & Y \\ \lambda_0 \downarrow & \nearrow \lambda_0 f & \\ \lambda_0 I(X) & & \end{array}$$

Conversely, if $f: I \rightsquigarrow Y$ and $f^*: \lambda_0 I(X) \rightarrow Y$ such that the corresponding diagram commutes, then f is a function and f^* is equal to the function from $\lambda_0 I(X)$ to Y generated by f .

Although a family of equivalence classes is not, in general, a set of subsets, we can define functions on them, if we use appropriate functions on their index-set.

Definition 8.7. If X is a set and $R_X(x, x')$ is an extensional property on $X \times X$ that satisfies the conditions of an equivalence relation, we call the pair (X, R_X) an *equivalence structure*. If (Y, S_Y) is an equivalence structure, a function $f: X \rightarrow Y$ is an *equivalence preserving function*, or an (R_X, S_Y) -function, if

$$\forall_{x, x' \in X} (R(x, x') \Rightarrow S(f(x), f(x'))).$$

If, for every $x, x' \in X$, the converse implication holds, we say that f is an (R_X, S_Y) -embedding. Let $\mathbb{F}(R_X, S_Y)$ be the set of (R_X, S_Y) -functions¹⁵.

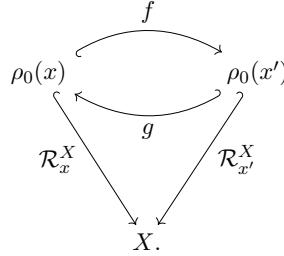
Proposition 8.8. If (X, R_X) is an equivalence structure, let $R(X) := (\rho_0, \mathcal{R}^X, \rho_1)$, where $\rho_0: X \rightsquigarrow \mathbb{V}_0$ is defined by $\rho_0(x) := \{y \in X \mid R_X(y, x)\}$, for every $x \in X$, and the dependent operations $\mathcal{R}^X: \lambda_{x \in X} \mathbb{F}(\rho_0(x), X)$, $\rho_1: \lambda_{(x, x') \in D(X)} \mathbb{F}(\rho_0(x), \rho_0(x'))$ are defined by

$$\mathcal{R}_x^X: \rho_0(x) \hookrightarrow X \quad y \mapsto y; \quad y \in \rho_0(x),$$

$$\rho_1(x, x') := \rho_{xx'}: \rho_0(x) \rightarrow \rho_0(x') \quad y \mapsto y; \quad y \in \rho_0(x).$$

Then $R(X) \in \mathbf{Fam}(X, X)$, and $\forall_{x, x' \in X} (\rho_0(x) =_{\mathcal{P}(X)} \rho_0(x') \Rightarrow R(x, x'))$.

Proof. By the extensionality of R_X the set $\rho_0(x)$ is a well-defined extensional subset of X . If $x =_X x'$ and $R_X(y, x)$, then by the extensionality of R_X we get $R_X(y, x')$, hence $\rho_{xx'}$ is well-defined. Let $(f, g): \rho_0(x) =_{\mathcal{P}(X)} \rho_0(x')$



If $y \in \rho_0(x) :\Leftrightarrow R_X(y, x)$, then $f(y) \in \rho_0(x') :\Leftrightarrow R_X(f(y), x')$, and by the commutativity of the corresponding above diagram we get $f(y) =_X y$. Hence by the extensionality of R_X we get $R_X(y, x')$. Since $R_X(y, x)$ implies $R_X(x, y)$, by transitivity we get $R_X(x, x')$. \square

Corollary 8.9. Let $\mathbf{Eq1}(X) := (\mathbf{eq1}_0^X, \mathcal{E}^X, \mathbf{eq1}_1^X)$ be the X -family of subsets of X induced by the equivalence relation $=_X$ i.e., $\mathbf{eq1}_0^X(x) := \{y \in X \mid y =_X x\}$. Then $\mathbf{Eq1}(X) \in \mathbf{Set}(X, X)$.

Proof. It follows immediately from Proposition 8.8. \square

¹⁵By the extensionality of S_Y the property of being an (R_X, S_Y) -function is extensional on $\mathbb{F}(X, Y)$.

9. DIRECT LIMIT OF BISHOP SPACES

If X is a set, by Corollary 8.9 the family $\mathbf{Eq1}(X) := (\mathbf{eq1}_0^X, \mathcal{E}^X, \mathbf{eq1}_1^X) \in \mathbf{Set}(X, X)$, where $\mathbf{eq1}_0^X(x) := \{y \in X \mid y =_X x\}$. Consequently, if $f: X \rightarrow Y$, there is unique $\mathbf{eq1}_0 f: \mathbf{eq1}_0 X(X) \rightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y, \\ \mathbf{eq1}_0^* \downarrow & \nearrow \mathbf{eq1}_0 f & \\ \mathbf{eq1}_0 X(X) & & \end{array}$$

where $\mathbf{eq1}_0 X(X)$ is the totality X with the equality $x =_{\mathbf{eq1}_0 X(X)} x' :\Leftrightarrow \mathbf{eq1}_0^X(x) =_{\mathcal{P}(X)} \mathbf{eq1}_0^X(x')$. As $\mathbf{Eq1}(X) \in \mathbf{Set}(X, X)$, we get $\mathbf{eq1}_0^X(x) =_{\mathcal{P}(X)} \mathbf{eq1}_0^X(x') \Leftrightarrow x =_X x'$. The map $\mathbf{eq1}_0^*: X \rightarrow \mathbf{eq1}_0 X(X)$ is defined by the identity map-rule, written in the form $x \mapsto \mathbf{eq1}_0^X(x)$, for every $x \in X$. We use the set $\mathbf{eq1}_0 X(X)$ to define the direct limit of a direct spectrum of Bishop spaces¹⁶. In what follows we avoid including the superscript X in our notation.

Definition 9.1. Let $S(\Lambda^\preceq) := (\lambda_0, \lambda_1^\preceq, \phi_0^{\Lambda^\preceq}, \phi_1^{\Lambda^\preceq}) \in \mathbf{Spec}(I, \preceq_I)$ and $\mathbf{eq1}_0: \sum_{i \in I}^\preceq \lambda_0(i) \rightsquigarrow \mathbb{V}_0$, defined by

$$\mathbf{eq1}_0(i, x) := \left\{ (j, y) \in \sum_{i \in I}^\preceq \lambda_0(i) \mid (j, y) =_{\sum_{i \in I}^\preceq \lambda_0(i)} (i, x) \right\}; \quad (i, x) \in \sum_{i \in I}^\preceq \lambda_0(i).$$

The *direct limit* $\mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i)$ of $S(\Lambda^\preceq)$ is the set

$$\mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i) := \mathbf{eq1}_0 \sum_{i \in I}^\preceq \lambda_0(i) \left(\sum_{i \in I}^\preceq \lambda_0(i) \right),$$

$$\mathbf{eq1}_0(i, x) =_{\mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i)} \mathbf{eq1}_0(j, y) :\Leftrightarrow \mathbf{eq1}_0(i, x) =_{\mathcal{P}(\sum_{i \in I}^\preceq \lambda_0(i))} \mathbf{eq1}_0(j, y) \Leftrightarrow (i, x) =_{\sum_{i \in I}^\preceq \lambda_0(i)} (j, y).$$

We write $\mathbf{eq1}_0^{\Lambda^\preceq}$ when we need to express the dependence of $\mathbf{eq1}_0$ from Λ^\preceq .

Remark 9.2. If $S(\Lambda^\preceq) := (\lambda_0, \lambda_1^\preceq, \phi_0^{\Lambda^\preceq}, \phi_1^{\Lambda^\preceq}) \in \mathbf{Spec}(I, \preceq_I)$ and $i \in I$, the operation $\mathbf{eq1}_i: \lambda_0(i) \rightsquigarrow \mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i)$, defined by $\mathbf{eq1}_i(x) := \mathbf{eq1}_0(i, x)$, for every $x \in \lambda_0(i)$, is a function.

Proof. If $x, x' \in \lambda_0(i)$ such that $x =_{\lambda_0(i)} x'$, then

$$\begin{aligned} \mathbf{eq1}_i(x) =_{\mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i)} \mathbf{eq1}_i(x') &:\Leftrightarrow \mathbf{eq1}_0(i, x) =_{\mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i)} \mathbf{eq1}_0(i, x') \\ &\Leftrightarrow (i, x) =_{\sum_{i \in I}^\preceq \lambda_0(i)} (i, x') \\ &:\Leftrightarrow \exists k \in I (i \preceq k \ \& \ \lambda_{ik}^\preceq(x) =_{\lambda_0(k)} \lambda_{ik}^\preceq(x')), \end{aligned}$$

which holds, since λ_{ik}^\preceq is a function, and hence if $x =_{\lambda_0(i)} x'$, then $\lambda_{ik}^\preceq(x) =_{\lambda_0(k)} \lambda_{ik}^\preceq(x')$, for every $k \in I$ such that $i \preceq k$. Such a $k \in I$ always exists e.g., one can take $k := i$. \square

¹⁶In this way, our definition of the direct limit of a direct spectrum of Bishop spaces is in complete analogy to the corresponding definition for topological spaces.

Definition 9.3. Let $S(\Lambda^{\preceq}) := (\lambda_0, \lambda_1^{\preceq}, \phi_0^{\Lambda^{\preceq}}, \phi_1^{\Lambda^{\preceq}}) \in \mathbf{Spec}(I, \preceq_I)$ with Bishop spaces $(F_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preceq})_{(i,j) \in D^{\preceq}(I)}$. The *direct limit* of $S(\Lambda^{\preceq})$ is the Bishop space

$$\mathop{\mathrm{Lim}}_{\rightarrow} \mathcal{F}_i := \left(\mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i), \mathop{\mathrm{Lim}}_{\rightarrow} F_i \right), \quad \text{where}$$

$$\mathop{\mathrm{Lim}}_{\rightarrow} F_i := \bigvee_{\Theta \in \Pi_{i \in I}^{\preceq} F_i} \mathbf{eq} \mathbf{l}_0 f_{\Theta},$$

$$\mathbf{eq} \mathbf{l}_0 f_{\Theta}(\mathbf{eq} \mathbf{l}_0(i, x)) := f_{\Theta}(i, x) := \Theta_i(x); \quad \mathbf{eq} \mathbf{l}_0(i, x) \in \mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i)$$

$$\begin{array}{ccc} \sum_{i \in I}^{\preceq} \lambda_0(i) & \xrightarrow{f_{\Theta}} & \mathbb{R}. \\ \mathbf{eq} \mathbf{l}_0^* \downarrow & \nearrow \mathbf{eq} \mathbf{l}_0 f_{\Theta} & \\ \mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i) & & \end{array}$$

Remark 9.4. If (I, \preceq) is a directed set, $\mathcal{G} := (Y, G)$ is a Bishop space, and $S(\Lambda^{\preceq, Y})$ is the constant direct spectrum over (I, \preceq_I) with Bishop space \mathcal{G} and Bishop morphism id_Y , the direct limit $\mathop{\mathrm{Lim}}_{\rightarrow} \mathcal{G}$ of $S(\Lambda^{\preceq, Y})$ is Bishop-isomorphic to \mathcal{G} . Moreover, every Bishop space is Bishop-isomorphic to the direct limit of a direct spectrum over any given directed set.

Proposition 9.5 (Universal property of the direct limit). *If $S(\Lambda^{\preceq}) := (\lambda_0, \lambda_1^{\preceq}, \phi_0^{\Lambda^{\preceq}}, \phi_1^{\Lambda^{\preceq}}) \in \mathbf{Spec}(I, \preceq_I)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preceq})_{(i,j) \in D^{\preceq}(I)}$, its direct limit $\mathop{\mathrm{Lim}}_{\rightarrow} \mathcal{F}_i$ satisfies the universal property of direct limits i.e.,*

(i) *For every $i \in I$, we have that $\mathbf{eq} \mathbf{l}_i \in \mathbf{Mor}(\mathcal{F}_i, \mathop{\mathrm{Lim}}_{\rightarrow} \mathcal{F}_i)$.*

(ii) *If $i \preceq_I j$, the following left diagram commutes*

$$\begin{array}{ccc} & \mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i) & \\ \mathbf{eq} \mathbf{l}_i \nearrow & & \nwarrow \mathbf{eq} \mathbf{l}_j \\ \lambda_0(i) & \xrightarrow{\lambda_{ij}^{\preceq}} & \lambda_0(j) \end{array} \quad \begin{array}{ccc} & Y & \\ \varepsilon_i \nearrow & & \nwarrow \varepsilon_j \\ \lambda_0(i) & \xrightarrow{\lambda_{ij}^{\preceq}} & \lambda_0(j). \end{array}$$

(iii) *If $\mathcal{G} := (Y, G)$ is a Bishop space and $\varepsilon_i : \lambda_0(i) \rightarrow Y \in \mathbf{Mor}(\mathcal{F}_i, \mathcal{G})$, for every $i \in I$, are such that if $i \preceq j$, the above right diagram commutes, there is a unique function $h : \mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i) \rightarrow Y \in \mathbf{Mor}(\mathop{\mathrm{Lim}}_{\rightarrow} \mathcal{F}_i, \mathcal{G})$ such that the following diagrams commute*

$$\begin{array}{ccc} & Y & \\ \varepsilon_i \nearrow & \uparrow h & \nwarrow \varepsilon_j \\ \lambda_0(i) & \xrightarrow{\lambda_{ij}^{\preceq}} & \lambda_0(j), \\ \mathbf{eq} \mathbf{l}_i \searrow & & \nearrow \mathbf{eq} \mathbf{l}_j \\ & \mathop{\mathrm{Lim}}_{\rightarrow} \lambda_0(i). & \end{array}$$

Proof. For the proof of (i), we use the \vee -lifting of morphisms. We have that

$$\mathbf{eq1}_i \in \text{Mor}(\mathcal{F}_i, \varinjlim \mathcal{F}_i) \Leftrightarrow \forall_{\Theta \in \prod_{i \in I}^{\preccurlyeq} F_i} (\mathbf{eq1}_0 f_{\Theta} \circ \mathbf{eq1}_i \in F_i).$$

If $x \in \lambda_0(i)$, then $(\mathbf{eq1}_0 f_{\Theta} \circ \mathbf{eq1}_i)(x) := \mathbf{eq1}_0 f_{\Theta}(\mathbf{eq1}_0(i, x)) := f_{\Theta}(i, x) := \Theta_i(x)$ hence $\mathbf{eq1}_0 f_{\Theta} \circ \mathbf{eq1}_i := \Theta_i \in F_i$. For the proof of (ii), if $x \in \lambda_0(i)$, then

$$\begin{aligned} \mathbf{eq1}_j(\lambda_{ij}^{\preccurlyeq}(x)) &=_{\varinjlim \lambda_0(i)} \mathbf{eq1}_i(x) \Leftrightarrow \mathbf{eq1}_0(j, \lambda_{ij}^{\preccurlyeq}(x)) =_{\varinjlim \lambda_0(i)} \mathbf{eq1}_0(i, x) \\ &\Leftrightarrow (j, \lambda_{ij}^{\preccurlyeq}(x)) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (i, x) \\ &\Leftrightarrow \exists_{k \in I} (i \preccurlyeq k \ \& \ j \preccurlyeq k \ \& \ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(\lambda_{ij}^{\preccurlyeq}(x))), \end{aligned}$$

which holds, since if $k \in I$ with $j \preccurlyeq k$, the equality $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(\lambda_{ij}^{\preccurlyeq}(x))$ holds by the definition of a direct family of sets, and by the definition of a directed set such a k always exists. To prove (iii) let the operation $h: \varinjlim \lambda_0(i) \rightsquigarrow Y$, defined by $h(\mathbf{eq1}_0(i, x)) := \varepsilon_i(x)$, for every $\mathbf{eq1}_0(i, x) \in \varinjlim \lambda_0(i)$. First we show that h is a function. Let

$$\mathbf{eq1}_0(i, x) =_{\varinjlim \lambda_0(i)} \mathbf{eq1}_0(j, y) \Leftrightarrow \exists_{k \in I} (i, j \preccurlyeq k \ \& \ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)).$$

By the supposed commutativity of the following diagrams

$$\begin{array}{ccc} & Y & \\ \varepsilon_i \nearrow & & \nwarrow \varepsilon_k \\ \lambda_0(i) & \xrightarrow{\lambda_{ik}^{\preccurlyeq}} & \lambda_0(k) \end{array} \quad \begin{array}{ccc} & Y & \\ \varepsilon_j \nearrow & & \nwarrow \varepsilon_k \\ \lambda_0(j) & \xrightarrow{\lambda_{jk}^{\preccurlyeq}} & \lambda_0(k) \end{array}$$

we get $h(\mathbf{eq1}_0(i, x)) := \varepsilon_i(x) = \varepsilon_k(\lambda_{ik}^{\preccurlyeq}(x)) = \varepsilon_k(\lambda_{jk}^{\preccurlyeq}(y)) = \varepsilon_j(y) := h(\mathbf{eq1}_0(j, y))$. Next we show that h is a Bishop morphism. By definition $h \in \text{Mor}(\varinjlim \mathcal{F}_i, \mathcal{G}) \Leftrightarrow \forall_{g \in G} (g \circ h \in \varinjlim F_i)$. If $g \in G$, we show that the dependent operation $\Theta_g: \prod_{i \in I}^{\preccurlyeq} F_i$, defined by $\Theta_g(i) := g \circ \varepsilon_i$, for every $i \in I$, is well-defined, since $\varepsilon_i \in \text{Mor}(\mathcal{F}_i, \mathcal{G})$, and that $\Theta_g \in \prod_{i \in I}^{\preccurlyeq} F_i$. To prove the latter, if $i \preccurlyeq k$, we show that $\Theta_g(i) = \Theta_g(k) \circ \lambda_{ik}^{\preccurlyeq}$. By the commutativity of the above left diagram we have that $\Theta_g(k) \circ \lambda_{ik}^{\preccurlyeq} := (g \circ \varepsilon_k) \circ \lambda_{ik}^{\preccurlyeq} := g \circ (\varepsilon_k \circ \lambda_{ik}^{\preccurlyeq}) = g \circ \varepsilon_i := \Theta_g(i)$. Hence $f_{\Theta_g} \in \varinjlim F_i$. Since $(g \circ h)(\mathbf{eq1}_0(i, x)) := g(\varepsilon_i(x)) := (g \circ \varepsilon_i)(x) := [\Theta_g(i)](x) := f_{\Theta_g}(\mathbf{eq1}_0(i, x))$, we get $g \circ h := \mathbf{eq1}_0 f_{\Theta_g} \in \varinjlim F_i$. The uniqueness of h , and the commutativity of the diagram in property (iii) follow immediately. \square

The uniqueness of $\varinjlim \lambda_0(i)$, up to Bishop isomorphism, is shown easily from its universal property. Note that if $i, j \in I$, $x \in \lambda_0(i)$ and $y \in \lambda_0(j)$, we have that

$$\begin{aligned} \mathbf{eq1}_i(x) =_{\varinjlim \lambda_0(i)} \mathbf{eq1}_j(y) &\Leftrightarrow \mathbf{eq1}_0(i, x) =_{\varinjlim \lambda_0(i)} \mathbf{eq1}_0(j, y) \\ &\Leftrightarrow (i, x) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (j, y) \\ &\Leftrightarrow \exists_{k \in I} (i \preccurlyeq k \ \& \ j \preccurlyeq k \ \& \ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)). \end{aligned}$$

Definition 9.6. Let $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ be a direct spectrum over (I, \preccurlyeq) . If $i \in I$, an element x of $\lambda_0(i)$ is a *representative* of $\mathbf{eq1}_0(z) \in \varinjlim \lambda_0(i)$, if $\mathbf{eq1}_i(x) =_{\varinjlim \lambda_0(i)} \mathbf{eq1}_0(z)$.

Although an element $\mathbf{eq}1_0(z) \in \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i)$ may not have a representative in every $\lambda_0(i)$, it surely has one at some $\lambda_0(i)$. Actually, the following holds.

Proposition 9.7. *For every $n \geq 1$ and every $\mathbf{eq}1_0(z_1), \dots, \mathbf{eq}1_0(z_n) \in \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i)$ there are $i \in I$ and $x_1, \dots, x_n \in \lambda_0(i)$ such that x_l represents $\mathbf{eq}1_0(z_l)$, for every $l \in \{1, \dots, n\}$.*

Proof. The proof is by induction on \mathbb{N}^+ . We present only the case $n := 2$. Let $z := (j, y), z' := (j', y') \in \sum_{i \in I} \lambda_0(i)$, and $k \in I$ with $j \preceq k$ and $j' \preceq k$. By definition we have that $\lambda_{jk}^{\preceq}(y) \in \lambda_0(k)$ and $\lambda_{j'k}^{\preceq}(y') \in \lambda_0(k)$. We show that $\lambda_{jk}^{\preceq}(y)$ represents $\mathbf{eq}1_0(z)$ and $\lambda_{j'k}^{\preceq}(y')$ represents $\mathbf{eq}1_0(z')$. By the equivalences right before Definition 9.6 for the first representation we need to show that

$$\mathbf{eq}1_k(\lambda_{jk}^{\preceq}(y)) =_{\mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i)} \mathbf{eq}1_j(y) \Leftrightarrow \exists k' \in I (k \preceq k' \ \& \ j \preceq k' \ \& \ \lambda_{kk'}^{\preceq}(\lambda_{jk}^{\preceq}(y)) =_{\lambda_0(k')} \lambda_{j'k'}^{\preceq}(y)).$$

By the composition of the transport maps it suffices to take any $k' \in I$ with $k \preceq k' \ \& \ j \preceq k'$, and for the second representation it suffices to take any $k'' \in I$ with $k \preceq k'' \ \& \ j' \preceq k''$. \square

Theorem 9.8. *Let $S(\Lambda^{\preceq}) := (\lambda_0, \lambda_1^{\preceq}, \phi_0^{\Lambda^{\preceq}}, \phi_1^{\Lambda^{\preceq}}) \in \mathbf{Spec}(I, \preceq_I)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preceq})_{(i,j) \in D^{\preceq}(I)}$, $S(M^{\preceq}) := (\mu_0, \mu_1^{\preceq}, \phi_0^{M^{\preceq}}, \phi_1^{M^{\preceq}}) \in \mathbf{Spec}(I, \preceq_I)$ with Bishop spaces $(\mathcal{G}_i)_{i \in I}$ and Bishop morphisms $(\mu_{ij}^{\preceq})_{(i,j) \in D^{\preceq}(I)}$, and $\Psi: S(\Lambda^{\preceq}) \Rightarrow S(M^{\preceq})$.*

(i) *There is a unique function $\Psi_{\rightarrow}: \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i) \rightarrow \mathop{\mathrm{Lim}}\limits_{\rightarrow} \mu_0(i)$ such that, for every $i \in I$, the following diagram commutes*

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\Psi_i} & \mu_0(i) \\ \mathbf{eq}1_i^{\Lambda^{\preceq}} \downarrow & & \downarrow \mathbf{eq}1_i^{M^{\preceq}} \\ \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i) & \xrightarrow{\Psi_{\rightarrow}} & \mathop{\mathrm{Lim}}\limits_{\rightarrow} \mu_0(i). \end{array}$$

(ii) *If Ψ is continuous, then $\Psi_{\rightarrow} \in \mathbf{Mor}(\mathop{\mathrm{Lim}}\limits_{\rightarrow} \mathcal{F}_i, \mathop{\mathrm{Lim}}\limits_{\rightarrow} \mathcal{G}_i)$.*

(iii) *If Ψ_i is an embedding for every $i \in I$, then Ψ_{\rightarrow} is an embedding.*

Proof. (i) The following well-defined operation $\Psi_{\rightarrow}: \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i) \rightsquigarrow \mathop{\mathrm{Lim}}\limits_{\rightarrow} \mu_0(i)$, given by

$$\Psi_{\rightarrow}(\mathbf{eq}1_0^{\Lambda^{\preceq}}(i, x)) := \mathbf{eq}1_0^{M^{\preceq}}(i, \Psi_i(x)); \quad \mathbf{eq}1_0^{\Lambda^{\preceq}}(i, x) \in \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i)$$

is a function, since, if $\mathbf{eq}1_0^{\Lambda^{\preceq}}(i, x) =_{\mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i)} \mathbf{eq}1_0^{\Lambda^{\preceq}}(j, y) \Leftrightarrow (i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y)$, which is equivalent to $\exists k \in I (i \preceq k \ \& \ j \preceq k \ \& \ \lambda_{ik}^{\preceq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preceq}(y))$, we show that

$$\begin{aligned} \Psi_{\rightarrow}(\mathbf{eq}1_0^{\Lambda^{\preceq}}(i, x)) =_{\mathop{\mathrm{Lim}}\limits_{\rightarrow} \mu_0(i)} \Psi_{\rightarrow}(\mathbf{eq}1_0^{\Lambda^{\preceq}}(j, y)) &: \Leftrightarrow \mathbf{eq}1_0^{M^{\preceq}}(i, \Psi_i(x)) =_{\mathop{\mathrm{Lim}}\limits_{\rightarrow} \mu_0(i)} \mathbf{eq}1_0^{M^{\preceq}}(j, \Psi_j(y)) \\ &\Leftrightarrow (i, \Psi_i(x)) =_{\sum_{i \in I} \mu_0(i)} (j, \Psi_j(y)) \\ &\Leftrightarrow \exists i \in I (i, j \preceq k \ \& \ \mu_{ik}^{\preceq}(\Psi_i(x)) =_{\mu_0(k)} \mu_{jk}^{\preceq}(\Psi_j(y))). \end{aligned}$$

By the commutativity of the following diagrams, and since Ψ_k is a function,

$e, \phi_0^{\Lambda^\preceq} \circ e, \phi_1^{\Lambda^\preceq} \circ e$) the relative spectrum of $S(\Lambda^\preceq)$ to J .

(i) If $\Theta \in \prod_{i \in I}^{\preceq} F_i$, then $\Theta^J \in \prod_{j \in J}^{\preceq} F_j$, where for every $j \in J$ we define $\Theta_j^J := \Theta_{e(j)}$.

(ii) If $H^J \in \prod_{j \in J}^{\preceq} F_j$, then $H \in \prod_{i \in I}^{\preceq} F_i$, where, for every $i \in I$, let $H_i := H_{\text{cof}_J(i)}^J \circ \lambda_{ie(\text{cof}_J(i))}^{\preceq}$

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\lambda_{ie(\text{cof}_J(i))}^{\preceq}} & \lambda_0(e(\text{cof}_J(i))) \\ & \searrow H_i & \downarrow H_{\text{cof}_J(i)}^J \\ & & \mathbb{R}. \end{array}$$

Proof. (i) It suffices to show that if $j \preceq j' \Leftrightarrow e(j) \preceq e(j')$, then $\Theta_j^J = \Theta_{j'}^J \circ \lambda_{jj'}^{\preceq}$. Since $\Theta \in \prod_{i \in I}^{\preceq} F_i$ we have that $\Theta_j^J := \Theta_{e(j)} = \Theta_{e(j')} \circ \lambda_{e(j)e(j')}^{\preceq} =: \Theta_{j'}^J \circ \lambda_{jj'}^{\preceq}$.

(ii) By definition $H_{\text{cof}_J(i)}^J \in F_{\text{cof}_J(i)} := F_{e(\text{cof}_J(i))}$, and since $i \preceq e(\text{cof}_J(i))$, we get $H_i \in \text{Mor}(\mathcal{F}_i, \mathcal{R}) = \mathcal{F}_i$ i.e., $H : \bigwedge_{i \in I} F_i$. Next we show that if $i \preceq i'$, then $H_i = H_{i'} \circ \lambda_{ii'}^{\preceq}$. By (Cof_3) and (Cof_2) we have that

$$i \preceq i' \preceq e(\text{cof}_J(i')), \quad (9.1)$$

and $i \preceq i' \Rightarrow \text{cof}_J(i) \preceq \text{cof}_J(i') \Leftrightarrow e(\text{cof}_J(i)) \preceq e(\text{cof}_J(i'))$, hence we also get

$$i \preceq e(\text{cof}_J(i)) \preceq e(\text{cof}_J(i')). \quad (9.2)$$

Since $H^J \in \prod_{j \in J}^{\preceq} F_j$, we have that

$$\begin{aligned} H_{i'} \circ \lambda_{ii'}^{\preceq} &:= [H_{\text{cof}_J(i')}^J \circ \lambda_{i'e(\text{cof}_J(i'))}^{\preceq}] \circ \lambda_{ii'}^{\preceq} \\ &:= H_{\text{cof}_J(i')}^J \circ [\lambda_{i'e(\text{cof}_J(i'))}^{\preceq} \circ \lambda_{ii'}^{\preceq}] \\ &\stackrel{(9.1)}{=} H_{\text{cof}_J(i')}^J \circ \lambda_{ie(\text{cof}_J(i'))}^{\preceq} \\ &\stackrel{(9.2)}{=} H_{\text{cof}_J(i')}^J \circ [\lambda_{e(\text{cof}_J(i))e(\text{cof}_J(i'))}^{\preceq} \circ \lambda_{ie(\text{cof}_J(i))}^{\preceq}] \\ &:= [H_{\text{cof}_J(i')}^J \circ \lambda_{e(\text{cof}_J(i))e(\text{cof}_J(i'))}^{\preceq}] \circ \lambda_{ie(\text{cof}_J(i))}^{\preceq} \\ &:= [H_{\text{cof}_J(i')}^J \circ \lambda_{\text{cof}_J(i)\text{cof}_J(i')}^{\preceq}] \circ \lambda_{ie(\text{cof}_J(i))}^{\preceq} \\ &:= H_{\text{cof}_J(i)}^J \circ \lambda_{ie(\text{cof}_J(i))}^{\preceq} \\ &=: H_i. \end{aligned} \quad \square$$

Theorem 9.12. Let $S(\Lambda^\preceq) := (\lambda_0, \lambda_1^{\preceq}, \phi_0^{\Lambda^\preceq}, \phi_1^{\Lambda^\preceq}) \in \text{Spec}(I, \preceq_I)$, $(J, e, \text{cof}_J) \subseteq^{\text{cof}} I$, and $S(\Lambda^\preceq) \circ e := (\lambda_0 \circ e, \lambda_1 \circ e, \phi_0^{\Lambda^\preceq} \circ e, \phi_1^{\Lambda^\preceq} \circ e)$ the relative spectrum of $S(\Lambda^\preceq)$ to J . Then

$$\text{Lim}_{\rightarrow} \mathcal{F}_j \simeq \text{Lim}_{\rightarrow} \mathcal{F}_i.$$

Proof. We define the operation $\phi : \text{Lim}_{\rightarrow} \lambda_0(j) \rightsquigarrow \text{Lim}_{\rightarrow} \lambda_0(i)$ by $\phi(\text{eq}1_0^{\Lambda^\preceq \circ e}(j, y)) := \text{eq}1_0^{\Lambda^\preceq}(e(j), y)$

$$\begin{array}{ccc}
& \lambda_0(j) & \\
\text{eq}1_j^{\Lambda^{\approx}oe} \swarrow & & \searrow \text{eq}1_{e(j)}^{\Lambda^{\approx}} \\
\text{Lim}\lambda_0(j) & \xrightarrow{\phi} & \text{Lim}\lambda_0(i), \\
\downarrow & & \downarrow
\end{array}$$

for every $\text{eq}1_0^{\Lambda^{\approx}oe}(j, y) \in \text{Lim}\lambda_0(j)$, where, if $j \in J$ and $y \in \lambda_0(j)$, we have that

$$\begin{aligned}
\text{eq}1_0^{\Lambda^{\approx}oe}(j, y) &:= \left\{ (j', y') \in \sum_{j \in J}^{\approx} \lambda_0(j) \mid (j', y') =_{\sum_{j \in J}^{\approx} \lambda_0(j)} (j, y) \right\}, \\
\text{eq}1_0^{\Lambda^{\approx}}(e(j), y) &:= \left\{ (i, x) \in \sum_{i \in I}^{\approx} \lambda_0(i) \mid (i, x) =_{\sum_{i \in I}^{\approx} \lambda_0(i)} (e(j), y) \right\}.
\end{aligned}$$

First we show that ϕ is a function. By definition we have that

$$\begin{aligned}
\text{eq}1_0^{\Lambda^{\approx}oe}(j, y) =_{\text{Lim}\lambda_0(j)} \text{eq}1_0^{\Lambda^{\approx}oe}(j', y') &\Leftrightarrow (j, y) =_{\sum_{j \in J}^{\approx} \lambda_0(j)} (j', y') \\
&\Leftrightarrow \exists j'' \in J (j, j' \approx j'' \ \& \ \lambda_{j j''}^{\approx}(y) =_{\lambda_0(j'')} \lambda_{j' j''}^{\approx}(y')) \quad (1)
\end{aligned}$$

$$\begin{aligned}
\text{eq}1_0^{\Lambda^{\approx}}(e(j), y) =_{\text{Lim}\lambda_0(i)} \text{eq}1_0^{\Lambda^{\approx}}(e(j'), y') &\Leftrightarrow (e(j), y) =_{\sum_{i \in I}^{\approx} \lambda_0(i)} (e(j'), y') \\
&\Leftrightarrow \exists k \in I (e(j), e(j') \approx k \ \& \ \lambda_{e(j)k}^{\approx}(y) =_{\lambda_0(k)} \lambda_{e(j')k}^{\approx}(y')). \quad (2)
\end{aligned}$$

Taking $k := e(j'')$, we see that (1) implies (2), and hence ϕ is a function. To show that ϕ is an embedding, we show that (2) implies (1). Since $e(j), e(j') \approx k \approx e(\text{cof}_J(k))$, we get $j, j' \approx \text{cof}_J(k) := j''$. By the commutativity of the following diagrams

$$\begin{array}{ccc}
\lambda_0(e(j)) & & \lambda_0(e(j')) \\
\lambda_{e(j)k}^{\approx} \downarrow & \searrow \lambda_{e(j)e(\text{cof}_J(k))}^{\approx} & \lambda_{e(j')k}^{\approx} \downarrow \\
\lambda_0(k) & \xrightarrow{\lambda_{ke(\text{cof}_J(k))}^{\approx}} & \lambda_0(e(\text{cof}_J(k))) & \lambda_0(k) & \xrightarrow{\lambda_{ke(\text{cof}_J(k))}^{\approx}} & \lambda_0(e(\text{cof}_J(k))) \\
& & & \lambda_{e(j')e(\text{cof}_J(k))}^{\approx} \searrow & &
\end{array}$$

$$\begin{aligned}
\lambda_{j j''}^{\approx}(y) &:= \lambda_{e(j)e(\text{cof}_J(k))}^{\approx}(y) \\
&= [\lambda_{ke(\text{cof}_J(k))} \circ \lambda_{e(j)k}^{\approx}](y) \\
&= \lambda_{ke(\text{cof}_J(k))}(\lambda_{e(j)k}^{\approx}(y)) \\
&= \lambda_{ke(\text{cof}_J(k))}(\lambda_{e(j')k}^{\approx}(y')) \\
&=: [\lambda_{ke(\text{cof}_J(k))} \circ \lambda_{e(j')k}^{\approx}](y') \\
&= \lambda_{e(j')e(\text{cof}_J(k))}^{\approx}(y') \\
&:= \lambda_{j' j''}(y').
\end{aligned}$$

By the ∇ -lifting of morphisms we have that

$$\phi \in \text{Mor}(\text{Lim}\mathcal{F}_j, \text{Lim}\mathcal{F}_i) := \Leftrightarrow \forall_{\Theta \in \prod_{i \in I}^{\approx} F_i} (\text{eq}1_0 f \Theta \circ \phi \in \text{Lim}\mathcal{F}_j).$$

If $\Theta \in \prod_{i \in I}^{\rightsquigarrow} F_i$, we have that

$$\begin{aligned} (\mathbf{eql}_0 f_\Theta \circ \phi)(\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}oe}(j, y)) &:= (\mathbf{eql}_0 f_\Theta)(\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}}(e(j), y)) \\ &:= \Theta_{e(j)}(y) := \Theta_j^J(y) := (\mathbf{eql}_0 f_{\Theta^J})(\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}oe}(j, y)), \end{aligned}$$

where $\Theta^J \in \prod_{j \in J}^{\rightsquigarrow} F_j$ is defined in Lemma 9.11(i). Hence, $\mathbf{eql}_0 f_\Theta \circ \phi = \mathbf{eql}_0 f_{\Theta^J} \in \mathop{\mathrm{Lim}}\limits_{\rightarrow} F_j$.

Next we show that ϕ is a surjection. If $\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}}(i, x) \in \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i)$, we find $\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}oe}(j, y) \in \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(j)$ such that $\phi(\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}oe}(j, y)) := \mathbf{eql}_0^{\Lambda^{\rightsquigarrow}}(e(j), y) =_{\mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i)} \mathbf{eql}_0^{\Lambda^{\rightsquigarrow}}(i, x)$ i.e., we find $k \in I$ such that $i, e(j) \rightsquigarrow k$ and $\lambda_{ik}^{\rightsquigarrow}(x) =_{\lambda_0(k)} \lambda_{e(j)k}^{\rightsquigarrow}(y)$. If $j := \mathbf{cof}_J(i)$, by (Cof₃) we have that $i \rightsquigarrow e(\mathbf{cof}_J(i))$, and by the reflexivity of \rightsquigarrow we have that $e(\mathbf{cof}_J(i)) \rightsquigarrow e(\mathbf{cof}_J(i)) =: k$. If $y := \lambda_{ie(\mathbf{cof}_J(i))}^{\rightsquigarrow}(x) \in \lambda_0(e(\mathbf{cof}_J(i))) := (\lambda_0 \circ e)(\mathbf{cof}_J(i))$, then

$$\lambda_{e(\mathbf{cof}_J(i))e(\mathbf{cof}_J(i))}^{\rightsquigarrow}(\lambda_{ie(\mathbf{cof}_J(i))}^{\rightsquigarrow}(x)) =_{\lambda_0(k)} \lambda_{ie(\mathbf{cof}_J(i))}^{\rightsquigarrow}(x).$$

We could use the \vee -lifting of openness to show that ϕ is an open morphism, and hence a Bishop isomorphism, but it is better to define directly its inverse Bishop morphism using the previous proof of the surjectivity of ϕ . Let the operation $\theta: \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i) \rightsquigarrow \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(j)$, defined by

$$\theta(\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}}(i, x)) := \mathbf{eql}_0^{\Lambda^{\rightsquigarrow}oe}(\mathbf{cof}_J(i), \lambda_{ie(\mathbf{cof}_J(i))}^{\rightsquigarrow}(x)); \quad \mathbf{eql}_0^{\Lambda^{\rightsquigarrow}}(i, x) \in \mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i).$$

First we show that θ is a function. We have that

$$\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}}(i, x) =_{\mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(i)} \mathbf{eql}_0^{\Lambda^{\rightsquigarrow}}(i', x') \Leftrightarrow \exists k \in I (i \rightsquigarrow k \ \& \ i' \rightsquigarrow k \ \& \ \lambda_{ik}^{\rightsquigarrow}(x) =_{\lambda_0(k)} \lambda_{i'k}^{\rightsquigarrow}(x')),$$

$$\mathbf{eql}_0^{\Lambda^{\rightsquigarrow}oe}(\mathbf{cof}_J(i), \lambda_{ie(\mathbf{cof}_J(i))}^{\rightsquigarrow}(x)) =_{\mathop{\mathrm{Lim}}\limits_{\rightarrow} \lambda_0(j)} \mathbf{eql}_0^{\Lambda^{\rightsquigarrow}oe}(\mathbf{cof}_J(i'), \lambda_{i'e(\mathbf{cof}_J(i'))}^{\rightsquigarrow}(x')) \Leftrightarrow$$

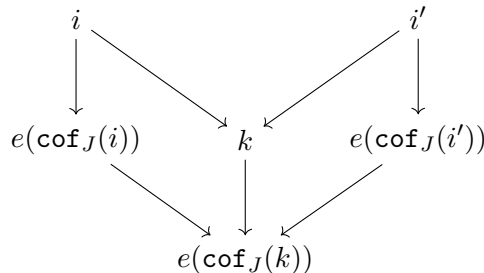
$$\exists j' \in J \left(\mathbf{cof}_J(i) \rightsquigarrow j' \ \& \ \mathbf{cof}_J(i') \rightsquigarrow j' \ \& \right.$$

$$\left. \lambda_{e(\mathbf{cof}_J(i))e(j')}^{\rightsquigarrow}(\lambda_{ie(\mathbf{cof}_J(i))}^{\rightsquigarrow}(x)) =_{\lambda_0(e(j'))} \lambda_{e(\mathbf{cof}_J(i'))e(j')}^{\rightsquigarrow}(\lambda_{i'e(\mathbf{cof}_J(i'))}^{\rightsquigarrow}(x')) \right).$$

If $j' := \mathbf{cof}_J(k)$, then by (Cof₂) we get $\mathbf{cof}_J(i) \rightsquigarrow j'$ and $\mathbf{cof}_J(i') \rightsquigarrow j'$. Next we show that

$$\lambda_{e(\mathbf{cof}_J(i))e(\mathbf{cof}_J(k))}^{\rightsquigarrow}(\lambda_{ie(\mathbf{cof}_J(i))}^{\rightsquigarrow}(x)) =_{\lambda_0(e(\mathbf{cof}_J(k)))} \lambda_{e(\mathbf{cof}_J(i'))e(\mathbf{cof}_J(k))}^{\rightsquigarrow}(\lambda_{i'e(\mathbf{cof}_J(i'))}^{\rightsquigarrow}(x')).$$

By the following order relations, the two terms of the required equality are written as



$\lambda_{ie(\mathbf{cof}_J(k))}^{\preceq}(x) = \lambda_{ke(\mathbf{cof}_J(k))}^{\preceq}(\lambda_{ik}^{\preceq}(x))$, and $\lambda_{i'e(\mathbf{cof}_J(k))}^{\preceq}(x') = \lambda_{ke(\mathbf{cof}_J(k))}^{\preceq}(\lambda_{i'k}^{\preceq}(x'))$. By the equality $\lambda_{ik}^{\preceq}(x) =_{\lambda_0(k)} \lambda_{i'k}^{\preceq}(x')$ we get the required equality. Next we show that

$$\theta \in \text{Mor}\left(\lim_{\rightarrow} \mathcal{F}_i, \lim_{\rightarrow} \mathcal{F}_j\right) \Leftrightarrow \forall_{H^J \in \prod_{j \in J}^{\preceq} F_j} \left(\mathbf{eq1}_0 f_{H^J} \circ \theta \in \bigvee_{\Theta \in \prod_{i \in I}^{\preceq} F_i} \mathbf{eq1}_0 f_{\Theta} \right).$$

If we fix $H^J \in \prod_{j \in J}^{\preceq} F_j$, and if $H \in \prod_{i \in I}^{\preceq} F_i$, defined in Lemma 9.11(ii), then

$$\begin{aligned} (\mathbf{eq1}_0 f_{H^J} \circ \theta)(\mathbf{eq1}_0^{\Lambda^{\preceq}}(i, x)) &:= \mathbf{eq1}_0 f_{H^J} \left(\mathbf{eq1}_0^{\Lambda^{\preceq} \circ e}(\mathbf{cof}_J(i), \lambda_{ie(\mathbf{cof}_J(i))}(x)) \right) \\ &:= f_{H^J}(\mathbf{cof}_J(i), \lambda_{ie(\mathbf{cof}_J(i))}(x)) \\ &:= H_{\mathbf{cof}_J(i)}^J(\lambda_{ie(\mathbf{cof}_J(i))}(x)) \\ &:= [H_{\mathbf{cof}_J(i)}^J \circ \lambda_{ie(\mathbf{cof}_J(i))}](x) \\ &:= H_i(x) \\ &:= f_H(i, x) \\ &:= \mathbf{eq1}_0 f_H(\mathbf{eq1}_0^{\Lambda^{\preceq}}(i, x)), \end{aligned}$$

hence $\mathbf{eq1}_0 f_{H^J} \circ \theta := \mathbf{eq1}_0 f_H \in \lim_{\rightarrow} F_i$. Next we show that ϕ and θ are inverse to each other.

$$\begin{aligned} \phi(\theta(\mathbf{eq1}_0^{\Lambda^{\preceq}}(i, x))) &:= \phi(\mathbf{eq1}_0^{\Lambda^{\preceq} \circ e}(\mathbf{cof}_J(i), \lambda_{ie(\mathbf{cof}_J(i))}(x))) \\ &:= \mathbf{eq1}_0^{\Lambda^{\preceq}}(e(\mathbf{cof}_J(i)), \lambda_{ie(\mathbf{cof}_J(i))}(x)), \end{aligned}$$

which is equal to $\mathbf{eq1}_0^{\Lambda^{\preceq}}(i, x)$ if and only if there is $k \in I$ with $i \preceq k$ and $e(\mathbf{cof}_J(i)) \preceq k$ and

$$\lambda_{ik}^{\preceq}(x) =_{\lambda_0(k)} \lambda_{e(\mathbf{cof}_J(i))k}^{\preceq}(\lambda_{ie(\mathbf{cof}_J(i))}(x)),$$

which holds for every such $k \in I$. As by (Cof₃) we have that $i \preceq e(\mathbf{cof}_J(i))$, the existence of such a $k \in I$ follows trivially. Similarly,

$$\begin{aligned} \theta(\phi(\mathbf{eq1}_0^{\Lambda^{\preceq} \circ e}(j, y))) &:= \theta(\mathbf{eq1}_0^{\Lambda^{\preceq}}(e(j), y)) \\ &:= \mathbf{eq1}_0^{\Lambda^{\preceq} \circ e}(\mathbf{cof}_J(e(j)), \lambda_{e(j)e(\mathbf{cof}_J(e(j)))}(y)), \end{aligned}$$

which is equal to $\mathbf{eq1}_0^{\Lambda^{\preceq} \circ e}(j, y)$ if and only if there is $j' \in J$ with $j \preceq j'$, $(\mathbf{cof}_J(e(j))) \preceq j'$ and

$$\lambda_{e(j)e(j')}^{\preceq}(y) =_{\lambda_0(e(j'))} \lambda_{e(\mathbf{cof}_J(e(j)))e(j')}^{\preceq}(\lambda_{e(j)e(\mathbf{cof}_J(e(j)))}(y)),$$

which holds for every such $j' \in J$. As by (Cof₁) we have that $j =_J \mathbf{cof}_J(e(j))$, the existence of such a $j' \in J$ follows trivially. \square

Notice that there is an alternative way for proving Theorem 9.12 by appealing to the universal property (Proposition 9.5). Next, we use for simplicity the same symbol for different orderings.

Proposition 9.13. *If $(I, \preceq), (J, \preceq)$ are directed sets, $i \in I$ and $j \in J$, let*

$$(i, j) \preceq (i', j') \Leftrightarrow i \preceq i' \ \& \ j \preceq j'.$$

If $(K, i_K, \mathbf{cof}_K) \subseteq^{\mathbf{cof}} I$ and $(L, i_L, \mathbf{cof}_L) \subseteq^{\mathbf{cof}} J$, let $i_{K \times L} : K \times L \hookrightarrow I \times J$ and $\mathbf{cof}_{K \times L} : I \times J \rightarrow K \times L$, defined, for every $k \in K$ and $l \in L$, by

$$i_{K \times L}(k, l) := (i_K(k), i_L(l)) \quad \& \quad \mathbf{cof}_{K \times L}(i, j) := (\mathbf{cof}_K(i), \mathbf{cof}_L(j)).$$

Let $\Lambda^{\preceq} := (\lambda_0, \lambda_1^{\preceq}) \in \mathbf{Fam}(I, \preceq)$ and $M^{\preceq} := (\mu_0, \mu_1^{\preceq}) \in \mathbf{Fam}(J, \preceq)$. Let also $S(\Lambda^{\preceq}) := (\lambda_0, \lambda_1^{\preceq}, \phi_0^{\Lambda^{\preceq}}, \phi_1^{\Lambda^{\preceq}}) \in \mathbf{Spec}(I, \preceq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and morphisms $(\lambda_{ii'})_{(i,i') \in \preceq(I)}$, and $S(M^{\preceq}) := (\mu_0, \mu_1^{\preceq}, \phi_0^{M^{\preceq}}, \phi_1^{M^{\preceq}}) \in \mathbf{Spec}(J, \preceq)$ with Bishop spaces $(\mathcal{G}_j)_{j \in J}$ and morphisms $(\mu_{jj'})_{(j,j') \in \preceq(J)}$.

(i) $(I \times J, \preceq)$ is a directed set, and $(K \times L, i_{K \times L}, \mathbf{cof}_{K \times L}) \subseteq^{\mathbf{cof}} I \times J$.

(ii) The pair $\Lambda^{\preceq} \times M^{\preceq} := (\lambda_0 \times \mu_0, (\lambda_1 \times \mu_1)^{\preceq}) \in \mathbf{Fam}(I \times J, \preceq)$, where

$$(\lambda_0 \times \mu_0)((i, j)) := \lambda_0(i) \times \mu_0(j),$$

$$(\lambda_1 \times \mu_1)^{\preceq}((i, j), (i', j')) := (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq},$$

$$(\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq}((x, y)) := (\lambda_{ii'}^{\preceq}(x), \mu_{jj'}^{\preceq}(y)).$$

(iii) The structure $S(\Lambda^{\preceq} \times M^{\preceq}) := (\lambda_0 \times \mu_0, \lambda_1^{\preceq} \times \mu_1^{\preceq}; \phi_0^{\Lambda^{\preceq} \times M^{\preceq}}, \phi_1^{\Lambda^{\preceq} \times M^{\preceq}}) \in \mathbf{Spec}(I \times J, \preceq)$ with Bishop spaces $(F_i \times G_j)_{(i,j) \in I \times J}$ and Bishop morphisms $(\lambda_1 \times \mu_1)_{((i,j),(i',j')) \in D^{\preceq}(I \times J)}$ where

$$\phi_0^{\Lambda^{\preceq} \times M^{\preceq}}(i, j) := F_i \times G_j,$$

$$\phi_1^{\Lambda^{\preceq} \times M^{\preceq}}((i, j), (i', j')) := [(\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq}]^* : F_{i'} \times G_{j'} \rightarrow F_i \times G_j.$$

Proof. (i) is immediate to show. For the proof of (ii) we have that $(\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq}((x, y)) := (\lambda_{ii'}^{\preceq}(x), \mu_{jj'}^{\preceq}(y)) := (x, y)$, and if $(i, j) \preceq (i', j') \preceq (i'', j'')$, then the commutativity of the

$$\begin{array}{ccc} \lambda_0(i) \times \mu_0(j) & & \\ \downarrow (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq} & \searrow (\lambda_1 \times \mu_1)_{(i,j),(i'',j'')}^{\preceq} & \\ \lambda_0(i') \times \mu_0(j') & \xrightarrow{(\lambda_1 \times \mu_1)_{(i',j'),(i'',j'')}^{\preceq}} & \lambda_0(i'') \times \mu_0(j'') \end{array}$$

above diagram follows from the equalities $\lambda_{ii''}^{\preceq} = \lambda_{i'i''}^{\preceq} \circ \lambda_{ii'}^{\preceq}$ and $\mu_{jj''}^{\preceq} = \mu_{j'j''}^{\preceq} \circ \mu_{jj'}^{\preceq}$.

(iii) We show that $(\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq} \in \mathbf{Mor}(\mathcal{F}_i \times \mathcal{G}_j, \mathcal{F}_{i'} \times \mathcal{G}_{j'})$. By the \vee -lifting of morphisms it suffices to show that $\forall f \in F_{i'} ((f \circ \pi_1) \circ (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq} \in F_i \times G_j)$ and $\forall g \in G_{j'} ((g \circ \pi_2) \circ (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq} \in F_i \times G_j)$. If $f \in F_{i'}$, then $(f \circ \pi_1) \circ (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq} := (f \circ \lambda_{ii'}^{\preceq}) \circ \pi_1 \in F_i \times G_j$, as $f \circ \lambda_{ii'}^{\preceq} \in F_i$ and $[(f \circ \pi_1) \circ (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq}](x, y) := (f \circ \pi_1)(\lambda_{ii'}^{\preceq}(x), \mu_{jj'}^{\preceq}(y)) := f(\lambda_{ii'}^{\preceq}(x)) := [(f \circ \lambda_{ii'}^{\preceq}) \circ \pi_1](x, y)$. If $g \in G_{j'}$, we get $(g \circ \pi_2) \circ (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\preceq} := (g \circ \lambda_{jj'}^{\preceq}) \circ \pi_2 \in F_i \times G_j$. \square

Lemma 9.14. Let $S(\Lambda^{\preceq}) := (\lambda_0, \lambda_1^{\preceq}, \phi_0^{\Lambda^{\preceq}}, \phi_1^{\Lambda^{\preceq}}) \in \mathbf{Spec}(I, \preceq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ii'})_{(i,i') \in D^{\preceq}(I)}$, $S(M^{\preceq}) := (\mu_0, \mu_1^{\preceq}, \phi_0^{M^{\preceq}}, \phi_1^{M^{\preceq}}) \in \mathbf{Spec}(J, \preceq)$ with Bishop spaces $(\mathcal{G}_j)_{j \in J}$ and Bishop morphisms $(\mu_{jj'})_{(j,j') \in D^{\preceq}(J)}$, $\Theta \in \prod_{i \in I}^{\preceq} F_i$ and $\Phi \in \prod_{j \in J}^{\preceq} G_j$. Then

$$\Theta_1 \in \prod_{(i,j) \in I \times J}^{\preceq} F_i \times G_j \quad \& \quad \Phi_2 \in \prod_{(i,j) \in I \times J}^{\preceq} F_i \times G_j,$$

$$\Theta_1(i, j) := \Theta_i \circ \pi_1 \in F_i \times G_j \quad \& \quad \Phi_2(i, j) := \Phi_j \circ \pi_2 \in F_i \times G_j; \quad (i, j) \in I \times J.$$

Proof. We prove that $\Theta_1 \in \prod_{(i,j) \in I \times J}^{\approx} F_i \times G_j$, and for Φ_2 we proceed similarly. If $(i, j) \preccurlyeq (i', j')$, we need to show that $\Theta_1(i, j) = \Theta_1(i', j') \circ (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\approx}$. Since $\Theta \in \prod_{i \in I}^{\approx} F_i$, we have that $\Theta_i = \Theta_{i'} \circ \lambda_{ii'}^{\approx}$. If $x \in \lambda_0(i)$ and $y \in \mu_0(j)$, we have that

$$\begin{aligned} [\Theta_1(i', j') \circ (\lambda_1 \times \mu_1)_{(i,j),(i',j')}^{\approx}](x, y) &:= [\Theta_{i'} \circ \pi_1](\lambda_{ii'}^{\approx}(x), \mu_{jj'}^{\approx}(y)) \\ &:= \Theta_{i'}(\lambda_{ii'}^{\approx}(x)) \\ &:= [(\Theta_{i'} \circ \lambda_{ii'}^{\approx}) \circ \pi_1](x, y) \\ &:= (\Theta_i \circ \pi_1)(x, y) \\ &:= [\Theta_1(i, j)](x, y). \quad \square \end{aligned}$$

Proposition 9.15. *If $S(\Lambda^{\approx}) := (\lambda_0, \lambda_1^{\approx}, \phi_0^{\Lambda^{\approx}}, \phi_1^{\Lambda^{\approx}}) \in \mathbf{Spec}(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ii'})_{(i,i') \in D^{\approx}(I)}$, and $S(M^{\approx}) := (\mu_0, \mu_1^{\approx}, \phi_0^{M^{\approx}}, \phi_1^{M^{\approx}}) \in \mathbf{Spec}(J, \preccurlyeq)$ with Bishop spaces $(\mathcal{G}_j)_{j \in J}$ and Bishop morphisms $(\mu_{jj'})_{(j,j') \in D^{\approx}(J)}$, there is a bijection*

$$\theta : \varinjlim (\lambda_0(i) \times \mu_0(j)) \rightarrow \varinjlim \lambda_0(i) \times \varinjlim \mu_0(j) \in \mathbf{Mor}(\varinjlim (\mathcal{F}_i \times \mathcal{G}_j), \varinjlim \mathcal{F}_i \times \varinjlim \mathcal{G}_j).$$

Proof. Let the operation $\theta : \varinjlim (\lambda_0(i) \times \mu_0(j)) \rightsquigarrow \varinjlim \lambda_0(i) \times \varinjlim \mu_0(j)$ be defined by

$$\theta(\mathbf{eq1}_0^{\Lambda^{\approx} \times M^{\approx}}((i, j), (x, y))) := (\mathbf{eq1}_0^{\Lambda^{\approx}}(i, x), \mathbf{eq1}_0^{M^{\approx}}(j, y)).$$

First we show that θ is an embedding as follows:

$$\begin{aligned} \mathbf{eq1}_0^{\Lambda^{\approx} \times M^{\approx}}((i, j), (x, y)) = \mathbf{eq1}_0^{\Lambda^{\approx} \times M^{\approx}}((i', j'), (x', y')) &:\Leftrightarrow \\ :\Leftrightarrow \exists (k, l) \in I \times J ((i, j), (i', j') \preccurlyeq (k, l) \ \& \ (\lambda_1 \times \mu_1)_{(i,j),(k,l)}^{\approx}(x, y) = (\lambda_1 \times \mu_1)_{(i',j'),(k,l)}^{\approx}(x', y')) & \\ :\Leftrightarrow \exists (k, l) \in I \times J ((i, j), (i', j') \preccurlyeq (k, l) \ \& \ (\lambda_{ik}^{\approx}(x), \mu_{jl}^{\approx}(y)) = (\lambda_{i'k}^{\approx}(x'), \mu_{j'l}^{\approx}(y'))) & \\ :\Leftrightarrow \exists k \in I (i, i' \preccurlyeq k \ \& \ \lambda_{ik}^{\approx}(x) = \lambda_{i'k}^{\approx}(x')) \ \& \ \exists l \in J (j, j' \preccurlyeq l \ \& \ \mu_{jl}^{\approx}(y) = \mu_{j'l}^{\approx}(y')) & \\ :\Leftrightarrow \mathbf{eq1}_0^{\Lambda^{\approx}}(i, x) = \mathbf{eq1}_0^{\Lambda^{\approx}}(i', x') \ \& \ \mathbf{eq1}_0^{M^{\approx}}(j, y) = \mathbf{eq1}_0^{M^{\approx}}(j', y') & \\ :\Leftrightarrow (\mathbf{eq1}_0^{\Lambda^{\approx}}(i, x), \mathbf{eq1}_0^{M^{\approx}}(j, y)) = (\mathbf{eq1}_0^{\Lambda^{\approx}}(i', x'), \mathbf{eq1}_0^{M^{\approx}}(j', y')) & \\ :\Leftrightarrow \theta(\mathbf{eq1}_0^{\Lambda^{\approx} \times M^{\approx}}((i, j), (x, y))) = \theta(\mathbf{eq1}_0^{\Lambda^{\approx} \times M^{\approx}}((i', j'), (x', y'))). & \end{aligned}$$

The fact that θ is a surjection is immediate to show. By definition of the direct limit and by the \vee -lifting of the product Bishop topology we have that

$$\begin{aligned} \varinjlim (\mathcal{F}_i \times \mathcal{G}_j) &:= \left(\varinjlim (\lambda_0(i) \times \mu_0(j)), \bigvee_{\Xi \in \prod_{(i,j) \in I \times J}^{\approx} F_i \times G_j} \mathbf{eq1}_0 f_{\Xi} \right), \\ \varinjlim \mathcal{F}_i \times \varinjlim \mathcal{G}_j &:= \left(\varinjlim \lambda_0(i) \times \varinjlim \mu_0(j), \bigvee_{\Theta \in \prod_{i \in I}^{\approx} F_i} \bigvee_{H \in \prod_{j \in J}^{\approx} G_j} \mathbf{eq1}_0 f_{\Theta} \circ \pi_1, \mathbf{eq1}_0 f_H \circ \pi_2 \right). \end{aligned}$$

To show that $\theta \in \mathbf{Mor}(\varinjlim (\mathcal{F}_i \times \mathcal{G}_j), \varinjlim \mathcal{F}_i \times \varinjlim \mathcal{G}_j)$ it suffices to show that

$$\bigvee_{\Theta \in \prod_{i \in I}^{\approx} F_i} \bigvee_{H \in \prod_{j \in J}^{\approx} G_j} (\mathbf{eq1}_0 f_{\Theta} \circ \pi_1) \circ \theta \in \varinjlim (F_i \times G_j) \ \& \ (\mathbf{eq1}_0 f_H \circ \pi_2) \circ \theta \in \varinjlim (F_i \times G_j).$$

If $\Theta \in \prod_{i \in I}^{\succ} F_i$, we show that $(\text{eq}1_0 f_{\Theta} \circ \pi_1) \circ \theta \in \lim_{\rightarrow} (F_i \times G_j)$. From the equalities

$$\begin{aligned} [(\text{eq}1_0 f_{\Theta} \circ \pi_1) \circ \theta](\text{eq}1_0^{\Lambda^{\preceq} \times M^{\preceq}}((i, j), (x, y))) &:= (\text{eq}1_0 f_{\Theta} \circ \pi_1)(\text{eq}1_0^{\Lambda^{\preceq}}(i, x), \text{eq}1_0^{M^{\preceq}}(j, y)) \\ &:= \text{eq}1_0 f_{\Theta}(\text{eq}1_0^{\Lambda^{\preceq}}(i, x)) \\ &:= \Theta_i(x) \\ &=: (\Theta_i \circ \pi_1)(x, y) \\ &:= [\Theta_1(i, j)](x, y) \\ &:= \text{eq}1_0 f_{\Theta_1}(\text{eq}1_0^{\Lambda^{\preceq} \times M^{\preceq}}((i, j), (x, y))), \end{aligned}$$

where $\Theta_1 \in \prod_{i \in I}^{\succ} F_i \times G_j$ is defined in Lemma 9.14, we conclude that $(\text{eq}1_0 f_{\Theta} \circ \pi_1) \circ \theta := \text{eq}1_0 f_{\Theta_1} \in \lim_{\rightarrow} (F_i \times G_j)$. For the second case we work similarly. \square

10. INVERSE LIMIT OF BISHOP SPACES

Definition 10.1. If $S(\Lambda^{\succ}) := (\lambda_0, \lambda_1^{\succ}, \phi_0^{\Lambda^{\succ}}, \phi_1^{\Lambda^{\succ}})$ is in $\text{Spec}(I, \succ)$ i.e., a contravariant (I, \preceq) -spectrum, with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\preceq})_{(i, j) \in D^{\preceq}(I)}$, the *inverse limit* of $S(\Lambda^{\succ})$ is the Bishop space

$$\begin{aligned} \lim_{\leftarrow} \mathcal{F}_i &:= (\lim_{\leftarrow} \lambda_0(i), \lim_{\leftarrow} \mathcal{F}_i), \\ \lim_{\leftarrow} \lambda_0(i) &:= \prod_{i \in I}^{\succ} \lambda_0(i) \quad \& \quad \lim_{\leftarrow} \mathcal{F}_i := \bigvee_{i \in I}^{f \in \mathcal{F}_i} f \circ \pi_i^{\Lambda^{\succ}}. \end{aligned}$$

We write π_i instead of $\pi_i^{\Lambda^{\succ}}$ for the function $\pi_i^{\Lambda^{\succ}} : \prod_{i \in I}^{\succ} \lambda_0(i) \rightarrow \lambda_0(i)$, which is defined, as its dual $\pi_i^{\Lambda^{\preceq}}$ in the Proposition 5.4(iv), by the rule $\Phi \mapsto \Phi_i$, for every $i \in I$.

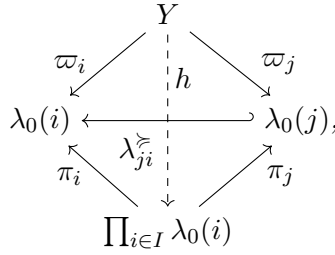
Proposition 10.2 (Universal property of the inverse limit). *If $S(\Lambda^{\succ}) := (\lambda_0, \lambda_1^{\succ}, \phi_0^{\Lambda^{\succ}}, \phi_1^{\Lambda^{\succ}})$ is in $\text{Spec}(I, \succ)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\preceq})_{(i, j) \in \preceq(I)}$, its inverse limit $\lim_{\leftarrow} \mathcal{F}_i$ satisfies the universal property of inverse limits:*

(i) For every $i \in I$, we have that $\pi_i \in \text{Mor}(\lim_{\leftarrow} \mathcal{F}_i, \mathcal{F}_i)$.

(ii) If $i \preceq j$, the following left diagram commutes

$$\begin{array}{ccc} \prod_{i \in I}^{\succ} \lambda_0(i) & & Y \\ \pi_i \swarrow & & \swarrow \varpi_i \\ \lambda_0(i) & \xleftarrow{\lambda_{ji}^{\preceq}} & \lambda_0(j) \\ & \searrow \pi_j & \searrow \varpi_j \\ & & \lambda_0(j) \end{array}$$

(iii) If $\mathcal{G} := (Y, G)$ is a Bishop space and $\varpi_i : Y \rightarrow \lambda_0(i) \in \text{Mor}(\mathcal{G}, \mathcal{F}_i)$, for every $i \in I$, are such that if $i \preceq j$, the above right diagram commutes, there is a unique function $h : Y \rightarrow \prod_{i \in I}^{\succ} \lambda_0(i) \in \text{Mor}(\mathcal{G}, \lim_{\leftarrow} \mathcal{F}_i)$ such that the following diagrams commute



Proof. The condition $\pi_i \in \text{Mor}(\text{Lim}_{\leftarrow} \mathcal{F}_i, \mathcal{F}_i) :\Leftrightarrow \forall f \in F_i \left(f \circ \pi_i \in \bigvee_{i \in I}^{f \in F_i} f \circ \pi_i \right)$ is trivially satisfied, and (i) follows. For (ii), the required equality $\lambda_{ji}^\succ(\pi_j(\Phi)) =_{\lambda_0(i)} \pi_i(\Phi) :\Leftrightarrow \lambda_{ji}^\succ(\Phi_j) =_{\lambda_0(i)} \Phi_i$ holds by the definition of $\prod_{i \in I}^\succ \lambda_0(i)$. To show (iii), let the operation $h : Y \rightsquigarrow \prod_{i \in I}^\succ \lambda_0(i)$, defined by $h(y) := \Phi_y$, where $\Phi_y(i) := \varpi_i(y)$, for every $y \in Y$ and $i \in I$. First we show that h is well-defined i.e., $h(y) \in \prod_{i \in I}^\succ \lambda_0(i)$. If $i \leq j$, by the supposed commutativity of the above right diagram we have that $\lambda_{ji}^\succ(\Phi_y(j)) := \lambda_{ji}^\succ(\varpi_j(y)) = \varpi_i(y) := \Phi_y(i)$. Next we show that h is a function. If $y =_Y y'$, the last formula in the following equivalences

$$\Phi_y =_{\prod_{i \in I}^\succ \lambda_0(i)} \Phi_{y'} :\Leftrightarrow \forall i \in I (\Phi_y(i) =_{\lambda_0(i)} \Phi_{y'}(i)) :\Leftrightarrow \forall i \in I (\varpi_i(y) =_{\lambda_0(i)} \varpi_i(y'))$$

holds by the fact that ϖ_i is a function, for every $i \in I$. By the \bigvee -lifting of morphisms we have that $h \in \text{Mor}(\mathcal{G}, \text{Lim}_{\leftarrow} \mathcal{F}_i) \Leftrightarrow \forall i \in I \forall f \in F_i ((f \circ \pi_i) \circ h \in G)$. If $i \in I$, $f \in F_i$, and $y \in Y$, then

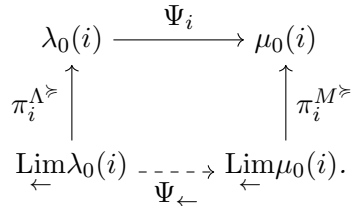
$$[(f \circ \pi_i) \circ h](y) := (f \circ \pi_i)(\Phi_y) := f(\Phi_y(i)) := f(\varpi_i(y)) =: (f \circ \varpi_i)(y),$$

hence $(f \circ \pi_i) \circ h := f \circ \varpi_i \in G$, since $\varpi_i \in \text{Mor}(\mathcal{G}, \mathcal{F}_i)$. The required commutativity of the last diagram above, and the uniqueness of h follow immediately. \square

The uniqueness of $\text{Lim}_{\leftarrow} \lambda_0(i)$, up to Bishop isomorphism, follows easily from its universal property. Next follows the inverse analogue to the Theorem 9.8.

Theorem 10.3. *Let $S(\Lambda^\succ) := (\lambda_0, \lambda_1^\succ, \phi_0^{\Lambda^\succ}, \phi_1^{\Lambda^\succ})$ be in $\text{Spec}(I, \succ)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^\succ)_{(i,j) \in D^\succ(I)}$, and $S(M^\succ) := (\mu_0, \mu_1, \phi_0^{M^\succ}, \phi_1^{M^\succ})$ in $\text{Spec}(I, \succ)$ with Bishop spaces $(\mathcal{G}_i)_{i \in I}$ and Bishop morphisms $(\mu_{ji}^\succ)_{(i,j) \in D^\succ(I)}$, and $\Psi : S(\Lambda^\succ) \Rightarrow S(M^\succ)$.*

(i) *There is a unique function $\Psi_{\leftarrow} : \text{Lim}_{\leftarrow} \lambda_0(i) \rightarrow \text{Lim}_{\leftarrow} \mu_0(i)$ such that, for every $i \in I$, the following diagram commutes*



(ii) *If Ψ is continuous, then $\Psi_{\leftarrow} \in \text{Mor}(\text{Lim}_{\leftarrow} \mathcal{F}_i, \text{Lim}_{\leftarrow} \mathcal{G}_i)$.*

(iii) *If Ψ_i is an embedding, for every $i \in I$, then Ψ_{\leftarrow} is an embedding.*

Proof. (i) Let the assignment routine $\Psi_{\leftarrow} : \text{Lim}_{\leftarrow} \lambda_0(i) \rightsquigarrow \text{Lim}_{\leftarrow} \mu_0(i)$, defined by

$$\Theta \mapsto \Psi_{\leftarrow}(\Theta), \quad [\Psi_{\leftarrow}(\Theta)]_i := \Psi_i(\Theta_i); \quad \Theta \in \text{Lim}_{\leftarrow} \lambda_0(i), \quad i \in I.$$

First we show that Ψ_{\leftarrow} is well-defined i.e., $\Psi_{\leftarrow}(\Theta) \in \prod_{i \in I}^{\succ} \mu_0(i)$. If $i \preccurlyeq j$, since $\Theta \in \prod_{i \in I}^{\succ} \lambda_0(i)$, we have that $\Theta_i = \lambda_{ji}^{\succ}(\Theta_j)$, and since $\Psi: S(\Lambda^{\succ}) \Rightarrow S(M^{\succ})$

$$\begin{array}{ccc} \lambda_0(i) & \xleftarrow{\lambda_{ji}^{\succ}} & \lambda_0(j) \\ \Psi_i \downarrow & & \downarrow \Psi_j \\ \mu_0(i) & \xleftarrow{\mu_{ji}^{\succ}} & \mu_0(j), \end{array}$$

$[\Psi_{\leftarrow}(\Theta)]_i := \Psi_i(\Theta_i) = \Psi_i(\lambda_{ji}^{\succ}(\Theta_j)) =: (\Psi_i \circ \lambda_{ji}^{\succ})(\Theta_j) = (\mu_{ji}^{\succ} \circ \Psi_j)(\Theta_j) := \mu_{ji}^{\succ}(\Psi_j(\Theta_j)) =: \mu_{ji}^{\succ}([\Psi_{\leftarrow}(\Theta)]_j)$. Ψ_{\leftarrow} is a function: $\Theta = \lim_{\leftarrow} \lambda_0(i) \Phi \Leftrightarrow \forall i \in I (\Theta_i = \lambda_0(i) \Phi_i) \Rightarrow \forall i \in I (\Psi_i(\Theta_i) = \mu_0(i) \Psi_i(\Phi_i)) \Leftrightarrow \forall i \in I ([\Psi_{\leftarrow}(\Theta)]_i = \mu_0(i) [\Psi_{\leftarrow}(\Phi)]_i) \Leftrightarrow \Psi_{\leftarrow}(\Theta) = \lim_{\leftarrow} \mu_0(i) \Psi_{\leftarrow}(\Phi)$. The commutativity of the diagram and the uniqueness of Ψ_{\leftarrow} are immediate to show.

(ii) By the \vee -lifting of morphisms $\Psi_{\leftarrow} \in \text{Mor}(\text{Lim}_{\leftarrow} \mathcal{F}_i, \text{Lim}_{\leftarrow} \mathcal{G}_i) \Leftrightarrow \forall i \in I \forall g \in G_i ((g \circ \pi_i^{M^{\succ}}) \circ \Psi_{\leftarrow} \in \text{Lim}_{\leftarrow} F_i)$. If $i \in I$ and $g \in G_i$, then $[(g \circ \pi_i^{M^{\succ}}) \circ \Psi_{\leftarrow}](\Theta) := g([\Psi_{\leftarrow}(\Theta)]_i) := g(\Psi_i(\Theta_i)) =: (g \circ \Psi_i)(\Theta_i) := [(g \circ \Psi_i) \circ \pi_i^{\Lambda^{\succ}}](\Theta)$, and $g \circ \Psi_i \in F_i$, by the continuity of Ψ , hence $(g \circ \pi_i^{M^{\succ}}) \circ \Psi_{\leftarrow} := (g \circ \Psi_i) \circ \pi_i^{\Lambda^{\succ}} \in \text{Lim}_{\leftarrow} F_i$.

(iii) By definition we have that $\Psi_{\leftarrow}(\Theta) = \lim_{\leftarrow} \mu_0(i) \Psi_{\leftarrow}(\Phi) \Leftrightarrow \forall i \in I (\Psi_i(\Theta_i) = \mu_0(i) \Psi_i(\Phi_i)) \Rightarrow \forall i \in I (\Theta_i = \lambda_0(i) \Phi_i) \Leftrightarrow \Theta = \lim_{\leftarrow} \lambda_0(i) \Phi$. \square

Proposition 10.4. *If $S(\Lambda^{\succ}) := (\lambda_0, \lambda_1^{\succ}, \phi_0^{\Lambda^{\succ}}, \phi_1^{\Lambda^{\succ}})$, $S(M^{\succ}) := (\mu_0, \mu_1^{\succ}, \phi_0^{M^{\succ}}, \phi_1^{M^{\succ}})$ and $S(N^{\succ}) := (\nu_0, \nu_1^{\succ}, \phi_0^{N^{\succ}}, \phi_1^{N^{\succ}})$ are in $\text{Spec}(I, \succ)$, and if $\Psi: S(\Lambda^{\succ}) \Rightarrow S(M^{\succ})$ and $\Xi: S(M^{\succ}) \Rightarrow S(N^{\succ})$, then $(\Xi \circ \Psi)_{\leftarrow} := \Xi_{\leftarrow} \circ \Psi_{\leftarrow}$*

$$\begin{array}{ccccc} & & (\Xi \circ \Psi)_i & & \\ & \searrow & \text{---} & \nearrow & \\ \lambda_0(i) & \xrightarrow{\Psi_i} & \mu_0(i) & \xrightarrow{\Xi_i} & \nu_0(i) \\ \pi_i^{\Lambda^{\succ}} \uparrow & & \pi_i^{M^{\succ}} \uparrow & & \pi_i^{N^{\succ}} \uparrow \\ \text{Lim}_{\leftarrow} \lambda_0(i) & \xrightarrow{\Psi_{\leftarrow}} & \text{Lim}_{\leftarrow} \mu_0(i) & \xrightarrow{\Xi_{\leftarrow}} & \text{Lim}_{\leftarrow} \nu_0(i) \\ & \searrow & \text{---} & \nearrow & \\ & & (\Xi \circ \Psi)_{\leftarrow} & & \end{array}$$

Theorem 10.5. *Let $S(\Lambda^{\succ}) := (\lambda_0, \lambda_1^{\succ}, \phi_0^{\Lambda^{\succ}}, \phi_1^{\Lambda^{\succ}})$ in $\text{Spec}(I, \succ)$, (J, e, cof_J) a cofinal subset of I , and $S(\Lambda^{\succ}) \circ e := (\lambda_0 \circ e, \lambda_1 \circ e, \phi_0^{\Lambda^{\succ} \circ e}, \phi_1^{\Lambda^{\succ} \circ e})$ the relative spectrum of $S(\Lambda^{\succ})$ to J . Then $\text{Lim}_{\leftarrow} \mathcal{F}_j \simeq \text{Lim}_{\leftarrow} \mathcal{F}_i$.*

Proof. If $\Theta \in \prod_{j \in J}^{\succ} \lambda_0(j)$, then, if $j \preccurlyeq j'$, we have that $\Theta_j = \lambda_{j'j}^{\succ}(\Theta_{j'}) := \lambda_{e(j')e(j)}^{\succ}(\Theta_{j'})$. If $i \in I$, then $\text{cof}_J(i) \in J$ and $\Theta_{\text{cof}_J(i)} \in \lambda_0(e(\text{cof}_J(i)))$. Since $i \preccurlyeq e(\text{cof}_J(i))$, we define the

operation $\phi : \text{Lim}_{\leftarrow} \lambda_0(j) \rightsquigarrow \text{Lim}_{\leftarrow} \lambda_0(i)$, by the rule $\Theta \mapsto \phi(\Theta)$, for every $\Theta \in \text{Lim}_{\leftarrow} \lambda_0(j)$, where

$$[\phi(\Theta)]_i := \lambda_{e(\text{cof}_J(i))i}^{\succ}(\Theta_{\text{cof}_J(i)}) \in \lambda_0(i); \quad i \in I.$$

First we show that ϕ is well-defined i.e., $\phi(\Theta) \in \prod_{i \in I}^{\succ} \lambda_0(i)$ i.e., for every $i, i' \in I$, $i \preceq i' \Rightarrow [\phi(\Theta)]_i = \lambda_{i'i}^{\succ}([\phi(\Theta)]_{i'})$. Working as in the proof of Lemma 9.11(ii), we get

$$\begin{aligned} \lambda_{i'i}^{\succ}([\phi(\Theta)]_{i'}) &:= \lambda_{i'i}^{\succ}(\lambda_{e(\text{cof}_J(i'))i'}^{\succ}(\Theta_{\text{cof}_J(i')})) \\ &=: [\lambda_{i'i}^{\succ} \circ \lambda_{e(\text{cof}_J(i'))i'}^{\succ}](\Theta_{\text{cof}_J(i')}) \\ &\stackrel{(9.1)}{=} \lambda_{e(\text{cof}_J(i'))i}^{\succ}(\Theta_{\text{cof}_J(i')}) \\ &\stackrel{(9.2)}{=} [\lambda_{e(\text{cof}_J(i))i}^{\succ} \circ \lambda_{e(\text{cof}_J(i'))e(\text{cof}_J(i))}^{\succ}](\Theta_{\text{cof}_J(i')}) \\ &:= \lambda_{e(\text{cof}_J(i))i}^{\succ}(\lambda_{e(\text{cof}_J(i'))e(\text{cof}_J(i))}^{\succ}(\Theta_{\text{cof}_J(i')})) \\ &= \lambda_{e(\text{cof}_J(i))i}^{\succ}(\Theta_{\text{cof}_J(i)}) \\ &:= [\phi(\Theta)]_i. \end{aligned}$$

To show that ϕ is a function we consider the following equivalences:

$$\begin{aligned} \phi(\Theta) =_{\text{Lim}_{\leftarrow} \lambda_0(i)} \phi(H) &:\Leftrightarrow \forall i \in I ([\phi(\Theta)]_i =_{\lambda_0(i)} [\phi(H)]_i) \\ &:\Leftrightarrow \forall i \in I (\lambda_{e(\text{cof}_J(i))i}^{\succ}(\Theta_{\text{cof}_J(i)}) =_{\lambda_0(i)} \lambda_{e(\text{cof}_J(i))i}^{\succ}(H_{\text{cof}_J(i)})), \quad (1) \end{aligned}$$

$$\Theta =_{\text{Lim}_{\leftarrow} \lambda_0(j)} H :\Leftrightarrow \forall j \in J (\Theta_j =_{\lambda_0(e(j))} H_j) \quad (2).$$

To show that (1) \Rightarrow (2) we use the fact that $e(\text{cof}_J(j)) = j$, and since $j \preceq j$, by the extensionality of \preceq we get $j \preceq e(\text{cof}_J(j))$. Since $\Theta_j = \lambda_{e(\text{cof}_J(j))j}^{\succ}(\Theta_{\text{cof}_J(j)})$, and $H_j = \lambda_{e(\text{cof}_J(j))j}^{\succ}(H_{\text{cof}_J(j)})$, we get (2). By the \vee -lifting of morphisms $\phi \in \text{Mor}(\text{Lim}_{\leftarrow} \mathcal{F}_j, \text{Lim}_{\leftarrow} \mathcal{F}_i) \Leftrightarrow \forall i \in I \forall f \in \mathcal{F}_i ((f \circ \pi_i^{S(\Lambda^{\succ})}) \circ \phi \in \text{Lim}_{\leftarrow} F_j)$. If $\Theta \in \prod_{j \in J}^{\succ} \lambda_0(j)$, we have that

$$\begin{aligned} [(f \circ \pi_i^{S(\Lambda^{\succ})}) \circ \phi](\Theta) &:= f(\pi_i^{S(\Lambda^{\succ})}(\phi(\Theta))) \\ &:= f([\phi(\Theta)]_i) \\ &:= f(\lambda_{e(\text{cof}_J(i))i}^{\succ}(\Theta_{\text{cof}_J(i)})) \\ &=: (f \circ \lambda_{e(\text{cof}_J(i))i}^{\succ})(\Theta_{\text{cof}_J(i)}) \\ &=: [(f \circ \lambda_{e(\text{cof}_J(i))i}^{\succ}) \circ \pi_{\text{cof}_J(i)}^{S(\Lambda^{\succ}) \circ e}](\Theta), \end{aligned}$$

hence $(f \circ \pi_i^{S(\Lambda^{\succ})}) \circ \phi := (f \circ \lambda_{e(\text{cof}_J(i))i}^{\succ}) \circ \pi_{\text{cof}_J(i)}^{S(\Lambda^{\succ}) \circ e} \in \text{Lim}_{\leftarrow} F_j$, as by definition $\lambda_{e(\text{cof}_J(i))i}^{\succ} \in \text{Mor}(\mathcal{F}_{e(\text{cof}_J(i))}, \mathcal{F}_i)$, and hence

$$\begin{array}{ccc} \lambda_0(e(\text{cof}_J(i))) & \xrightarrow{\lambda_{e(\text{cof}_J(i))i}^{\succ}} & \lambda_0(i) \\ & \searrow f \circ \lambda_{e(\text{cof}_J(i))i}^{\succ} & \downarrow f \\ & & \mathbb{R} \end{array}$$

$f \circ \lambda_{e(\text{cof}_J(i))}^{\succ} \in F_{e(\text{cof}_J(i))} := F_{\text{cof}_J(i)}$. Let the operation $\theta: \text{Lim}_{\leftarrow} \lambda_0(i) \rightsquigarrow \text{Lim}_{\leftarrow} \lambda_0(j)$, defined by the rule $H \mapsto \theta(H) := H^J$, for every $H \in \prod_{i \in I}^{\succ} \lambda_0(i)$, where $H_j^J := H_{e(j)} \in \lambda_0(e(j))$, for every $j \in J$. We show that $H^J \in \prod_{j \in J} \lambda_0(j)$. If $j \preccurlyeq j'$, then

$$H_j^J = \lambda_{e(j')e(j)}^{\succ}(H_{j'}^J) \Leftrightarrow H_{e(j)} = \lambda_{e(j')e(j)}^{\succ}(H_{e(j')}),$$

which holds by the hypothesis $H \in \prod_{i \in I}^{\succ} \lambda_0(i)$. Moreover, we have that $\phi(H^J) = H \Leftrightarrow \forall i \in I ([\phi(H^J)]_i =_{\lambda_0(i)} H_i)$. If $i \in I$, and since $i \preccurlyeq e(\text{cof}_J(i))$, we have that

$$[\phi(H^J)]_i := \lambda_{e(\text{cof}_J(i))i}^{\succ}(H_{\text{cof}_J(i)}^J) := \lambda_{e(\text{cof}_J(i))i}^{\succ}(H_{e(\text{cof}_J(i))}) = H_i.$$

It is immediate to show that θ is a function. Moreover, $\theta(\phi(\Theta)) = \Theta$, as if $j \in J$, then

$$\phi(\Theta)_j^J := \phi(\Theta)_{e(j)} := \lambda_{e(\text{cof}_J(e(j))e(j)}^{\succ}(\Theta_{\text{cof}_J(e(j))}) = \Theta_j,$$

as by hypothesis $\Theta_j = \lambda_{e(j')e(j)}^{\succ}(\Theta_{j'})$, with $j \preccurlyeq j'$, and by (Cof_1) we have that $j =_J (\text{cof}_J(e(j)))$, hence by the extensionality of \preccurlyeq we get $j \preccurlyeq (\text{cof}_J(e(j)))$. Finally, $\theta \in \text{Mor}(\text{Lim}_{\leftarrow} \mathcal{F}_i, \text{Lim}_{\leftarrow} \mathcal{F}_j) \Leftrightarrow \forall j \in J \forall f \in F_j ((f \circ \pi_j^{S(\Lambda^{\succ})}) \circ \theta \in \text{Lim}_{\leftarrow} F_i)$, which follows from

$$\begin{aligned} [(f \circ \pi_j^{S(\Lambda^{\succ})}) \circ \theta](H) &:= (f \circ \pi_j^{S(\Lambda^{\succ})})(H^J) \\ &:= f(H_j^J) \\ &:= f(H_{e(j)}) \\ &:= (f \circ \pi_{e(j)}^{S(\Lambda^{\succ})})(H). \end{aligned} \quad \square$$

Proposition 10.6. *If $(I, \preccurlyeq), (J, \preccurlyeq)$ are directed sets, $S(\Lambda^{\succ}) := (\lambda_0, \lambda_1^{\succ}, \phi_0^{\Lambda^{\succ}}, \phi_1^{\Lambda^{\succ}})$ is in $\text{Spec}(I, \succ)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{i'i}^{\succ})_{(i,i') \in D^{\preccurlyeq}(I)}$, and $S(M^{\succ}) := (\mu_0, \mu_1^{\succ}, \phi_0^{M^{\succ}}, \phi_1^{M^{\succ}})$ is in $\text{Spec}(J, \succ)$ with Bishop spaces $(\mathcal{G}_j)_{j \in J}$ and Bishop morphisms $(\mu_{j'j}^{\succ})_{(j,j') \in D^{\preccurlyeq}(J)}$, there is a function*

$$\times : \prod_{i \in I}^{\succ} \lambda_0(i) \times \prod_{j \in J}^{\succ} \mu_0(j) \rightarrow \prod_{(i,j) \in I \times J}^{\succ} \lambda_0(i) \times \mu_0(j) \in \text{Mor}(\text{Lim}_{\leftarrow} \mathcal{F}_i \times \text{Lim}_{\leftarrow} \mathcal{G}_j, \text{Lim}_{\leftarrow} (\mathcal{F}_i \times \mathcal{G}_j)).$$

Proof. We proceed as in the proof of Proposition 9.15. □

11. DUALITY BETWEEN DIRECT AND INVERSE LIMITS

Proposition 11.1. *Let $\mathcal{F} := (X, F), \mathcal{G} := (Y, G)$ and $\mathcal{H} := (Z, H)$ be Bishop spaces, and let $\lambda \in \text{Mor}(\mathcal{G}, \mathcal{H}), \mu \in \text{Mor}(\mathcal{H}, \mathcal{F})$. We define the mappings*

$$\begin{aligned} \lambda^+ : \text{Mor}(\mathcal{H}, \mathcal{F}) &\rightarrow \text{Mor}(\mathcal{G}, \mathcal{F}), & \lambda^+(\phi) &:= \phi \circ \lambda; & \phi &\in \text{Mor}(\mathcal{H}, \mathcal{F}), \\ \mu^- : \text{Mor}(\mathcal{F}, \mathcal{H}) &\rightarrow \text{Mor}(\mathcal{F}, \mathcal{G}), & \mu^-(\theta) &:= \mu \circ \theta; & \theta &\in \text{Mor}(\mathcal{F}, \mathcal{H}), \end{aligned}$$

$$\begin{array}{ccccc} Z & \xrightarrow{\phi} & X & \xrightarrow{\theta} & Z \\ \lambda \uparrow & & \nearrow \phi \circ \lambda & \mu \circ \theta \searrow & \downarrow \mu \\ Y & & & & Y. \end{array}$$

$$\begin{aligned} + &: \text{Mor}(\mathcal{G}, \mathcal{H}) \rightarrow \text{Mor}(\mathcal{H} \rightarrow \mathcal{F}, \mathcal{G} \rightarrow \mathcal{F}), & \lambda &\mapsto \lambda^+; & \lambda &\in \text{Mor}(\mathcal{G}, \mathcal{H}), \\ - &: \text{Mor}(\mathcal{H}, \mathcal{G}) \rightarrow \text{Mor}(\mathcal{F} \rightarrow \mathcal{H}, \mathcal{F} \rightarrow \mathcal{G}), & \mu &\mapsto \mu^-; & \mu &\in \text{Mor}(\mathcal{H}, \mathcal{G}). \end{aligned}$$

Then $^+ \in \text{Mor}(\mathcal{G} \rightarrow \mathcal{H}, (\mathcal{H} \rightarrow \mathcal{F}) \rightarrow (\mathcal{G} \rightarrow \mathcal{F}))$ and $^- \in \text{Mor}(\mathcal{H} \rightarrow \mathcal{G}, (\mathcal{F} \rightarrow \mathcal{H}) \rightarrow (\mathcal{F} \rightarrow \mathcal{G}))$.

Proof. By definition and the ∇ -lifting of the exponential topology we have that

$$\begin{aligned} \mathcal{G} \rightarrow \mathcal{H} &:= \left(\text{Mor}(\mathcal{G}, \mathcal{H}), \bigvee_{y \in Y}^{h \in H} \phi_{y,h} \right), & \mathcal{H} \rightarrow \mathcal{F} &:= \left(\text{Mor}(\mathcal{H}, \mathcal{F}), \bigvee_{z \in Z}^{f \in F} \phi_{z,f} \right), \\ \mathcal{G} \rightarrow \mathcal{F} &:= \left(\text{Mor}(\mathcal{G}, \mathcal{F}), \bigvee_{y \in Y}^{f \in F} \phi_{y,f} \right), \\ (\mathcal{H} \rightarrow \mathcal{F}) \rightarrow (\mathcal{G} \rightarrow \mathcal{F}) &:= \left(\text{Mor}((\mathcal{H} \rightarrow \mathcal{F}), (\mathcal{G} \rightarrow \mathcal{F})), \bigvee_{\varphi \in \text{Mor}(\mathcal{H}, \mathcal{F})}^{e \in G \rightarrow F} \phi_{\varphi,e} \right), \\ \bigvee_{\varphi \in \text{Mor}(\mathcal{H}, \mathcal{F})}^{e \in G \rightarrow F} \phi_{\varphi,e} &= \bigvee_{\varphi \in \text{Mor}(\mathcal{H}, \mathcal{F})}^{y \in Y, f \in F} \phi_{\varphi, \phi_{y,f}}. \end{aligned}$$

By the ∇ -lifting of morphisms we have that

$$^+ \in \text{Mor}(\mathcal{G} \rightarrow \mathcal{H}, (\mathcal{H} \rightarrow \mathcal{F}) \rightarrow (\mathcal{G} \rightarrow \mathcal{F})) \Leftrightarrow \forall_{\varphi \in \text{Mor}(\mathcal{H}, \mathcal{F})} \forall_{y \in Y} \forall_{f \in F} (\phi_{\varphi, \phi_{y,f}} \circ ^+ \in G \rightarrow H).$$

If $\lambda \in \text{Mor}(\mathcal{G}, \mathcal{H})$, we have that $[\phi_{\varphi, \phi_{y,f}} \circ ^+](\lambda) := \phi_{\varphi, \phi_{y,f}}(\lambda^+) := (\phi_{y,f} \circ \lambda^+)(\varphi) := (\phi_{y,f}(\varphi \circ \lambda)) := [f \circ (\varphi \circ \lambda)](y) := [(f \circ \varphi) \circ \lambda](y) := \phi_{y, f \circ \varphi}(\lambda)$ i.e., $\phi_{\varphi, \phi_{y,f}} \circ ^+ := \phi_{y, f \circ \varphi} \in G \rightarrow H$, since $\varphi \in \text{Mor}(\mathcal{H}, \mathcal{F})$ and hence $f \circ \varphi \in H$. For the mapping $^-$ we work similarly. \square

With the use of the exponential Bishop topology we can get a contravariant spectrum from a covariant one, and vice versa.

Proposition 11.2. (A) Let $S(\Lambda^{\Leftarrow}) := (\lambda_0, \lambda_1^{\Leftarrow}, \phi_0^{\Lambda^{\Leftarrow}}, \phi_1^{\Lambda^{\Leftarrow}}) \in \text{Spec}(I, \Leftarrow)$ and $\mathcal{F} := (X, F)$ a Bishop space.

(i) If $S(\Lambda^{\Leftarrow}) \rightarrow \mathcal{F} := (\mu_0, \mu_1^{\Leftarrow}, \phi_0^{M^{\Leftarrow}}, \phi_1^{M^{\Leftarrow}})$, where $M^{\Leftarrow} := (\mu_0, \mu_1^{\Leftarrow})$ is a contravariant direct family of sets over (I, \Leftarrow) with $\mu_0(i) := \text{Mor}(\mathcal{F}_i, \mathcal{F})$ and

$$\mu_1^{\Leftarrow}(i, j) := (\lambda_{ij}^{\Leftarrow})^+ : \text{Mor}(\mathcal{F}_j, \mathcal{F}) \rightarrow \text{Mor}(\mathcal{F}_i, \mathcal{F}),$$

and if $\phi_0^{M^{\Leftarrow}}(i) := F_i \rightarrow F$ and $\phi_1^{M^{\Leftarrow}}(i, j) := (F_i \rightarrow F, F_j \rightarrow F, [(\lambda_{ij}^{\Leftarrow})^+])^*$, then $S^{\Leftarrow} \rightarrow \mathcal{F}$ is a contravariant (I, \Leftarrow) -spectrum with Bishop spaces $(\mathcal{F}_i \rightarrow \mathcal{F})_{i \in I}$ and Bishop morphisms $((\lambda_{ij}^{\Leftarrow})^+)_{(i,j) \in D^{\Leftarrow}(I)}$.

(ii) If $\mathcal{F} \rightarrow S(N^{\Leftarrow}) := (\nu_0, \nu_1^{\Leftarrow}, \phi_0^{N^{\Leftarrow}}, \phi_1^{N^{\Leftarrow}})$, where $N^{\Leftarrow} := (\nu_0, \nu_1^{\Leftarrow})$ is a direct family of sets over (I, \Leftarrow) with $\nu_0(i) := \text{Mor}(\mathcal{F}, \mathcal{F}_i)$ and

$$\nu_1^{\Leftarrow}(i, j) := (\lambda_{ij}^{\Leftarrow})^- : \text{Mor}(\mathcal{F}, \mathcal{F}_i) \rightarrow \text{Mor}(\mathcal{F}, \mathcal{F}_j),$$

and if $\phi_0^{N^{\Leftarrow}}(i) := F \rightarrow F_i$ and $\phi_1^{N^{\Leftarrow}}(i, j) := (F \rightarrow F_j, F \rightarrow F_i, [(\lambda_{ij}^{\Leftarrow})^-])^*$, then $\mathcal{F} \rightarrow S^{\Leftarrow}$ is a covariant (I, \Leftarrow) -spectrum with Bishop spaces $(\mathcal{F} \rightarrow \mathcal{F}_i)_{i \in I}$ and Bishop morphisms $((\lambda_{ij}^{\Leftarrow})^-)_{(i,j) \in D^{\Leftarrow}(I)}$.

(B) Let $S(\Lambda^{\Leftarrow}) := (\lambda_0, \lambda_1^{\Leftarrow}, \phi_0^{\Lambda^{\Leftarrow}}, \phi_1^{\Lambda^{\Leftarrow}})$ be a contravariant (I, \Leftarrow) -spectrum, and $\mathcal{F} := (X, F)$ a Bishop space.

(i) If $S(\Lambda^{\Leftarrow}) \rightarrow \mathcal{F} := (\mu_0, \mu_1^{\Leftarrow}, \phi_0^{M^{\Leftarrow}}, \phi_1^{M^{\Leftarrow}})$, where $M^{\Leftarrow} := (\mu_0, \mu_1^{\Leftarrow})$ is a direct family of sets over (I, \Leftarrow) with $\mu_0(i) := \text{Mor}(\mathcal{F}_i, \mathcal{F})$ and

$$\mu_1^{\Leftarrow}(i, j) := (\lambda_{ji}^{\Leftarrow})^+ : \text{Mor}(\mathcal{F}_i, \mathcal{F}) \rightarrow \text{Mor}(\mathcal{F}_j, \mathcal{F}),$$

and if $\phi_0^{M^{\Leftarrow}}(i) := F_i \rightarrow F$ and $\phi_1^{M^{\Leftarrow}}(i, j) := (F_j \rightarrow F, F_i \rightarrow F, [(\lambda_{ji}^{\Leftarrow})^+]^*)$, then $S^{\Leftarrow} \rightarrow \mathcal{F}$ is an (I, \Leftarrow) -spectrum with Bishop spaces $(\mathcal{F}_i \rightarrow \mathcal{F})_{i \in I}$ and Bishop morphisms $((\lambda_{ji}^{\Leftarrow})^+)_{(i,j) \in D^{\Leftarrow}(I)}$.

(ii) If $\mathcal{F} \rightarrow S(N^{\Leftarrow}) := (\nu_0, \nu_1^{\Leftarrow}, \phi_0^{N^{\Leftarrow}}, \phi_1^{N^{\Leftarrow}})$, where $N^{\Leftarrow} := (\nu_0, \nu_1^{\Leftarrow})$ is a contravariant direct family of sets over (I, \Leftarrow) with $\nu_0(i) := \text{Mor}(\mathcal{F}, \mathcal{F}_i)$ and

$$\nu_1^{\Leftarrow}(i, j) := (\lambda_{ji}^{\Leftarrow})^- : \text{Mor}(\mathcal{F}, \mathcal{F}_j) \rightarrow \text{Mor}(\mathcal{F}, \mathcal{F}_i),$$

and if $\phi_0^{N^{\Leftarrow}}(i) := F \rightarrow F_i$ and $\phi_1^{N^{\Leftarrow}}(i, j) := (F \rightarrow F_i, F \rightarrow F_j, [(\lambda_{ji}^{\Leftarrow})^-]^*)$, then $\mathcal{F} \rightarrow S^{\Leftarrow}$ is a contravariant (I, \Leftarrow) -spectrum with Bishop spaces $(\mathcal{F} \rightarrow \mathcal{F}_i)_{i \in I}$ and Bishop morphisms $((\lambda_{ij}^{\Leftarrow})^-)_{(i,j) \in D^{\Leftarrow}(I)}$.

Proof. We prove only the case (A)(i) and for the other cases we work similarly. It suffices to show that if $i \Leftarrow j \Leftarrow k$, then the following diagram commutes

$$\begin{array}{ccc} \text{Mor}(\mathcal{F}_i, \mathcal{F}) & & \\ (\lambda_{ij}^{\Leftarrow})^+ \uparrow & \swarrow (\lambda_{ik}^{\Leftarrow})^+ & \\ \text{Mor}(F_j, \mathcal{F}) & \xleftarrow{(\lambda_{jk}^{\Leftarrow})^+} & \text{Mor}(\mathcal{F}_k, \mathcal{F}). \end{array}$$

If $\phi \in \text{Mor}(\mathcal{F}_k, \mathcal{F})$, then $(\lambda_{ij}^{\Leftarrow})^+ [(\lambda_{jk}^{\Leftarrow})^+(\phi)] := (\lambda_{ij}^{\Leftarrow})^+ [\phi \circ \lambda_{jk}^{\Leftarrow}] := (\phi \circ \lambda_{jk}^{\Leftarrow}) \circ \lambda_{ij}^{\Leftarrow} := \phi \circ (\lambda_{jk}^{\Leftarrow} \circ \lambda_{ij}^{\Leftarrow}) = \phi \circ \lambda_{ik}^{\Leftarrow} := (\lambda_{ik}^{\Leftarrow})^+(\phi)$. \square

Similarly to the \vee -lifting of the product topology, if $S(\Lambda^{\Leftarrow}) := (\lambda_0, \lambda_1^{\Leftarrow}, \phi_0^{\Lambda^{\Leftarrow}}, \phi_1^{\Lambda^{\Leftarrow}})$ is a contravariant direct spectrum over (I, \Leftarrow) with Bishop spaces $(\mathcal{F}_i = (\lambda_0(i), \vee F_{0i}))_{i \in I}$, then

$$\prod_{i \in I}^{\Leftarrow} F_i = \bigvee_{i \in I}^{f \in F_{0i}} (f \circ \pi_i^{\Lambda^{\Leftarrow}}).$$

Theorem 11.3 (Duality principle). *Let $S(\Lambda^{\Leftarrow}) := (\lambda_0, \lambda_1^{\Leftarrow}, \phi_0^{\Lambda^{\Leftarrow}}, \phi_1^{\Lambda^{\Leftarrow}}) \in \text{Spec}(I, \Leftarrow)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\Leftarrow})_{(i,j) \in D^{\Leftarrow}(I)}$. If $\mathcal{F} := (X, F)$ is a Bishop space and $S(\Lambda^{\Leftarrow}) \rightarrow \mathcal{F} := (\mu_0, \mu_1^{\Leftarrow}, \phi_0^{M^{\Leftarrow}}, \phi_1^{M^{\Leftarrow}})$ is the contravariant direct spectrum over (I, \Leftarrow) defined in Proposition 11.2 (A)(i), then*

$$\lim_{\leftarrow} (\mathcal{F}_i \rightarrow \mathcal{F}) \simeq [(\lim_{\rightarrow} \mathcal{F}_i) \rightarrow \mathcal{F}].$$

Proof. First we determine the topologies involved in the required Bishop isomorphism. By definition and by the above remark on the \vee -lifting of the \prod^{\Leftarrow} -topology we have that

$$\lim_{\leftarrow} (\mathcal{F}_i \rightarrow \mathcal{F}) := \left(\prod_{i \in I}^{\Leftarrow} \mu_0(i), \bigvee_{i \in I}^{g \in F_i \rightarrow F} g \circ \pi_i^{S(\Lambda^{\Leftarrow}) \rightarrow \mathcal{F}} \right),$$

$$\begin{aligned}
F_i \rightarrow F &:= \bigvee_{x \in \lambda_0(i)}^{f \in F} \phi_{x,f}, \\
\bigvee_{i \in I}^{g \in F_i \rightarrow F} g \circ \pi_i^{S \Leftarrow} \rightarrow \mathcal{F} &= \bigvee_{i \in I}^{x \in \lambda_0(i), f \in F} \phi_{x,f} \circ \pi_i^{S(\Lambda \Leftarrow) \rightarrow \mathcal{F}}, \\
\text{Lim}_{\rightarrow} \mathcal{F}_i &:= \left(\text{Lim}_{\rightarrow} \lambda_0(i), \bigvee_{\Theta \in \prod_{i \in I}^{\Leftarrow} F_i} \text{eq1}_0 f_{\Theta} \right), \\
(\text{Lim}_{\rightarrow} \mathcal{F}_i) \rightarrow \mathcal{F} &:= \left(\text{Mor}(\text{Lim}_{\rightarrow} \mathcal{F}_i, \mathcal{F}), \bigvee_{\text{eq1}_0^{\Lambda \Leftarrow}(i,x) \in \text{Lim}_{\rightarrow} \lambda_0(i)}^{f \in F} \phi_{\text{eq1}_0^{\Lambda \Leftarrow}(i,x),f} \right), \\
\phi_{\text{eq1}_0^{\Lambda \Leftarrow}(i,x),f} &:= (f \circ h)(\text{eq1}_0^{\Lambda \Leftarrow}(i,x)) \\
\text{Lim}_{\rightarrow} \lambda_0(i) &\xrightarrow{h \in \text{Mor}(\text{Lim}_{\rightarrow} \mathcal{F}_i, \mathcal{F})} X \\
&\searrow f \circ h \quad \downarrow f \in F \\
&\quad \mathbb{R}.
\end{aligned}$$

If $H \in \prod_{i \in I}^{\Leftarrow} \text{Mor}(\mathcal{F}_i, \mathcal{F})$, let the operation $\theta(H) : \text{Lim}_{\rightarrow} \lambda_0(i) \rightsquigarrow X$, defined by

$$\theta(H)(\text{eq1}_0^{\Lambda \Leftarrow}(i,x)) := H_i(x); \quad \text{eq1}_0^{\Lambda \Leftarrow}(i,x) \in \text{Lim}_{\rightarrow} \lambda_0(i).$$

We show that $\theta(H)$ is a function. If

$$\text{eq1}_0^{\Lambda \Leftarrow}(i,x) =_{\text{Lim}_{\rightarrow} \lambda_0(i)} \text{eq1}_0^{\Lambda \Leftarrow}(j,y) \Leftrightarrow \exists k \in I (i, j \Leftarrow k \ \& \ \lambda_{ik}^{\Leftarrow}(x) =_{\lambda_0(k)} \lambda_{jk}^{\Leftarrow}(y)),$$

we show that $\theta(H)(\text{eq1}_0^{\Lambda \Leftarrow}(i,x)) := H_i(x) =_X H_j(y) := \theta(H)(\text{eq1}_0^{\Lambda \Leftarrow}(j,y))$. By the equalities $H_i = (\lambda_{ik}^{\Leftarrow})^+(H_k) = H_k \circ \lambda_{ik}^{\Leftarrow}$ and $H_j = (\lambda_{jk}^{\Leftarrow})^+(H_k) = H_k \circ \lambda_{jk}^{\Leftarrow}$ we get

$$H_i(x) = (H_k \circ \lambda_{ik}^{\Leftarrow})(x) := H_k(\lambda_{ik}^{\Leftarrow}(x)) =_X H_k(\lambda_{jk}^{\Leftarrow}(y)) := (H_k \circ \lambda_{jk}^{\Leftarrow})(y) := H_j(y).$$

Next we show that $\theta(H) \in \text{Mor}(\text{Lim}_{\rightarrow} \mathcal{F}_i, \mathcal{F}) \Leftrightarrow \forall f \in F (f \circ \theta(H) \in \text{Lim}_{\rightarrow} F_i)$. If $f \in F$, then the dependent operation $\Theta : \bigwedge_{i \in I} F_i$, defined by $\Theta_i := f \circ H_i$, for every $i \in I$

$$\begin{array}{ccc}
\lambda_0(i) & \xrightarrow{H_i \in \text{Mor}(\mathcal{F}_i, \mathcal{F})} & X \\
& \searrow f \circ H_i & \downarrow f \in F \\
& & \mathbb{R}
\end{array}$$

is in $\prod_{i \in I}^{\Leftarrow} F_i$ i.e., if $i \Leftarrow j$, then $\Theta_i = (\lambda_{ij}^{\Leftarrow})^*(\Theta_j) = \Theta_j \circ \lambda_{ij}^{\Leftarrow}$, since $\Theta_i := f \circ H_i = f \circ (H_j \circ \lambda_{ij}^{\Leftarrow}) = (f \circ H_j) \circ \lambda_{ij}^{\Leftarrow} := \Theta_j \circ \lambda_{ij}^{\Leftarrow}$. Hence $f \circ \theta(H) := \text{eq1}_0 f_{\Theta} \in \text{Lim}_{\rightarrow} F_i$, since

$$[f \circ \theta(H)](\text{eq1}_0^{\Lambda \Leftarrow}(i,x)) := f(H_i(x)) := (f \circ H_i)(x) := f_{\Theta}(i,x) := \text{eq1}_0 f_{\Theta}(\text{eq1}_0^{\Lambda \Leftarrow}(i,x)).$$

Consequently, the operation $\theta: \prod_{i \in I}^{\rightrightarrows} \text{Mor}(\mathcal{F}_i, \mathcal{F}) \rightsquigarrow \text{Mor}(\lim_{\rightarrow} \mathcal{F}_i, \mathcal{F})$, defined by the rule $H \mapsto \theta(H)$, is well-defined. Next we show that θ is an embedding.

$$\begin{aligned} \theta(H) = \theta(K) &:\Leftrightarrow \forall_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x) \in \lim_{\rightarrow} \lambda_0(i)} (\theta(H)(\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x)) = \theta(K)(\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x))) \\ &:\Leftrightarrow \forall_{i \in I} (H_i(x) =_X K_i(x)) \\ &:\Leftrightarrow H = K. \end{aligned}$$

Next we show that $\theta \in \text{Mor}(\lim_{\leftarrow} (\mathcal{F}_i \rightarrow \mathcal{F}), (\lim_{\rightarrow} \mathcal{F}_i) \rightarrow \mathcal{F})$ i.e.,

$$\forall_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x) \in \lim_{\rightarrow} \lambda_0(i)} \forall_{f \in F} \left(\phi_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x),f} \circ \theta \in \bigvee_{i \in I, x \in \lambda_0(i)}^{f \in F} \phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \rightarrow F} \right).$$

By the equalities

$$\begin{aligned} [\phi_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x),f} \circ \theta](H) &:= \phi_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x),f}(\theta(H)) := [f \circ \theta(H)](\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x)) := f(H_i(x)), \\ [\phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \rightarrow F}](H) &:= \phi_{x,f}(H_i) := f(H_i(x)), \end{aligned}$$

we get $\phi_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x),f} \circ \theta = \phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \rightarrow F}$. Let $\phi: \text{Mor}(\lim_{\rightarrow} \mathcal{F}_i, \mathcal{F}) \rightsquigarrow \prod_{i \in I}^{\rightrightarrows} \text{Mor}(\mathcal{F}_i, \mathcal{F})$ be defined by $h \mapsto \phi(h) := H^h$, where $H^h: \prod_{i \in I} \text{Mor}(\mathcal{F}_i, \mathcal{F})$ is defined by $H_i^h := h \circ \text{eq}1_i$, for every $i \in I$

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\text{eq}1_i} & \lim_{\rightarrow} \lambda_0(i) \\ & \searrow^{H_i^h} & \downarrow h \\ & & X. \end{array}$$

By Proposition 9.5(i) $H_i \in \text{Mor}(\mathcal{F}_i, \mathcal{F})$, as a composition of Bishop morphisms. To show that $H^h \in \prod_{i \in I} \text{Mor}(\mathcal{F}_i, \mathcal{F})$, let $i \preccurlyeq j$, and by Proposition 9.5(ii) we get $H_i^h := h \circ \text{eq}1_i = h \circ (\text{eq}1_j \circ \lambda_{ij}^{\preccurlyeq}) := (h \circ \text{eq}1_j) \circ \lambda_{ij}^{\preccurlyeq} := H_j \circ \lambda_{ij}^{\preccurlyeq}$. Moreover, $\theta(H_h) := h$, since $\theta(H_h)(\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x)) := H_i(x) := (h \circ \text{eq}1_i)(x) := h(\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x))$. Clearly, ϕ is a function. Moreover $H^{\theta(H)} := H$, as, for every $i \in I$ we have that $(H_i^{\theta(H)})(x) := (\theta(H) \circ \text{eq}1_i)(x) := \theta(H)(\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x)) := H_i(x)$. Finally we show that $\phi \in \text{Mor}((\lim_{\rightarrow} \mathcal{F}_i) \rightarrow \mathcal{F}, \lim_{\leftarrow} (\mathcal{F}_i \rightarrow \mathcal{F}))$ if and only if

$$\forall_{i \in I} \forall_{x \in \lambda_0(i)} \forall_{f \in F} \left(\phi_{x,f} \circ \phi \in \bigvee_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x) \in \lim_{\rightarrow} \lambda_0(i)}^{f \in F} \phi_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x),f} \right).$$

If $h \in \text{Mor}(\lim_{\rightarrow} \mathcal{F}_i, \mathcal{F})$, then

$$\begin{aligned} &[\phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \rightarrow F} \circ \phi](h) := (\phi_{x,f} \circ \pi_i^{S(\Lambda^{\preccurlyeq}) \rightarrow F})(H^h) := \phi_{x,f}(H_i^h) \\ &=: \phi_{x,f}(h \circ \text{eq}1_i) := f[(h \circ \text{eq}1_i)(x)] := (f \circ h)(\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x)) := \phi_{\text{eq}1_0^{\Lambda^{\preccurlyeq}}(i,x),f}(h). \quad \square \end{aligned}$$

With respect to the possible dual to the previous theorem i.e., the isomorphism $\lim_{\rightarrow} (\mathcal{F}_i \rightarrow \mathcal{F}) \simeq [(\lim_{\leftarrow} \mathcal{F}_i) \rightarrow \mathcal{F}]$, what we can show is the following proposition.

Proposition 11.4. *Let $S(\Lambda^{\succ}) := (\lambda_0, \lambda_1^{\succ}, \phi_0^{\Lambda^{\succ}}, \phi_1^{\Lambda^{\succ}})$ be in $\mathbf{Spec}(I, \succ)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\succ})_{(i,j) \in D^{\succ}(I)}$. If $\mathcal{F} := (X, F)$ is a Bishop space and $S(\Lambda^{\succ}) \rightarrow \mathcal{F} := (\mu_0, \mu_1^{\succ}, \phi_0^{M^{\succ}}, \phi_1^{M^{\succ}})$ is the (I, \succ) -directed spectrum defined in Proposition 11.2 (B)(i), there is a function $\hat{\cdot} : \mathop{\text{Lim}}_{\rightarrow}[\text{Mor}(\mathcal{F}_i, \mathcal{F})] \rightarrow \text{Mor}(\mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i, \mathcal{F})$ such that the following hold:*

(i) $\hat{\cdot} \in \text{Mor}(\mathop{\text{Lim}}_{\rightarrow}(\mathcal{F}_i \rightarrow \mathcal{F}), (\mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i) \rightarrow \mathcal{F})$.

(ii) *If for every $j \in J$ and every $y \in \lambda_0(j)$ there is $\Theta_y \in \prod_{i \in I}^{\succ} \lambda_0(i)$ such that $\Theta_y(j) =_{\lambda_0(j)} y$, then $\hat{\cdot}$ is an embedding of $\mathop{\text{Lim}}_{\rightarrow}[\text{Mor}(\mathcal{F}_i, \mathcal{F})]$ into $\text{Mor}(\mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i, \mathcal{F})$.*

Proof. We proceed similarly to the proof of Theorem 11.3. \square

Theorem 11.5. *Let $S(\Lambda^{\succ}) := (\lambda_0, \lambda_1^{\succ}, \phi_0^{\Lambda^{\succ}}, \phi_1^{\Lambda^{\succ}})$ be in $\mathbf{Spec}(I, \succ)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\succ})_{(i,j) \in D^{\succ}(I)}$. If $\mathcal{F} := (X, F)$ is a Bishop space and $\mathcal{F} \rightarrow S(\Lambda^{\succ}) := (\nu_0, \nu_1^{\succ}, \phi_0^{\succ}, \phi_1^{N^{\succ}})$ is the contravariant direct spectrum over (I, \succ) , defined in Proposition 11.2 (B)(ii), then*

$$\mathop{\text{Lim}}_{\leftarrow}(\mathcal{F} \rightarrow \mathcal{F}_i) \simeq [\mathcal{F} \rightarrow \mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i].$$

Proof. First we determine the topologies involved in the required Bishop isomorphism:

$$\begin{aligned} \mathop{\text{Lim}}_{\leftarrow}(\mathcal{F} \rightarrow \mathcal{F}_i) &:= \left(\prod_{i \in I}^{\succ} \text{Mor}(\mathcal{F}, \mathcal{F}_i), \bigvee_{i \in I}^{g \in \mathcal{F} \rightarrow \mathcal{F}_i} g \circ \pi_i^{\mathcal{F} \rightarrow S(\Lambda^{\succ})} \right), \\ \bigvee_{i \in I}^{g \in \mathcal{F} \rightarrow \mathcal{F}_i} g \circ \pi_i^{\mathcal{F} \rightarrow S^{\succ}} &= \bigvee_{i \in I, x \in \lambda_0(i)}^{f \in \mathcal{F}_i} \phi_{x,f} \circ \pi_i^{\mathcal{F} \rightarrow S(\Lambda^{\succ})}, \\ \mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i &:= \left(\prod_{i \in I}^{\succ} \lambda_0(i), \bigvee_{i \in I}^{f \in \mathcal{F}_i} f \circ \pi_i^{S(\Lambda^{\succ})} \right), \\ \mathcal{F} \rightarrow \mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i &:= \left(\text{Mor}(\mathcal{F}, \mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i), \bigvee_{x \in X}^{g \in \mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i} \phi_{x,g} \right), \\ \bigvee_{x \in X}^{g \in \mathop{\text{Lim}}_{\leftarrow} \mathcal{F}_i} \phi_{x,g} &= \bigvee_{x \in X, i \in I}^{f \in \mathcal{F}_i} \phi_{x,f \circ \pi_i^{S(\Lambda^{\succ})}}. \end{aligned}$$

If $H \in \prod_{i \in I}^{\succ} \text{Mor}(\mathcal{F}, \mathcal{F}_i)$, and if $i \preccurlyeq j$, then $H_i = \nu_{ji}^{\succ}(H_j) = (\lambda_{ji}^{\succ})^-(H_j) = \lambda_{ji}^{\succ} \circ H_j$

$$\begin{array}{ccc} X & \xrightarrow{H_j} & \lambda_0(j) \\ & \searrow H_i & \downarrow \lambda_{ji}^{\succ} \\ & & \lambda_0(i). \end{array}$$

Let the operation $e(H) : X \rightsquigarrow \prod_{i \in I}^{\succ} \lambda_0(i)$, defined by $x \mapsto [e(H)](x)$, where $[[e(H)](x)]_i := H_i(x)$, for every $i \in I$. First we show that $[e(H)](x) \in \prod_{i \in I}^{\succ} \lambda_0(i)$. If $i \preccurlyeq j$, then $[[e(H)](x)]_i := H_i(x) = (\lambda_{ji}^{\succ} \circ H_j)(x) := \lambda_{ji}^{\succ}(H_j(x)) := \lambda_{ji}^{\succ}([[e(H)](x)]_j)$. Next we show that

$e(H)$ is a function. If $x =_X x'$, then $\forall_{i \in I} (H_i(x) =_{\lambda_0(i)} H_i(x')) \Leftrightarrow \forall_{i \in I} ([e(H)](x)]_i =_{\lambda_0(i)} [[e(H)](x')]]_i \Leftrightarrow [e(H)](x) =_{\prod_{i \in I}^{\approx} \lambda_0(i)} [e(H)](x')$. By the \forall -lifting of morphisms $e(H) \in \text{Mor}(\mathcal{F}, \lim_{\leftarrow} \mathcal{F}_i) \Leftrightarrow \forall_{i \in I} \forall_{f \in F_i} ((f \circ \pi_i^{S(\Lambda^\approx)}) \circ e(H) \in F)$. Since $[(f \circ \pi_i^{S(\Lambda^\approx)}) \circ e(H)](x) := (f \circ \pi_i^{S(\Lambda^\approx)})([e(H)](x)) := f(H_i(x)) := (f \circ H_i)(x)$, we get $(f \circ \pi_i^{S(\Lambda^\approx)}) \circ e(H) := f \circ H_i \in F$, since $f \in F_i$ and $H_i \in \text{Mor}(\mathcal{F}, \mathcal{F}_i)$. Hence, the operation $e : \prod_{i \in I}^{\approx} \text{Mor}(\mathcal{F}, \mathcal{F}_i) \rightsquigarrow \text{Mor}(\mathcal{F}, \lim_{\leftarrow} \mathcal{F}_i)$, defined by the rule $H \mapsto e(H)$, is well-defined. Next we show that e is an embedding. If $H, K \in \prod_{i \in I}^{\approx} \text{Mor}(\mathcal{F}, \mathcal{F}_i)$, then

$$\begin{aligned} e(H) = e(K) &\Leftrightarrow \forall_{x \in X} ([e(H)](x) =_{\prod_{i \in I}^{\approx} \lambda_0(i)} [e(K)](x)) \\ &\Leftrightarrow \forall_{x \in X} \forall_{i \in I} (H_i(x) =_{\lambda_0(i)} K_i(x)) \\ &\Leftrightarrow \forall_{i \in I} \forall_{x \in X} (H_i(x) =_{\lambda_0(i)} K_i(x)) \\ &\Leftrightarrow \forall_{i \in I} (H_i =_{\text{Mor}(\mathcal{F}, \mathcal{F}_i)} K_i) \\ &\Leftrightarrow H =_{\prod_{i \in I}^{\approx} \text{Mor}(\mathcal{F}, \mathcal{F}_i)} K. \end{aligned}$$

By the \forall -lifting of morphisms we show that

$$e \in \text{Mor}(\lim_{\leftarrow} (\mathcal{F} \rightarrow \mathcal{F}_i), \mathcal{F} \rightarrow \lim_{\leftarrow} \mathcal{F}_i) \Leftrightarrow \forall_{i \in I} \forall_{f \in F_i} (\phi_{x, f \circ \pi_i^{S(\Lambda^\approx)}} \circ e \in \lim_{\leftarrow} (F \rightarrow F_i))$$

$$\begin{aligned} (\phi_{x, f \circ \pi_i^{S(\Lambda^\approx)}} \circ e)(H) &:= \phi_{x, f \circ \pi_i^{S(\Lambda^\approx)}}(e(H)) \\ &=: [(f \circ \pi_i^{S(\Lambda^\approx)}) \circ e(H)](x) \\ &=: (f \circ \pi_i^{S(\Lambda^\approx)})([e(H)](x)) \\ &=: f([e(H)](x)]_i) \\ &=: f(H_i(x)) \\ &=: (f \circ H_i)(x) \\ &=: \phi_{x, f}(H_i) \\ &=: [\phi_{x, f} \circ \pi_i^{\mathcal{F} \rightarrow S(\Lambda^\approx)}](H) \end{aligned}$$

we get $\phi_{x, f \circ \pi_i^{S(\Lambda^\approx)}} \circ e := \phi_{x, f} \circ \pi_i^{\mathcal{F} \rightarrow S(\Lambda^\approx)} \in \lim_{\leftarrow} (F \rightarrow F_i)$. Let $\phi : \text{Mor}(\mathcal{F}, \lim_{\leftarrow} \mathcal{F}_i) \rightsquigarrow \prod_{i \in I}^{\approx} \text{Mor}(\mathcal{F}, \mathcal{F}_i)$, defined by the rule $\mu \mapsto H^\mu$, where for every $\mu : X \rightarrow \prod_{i \in I}^{\approx} \lambda_0(i) \in \text{Mor}(\mathcal{F}, \lim_{\leftarrow} \mathcal{F}_i)$ i.e., $\forall_{i \in I} \forall_{f \in F_i} ((f \circ \pi_i^{S(\Lambda^\approx)}) \circ \mu \in F)$, let

$$H^\mu : \bigwedge_{i \in I} \text{Mor}(\mathcal{F}, \mathcal{F}_i), \quad [H_\mu]_i : X \rightarrow \lambda_0(i), \quad H_i^\mu(x) := [\mu(x)]_i; \quad x \in X, \quad i \in I.$$

First we show that $H_i^\mu \in \text{Mor}(\mathcal{F}, \mathcal{F}_i) \Leftrightarrow \forall_{f \in F_i} (f \circ H_i^\mu \in F)$. If $f \in F_i$, and $x \in X$, then $[f \circ H_i^\mu](x) := f(H_i^\mu(x)) := f([\mu(x)]_i) =: [(f \circ \pi_i^{S(\Lambda^\approx)}) \circ \mu](x)$ i.e., $f \circ H_i^\mu := (f \circ \pi_i^{S(\Lambda^\approx)}) \circ \mu \in F$, as $\mu \in \text{Mor}(\mathcal{F}, \lim_{\leftarrow} \mathcal{F}_i)$. Since $\mu(x) \in \prod_{i \in I}^{\approx} \lambda_0(i)$, $[\mu(x)]_i = \lambda_{j_i}^{\approx}([\mu(x)]_j)$, for every $i, j \in I$

such that $i \preceq j$. To show that $H^\mu \in \prod_{i \in I}^{\preceq} \text{Mor}(\mathcal{F}, \mathcal{F}_i)$, let $i \preceq j$. Then

$$\begin{aligned} H_i^\mu &= \lambda_{ji}^{\preceq} \circ H_j^\mu \Leftrightarrow \forall x \in X (H_i^\mu(x) =_{\lambda_0(i)} [\lambda_{ji}^{\preceq} \circ H_j^\mu](x)) \\ &:\Leftrightarrow \forall x \in X ([\mu(x)]_i =_{\lambda_0(i)} [\lambda_{ji}^{\preceq}([\mu(x))]_j]), \end{aligned}$$

which holds by the previous remark on $\mu(x)$. It is immediate to show that ϕ is a function. To show that $\phi \in \text{Mor}([\mathcal{F} \rightarrow \varprojlim \mathcal{F}_i], \varprojlim (\mathcal{F} \rightarrow \mathcal{F}_i))$, we show that

$$\begin{aligned} \forall i \in I \forall f \in F_i \forall x \in \lambda_0(i) \left([\phi_{x,f} \circ \pi_i^{\mathcal{F} \rightarrow S(\Lambda^\preceq)}] \circ \phi \in \bigvee_{x \in X, i \in I} \phi_{x,f \circ \pi_i^{S(\Lambda^\preceq)}} \right), \\ [[\phi_{x,f} \circ \pi_i^{\mathcal{F} \rightarrow S(\Lambda^\preceq)}] \circ \phi](\mu) &:= [\phi_{x,f} \circ \pi_i^{\mathcal{F} \rightarrow S(\Lambda^\preceq)}](H^\mu) \\ &:= \phi_{x,f}(H_i^\mu) \\ &=: (f \circ H_i^\mu)(x) \\ &:= f(\mu(x)_i) \\ &=: (f \circ \pi_i^{S(\Lambda^\preceq)} \circ \mu)(x) \\ &=: [\phi_{x,f \circ \pi_i^{S(\Lambda^\preceq)}}](\mu). \end{aligned}$$

Moreover, $\phi(e(H)) := H$, as $H_i^{e(H)}(x) := [e(H)(x)]_i := H_i(x)$, and $e(\phi(\mu)) = \mu$, as

$$\begin{aligned} e(H^\mu) = \mu &:\Leftrightarrow \forall x \in X ([e(H^\mu)](x) =_{\prod_{i \in I}^{\preceq} \lambda_0(i)} \mu(x)) \\ &:\Leftrightarrow \forall x \in X \forall i \in I (H_i^\mu(x) =_{\lambda_0(i)} [\mu(x)]_i) \\ &:\Leftrightarrow \forall x \in X \forall i \in I ([\mu(x)]_i =_{\lambda_0(i)} [\mu(x)]_i). \quad \square \end{aligned}$$

With respect to the possible dual to the previous theorem i.e., the isomorphism $\varinjlim (\mathcal{F} \rightarrow \mathcal{F}_i) \simeq (\mathcal{F} \rightarrow \varinjlim \mathcal{F}_i)$, what we can show is the following proposition.

Proposition 11.6. *Let $S(\Lambda^\preceq) := (\lambda_0, \lambda_1^\preceq, \phi_0^{\Lambda^\preceq}, \phi_1^{\Lambda^\preceq}) \in \text{Spec}(I, \preceq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^\preceq)_{(i,j) \in D \approx (I)}$. If $\mathcal{F} := (X, F)$ is a Bishop space and $\mathcal{F} \rightarrow S(\Lambda^\preceq) := (\nu_0, \nu_1^\preceq, \phi_0^{N^\preceq}, \phi_1^{N^\preceq})$ is the (I, \preceq) -direct spectrum defined in Proposition 11.2 (A)(ii), there is a map $\hat{} : \varinjlim [\text{Mor}(\mathcal{F}, \mathcal{F}_i)] \rightarrow \text{Mor}(\mathcal{F}, \varinjlim \mathcal{F}_i)$ with $\hat{} \in \text{Mor}((\varinjlim (\mathcal{F} \rightarrow \mathcal{F}_i), \mathcal{F} \rightarrow \varinjlim \mathcal{F}_i))$.*

Proof. We proceed similarly to the proof of Theorem 11.5. □

12. CONCLUDING REMARKS

In this paper we tried to show how some fundamental notions of Bishop Set Theory, a “completion” of Bishop’s theory of sets that underlies Bishop-style constructive mathematics BISH, can be applied to the constructive topology of Bishop spaces, and especially in the theory of limits of Bishop spaces. For that we defined the notion of a direct family of sets, and by associating to such a family of sets a family of Bishop spaces the notions of a direct and contravariant direct spectrum of Bishop spaces emerged. The definition of sum Bishop topology (Definition 7.5) on the corresponding direct sum shows the harmonious relation

between the notion of Bishop set and that of Bishop space. In the subsequent sections we proved constructively the translation of all fundamental results in the theory of limits of topological spaces into the theory of Bishop spaces.

All notions of families of sets studied here have their generalised counterpart i.e., we can define generalised I -families of sets, or generalised families of sets over a directed set (I, \preceq) , where more than one transport maps from $\lambda_0(i)$ to $\lambda_0(j)$ are permitted (see [35], section 3.9). The corresponding notions of generalised spectra of Bishop spaces and their limits can be studied, and all major results of the previous sections are expected to be extended to the case of generalised spectra of Bishop spaces. In [35], section 6.9, we introduce the notion of a direct spectrum of Bishop subspaces. As in the case of set-indexed families of subsets, the main properties of the direct spectra of Bishop subspaces are determined internally.

We hope to examine interesting applications of the theory of spectra of Bishop spaces in future work.

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