KERNELIZING MSO PROPERTIES OF TREES OF FIXED HEIGHT, AND SOME CONSEQUENCES [*]

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ABSTRACT. Fix an integer $h \geq 1$. In the universe of coloured trees of height at most h, we prove that for any graph decision problem defined by an MSO_1 formula with r quantifiers, there exists a set of kernels, each of size bounded by an elementary function of r and the number of colours. This yields two noteworthy consequences. Consider any graph class $\mathcal{G}$ having a one-dimensional MSO_1 interpretation in the universe of coloured trees of height h (equivalently, $\mathcal{G}$ being a class of shrub-depth h). First, $\mathcal{G}$ admits an MSO_1 model checking algorithm whose runtime has an elementary dependence on the formula size. Second, on $\mathcal{G}$ the expressive powers of FO and MSO_1 coincide (which extends a 2012 result of Elberfeld, Grohe, and Tantau).

1. INTRODUCTION

First order (FO) and monadic second-order (MSO) logics play an undoubtedly crucial role in computer science. Besides traditional tight relations to finite automata and regular languages, this is also witnessed by their frequent occurrence in the so called *algorithmic metatheorems* which have gained increasing popularity in the past few years. The term algorithmic metatheorem commonly refers to a general algorithmic toolbox ready to be applied onto a wide range of problems in specific situations, and MSO or FO logic is often used in the expression of this "range of problems".

One of the perhaps most celebrated algorithmic metatheorems (and the original motivation for our research) is Courcelle's theorem [2] stating that every graph property ϕ expressible in the MSO_2 *logic of graphs* (allowing for both vertex and edge set quantifiers) can be decided in linear fpt time on graphs of bounded tree-width. Courcelle, Makowsky, and Rotics [5] then have analogously addressed a wider class of graphs, namely those of bounded clique-width, at the expense of restricting ϕ to MSO_1 *logic* (i.e., with only vertex set quantification).

Regarding Courcelle's theorem [2] and closely related [1, 5], it is worth to remark that a solution can be obtained via translating of the respective graph problem to an MSO formula over coloured trees (which relates the topic all the way back to Rabin's S2S theorem [21] and works of Doner [8] and Thatcher and Wright [22]). However, a common drawback of these metatheorems is that, when their runtime is expressed as $\mathcal{O}\big(f(\phi, width(G)) \cdot |G|\big)$, this function f grows asymptotically as $2^{2^{\cdot^{\cdot^{2^{width(G)}}}}} \Big\}a$ where the height a depends on ϕ, precisely on the quantifier alternation depth of ϕ (i.e., f is a *non-elementary function* of the parameter ϕ). The latter is not surprising since Frick and Grohe [12] proved that it is not possible to avoid a non-elementary tower of exponents in deciding MSO properties on all trees or coloured paths (unless P=NP), and Lampis [18] proved an analogous negative result even for uncoloured paths (unless EXP=NEXP).

The aforementioned negative results leave room for possible improvement on suitably restricted subclass(es) of all coloured trees, namely on those avoiding long paths. In this respect, our first result (Theorem 3.2) gives a new algorithm for deciding MSO properties ϕ of rooted coloured trees T of fixed height h. The algorithm (Corollary 3.4) uses so called kernelization—which means it efficiently reduces the input tree into an equivalent one (the *kernel*) of elementarily bounded size; by

$$2^{2^{\cdot^{\cdot^{\mathcal{O}(|\phi|^2)}}}} \Big\}h+1.$$

In the complexity aspects, our result "trades" quantifier alternation depth of ϕ from Courcelle's theorem for bounded height of the tree. Again, the tower of exponents of (this time fixed) height $h + 1$ in the expression is unavoidable unless the Exponential Time Hypothesis fails, as proved by aforementioned Lampis [18]. We refer to Section 3 for an exact expression of runtime as well as for an extension to counting MSO logic.

From a more general perspective our algorithm can be straightforwardly applied to any suitable "depth-structured" graph class via efficient interpretability of logic theories, such as to graph classes of bounded tree-depth or of bounded shrub-depth [14]. This (asymptotically) includes previous results of Lampis [17] and Ganian [13] as special cases. Even more, the scope of our result can be extended to the so called LinEMSO optimization and enumeration framework, see e.g. in [5], over such graph classes as follows: the algorithmic metatheorems of [2, 1, 5] (and similar ones) can be treated using finite tree-automata (with non-elementary numbers of states in general), however, in the universe of a suitable "depth-structured" graph class only very few of the automaton states correspond to some of the kernels and so are actually reachable. This in Section 4 concludes the first half of our paper.

There are also other sides of the main result. First, the initial discovery of Theorem 3.2 was the prime motivation for defining shrub-depth in [14], and a key ingredient in the proof that shrub-depth is stable under MSO_1 interpretations, again in [14].

Second, Elberfeld, Grohe, and Tantau [11] prove that FO and MSO_2 have equal expressive power on the graphs of bounded tree-depth. Having Theorem 3.2 at hand, we can provide a relatively simple alternative proof of this result. Furthermore, using some more sophisticated combinatorial tools, namely well-quasi-ordering, we prove a new result (Theorem 5.14) that FO and MSO_1 have equal expressive power on any graph class of bounded shrub-depth. In the converse direction, Elberfeld, Grohe, and Tantau [11] also prove that on monotone graph classes of unbounded tree-depth, MSO_2 is strictly stronger than FO. Unfortunately, due to lack of a suitable "forbidden substructure" characterization of shrub-depth, we are not yet

able to prove the analogous converse claim, but we conjecture that a hereditary class on which FO and MSO_1 coincide must have bounded shrub-depth (Conjecture 6.1).

2. Preliminaries

We assume standard terminology and notation of graph theory, see e.g. Diestel [6]. Our graphs are finite, simple and undirected by default. When dealing with trees, we implicitly consider them as rooted (with the implicit parent-child tree order) and with unordered descendants. We use $\subseteq$ for the usual subgraph relation and $\subseteq_i$ for induced subgraphs. A graph class $\mathcal{C}$ is *hereditary* if it is closed under the induced subgraph relation, i.e., $G \in \mathcal{C}$ and $H \subseteq_i G$ implies $H \in \mathcal{C}$.

A class $\mathcal{C}$ of graphs is *well-quasi-ordered* (WQO) under an order $\preceq$ if, for any infinite sequence $(G_1, G_2, \dots) \subseteq \mathcal{C}$, it is $G_i \preceq G_j$ for some $i < j$. WQO techniques are very popular in structural graph theory, and we refer to [6, Chapter 12] for a brief overview. A graph property Π is *preserved under* $\preceq$ if the following holds; whenever $G \in \mathcal{C}$ has the property Π, every $G' \in \mathcal{C}$ such that $G' \preceq G$ has Π as well. The following simple folklore claim is crucial in this paper:

Proposition 2.1. *Let $\mathcal{C}$ be a graph class. If Π is a property preserved under a well-quasi-order $\preceq$ on $\mathcal{C}$, then there exists a finite set $Obst \subseteq \mathcal{C}$ such that; $G \in \mathcal{C}$ has Π if and only if $H \not\preceq G$ for all $H \in Obst$.* □

We informally say that Π from Proposition 2.1 has finitely many obstacles in $\mathcal{C}$. We are going to use the following well-quasi-ordered graph class.

Theorem 2.2 (Ding [7]). *Let $m \in \mathbb{N}$ be an integer and C be a finite set of colours. The class of the graphs not containing a path on m vertices as a subgraph and with vertices coloured by C is well-quasi-ordered under the colour-preserving induced subgraph order $\subseteq_i$.* □

Given a graph G, a *tree-decomposition of G* is an ordered pair $(T, \mathcal{W})$, where T is a tree and $\mathcal{W} = \{W_x \subseteq V(G) \mid x \in V(T)\}$ is a collection of *bags* (vertex sets of G), such that the following hold:

(1) $\bigcup_{x \in V(T)} W_x = V(G)$;
(2) for every edge $e = uv$ in G, there exists $x \in V(T)$ such that $u, v \in W_x$;
(3) for each $u \in V(G)$, the set $\{x \in V(T) \mid u \in W_x\}$ induces a subtree of T.

The width of a tree-decomposition $(T, \mathcal{W})$ is $(\max_{x \in V(T)} |W_x|) - 1$. The *tree-width* of G, denoted $tw(G)$, is the smallest width of a tree-decomposition of G.

Besides tree-width, another useful width measure of graphs is the *clique-width* of a graph G. This is defined for a graph G as the smallest number of labels $k = cw(G)$ such that some labelling of G can be defined by an algebraic *k-expression* using the following operations:

(1) create a new vertex with label i;
(2) take the disjoint union of two labelled graphs;
(3) add all edges between vertices of label i and label j; and
(4) relabel all vertices with label i to have label j.

Monadic second-order logic (MSO) is an extension of first-order logic (FO) by quantification over sets. The *quantifier rank* $qr(\phi)$ is the nesting depth of quantifiers in ϕ. The formulas of quantifier rank 0 are called *quantifier free*. *Counting monadic second-order logic* (CMSO) is an extension of MSO which allows use of predicates $mod_{a,b}(X)$, where X is a

set variable. The semantics of the predicate $mod_{a,b}(X)$ is that the set X has a modulo b elements.

Let σ and τ be relational vocabularies and let $L \in \{\mathrm{FO}, \mathrm{MSO}, \mathrm{CMSO}\}$. A *one-dimensional interpretation*[1] of τ in σ is a tuple $I = \big(\nu(x), \{\eta_R(\bar{x})\}_{R \in \tau}\big)$ of $L[\sigma]$-formulas where ν has one free variable and the number of free variables in each η_R is equal to the arity of R in τ.

- To every σ-structure A the interpretation I assigns a τ-structure A^I with the domain $A^I = \{a \mid A \models \nu(a)\}$ and the relations $R^I = \{\bar{a} \mid A \models \eta_R(\bar{a})\}$ for each $R \in \tau$. We say that a class $\mathcal{C}$ of τ-structures has an interpretation in a class $\mathcal{D}$ of σ-structures if there exists an interpretation I such that for each $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $C \simeq D^I$, and for every $D \in \mathcal{D}$ the structure D^I is isomorphic to a member of $\mathcal{C}$.
- The interpretation I of τ in σ defines a translation of every $L[\tau]$-formula ψ to an $L[\sigma]$-formula ψ^I as follows:
  - every $\exists x.\phi$ is replaced by $\exists x.(\nu(x) \wedge \phi^I)$,
  - every $\exists X.\phi$ is replaced by $\exists X.(\forall y(y \in X \to \nu(y)) \wedge \phi^I)$, and
  - every occurrence of a σ-atom $R(\bar{x})$ is replaced by the corresponding formula $\eta_R(\bar{x})$.

We have added the adjective "one-dimensional" (interpretation) to indicate that our ν is of arity one, i.e., that the domain of τ-structures is interpreted in singleton elements of the σ-structures. Since we use only one-dimensional interpretations throughout the paper, from now on we will say shortly an "*interpretation*" to mean a one-dimensional interpretation.

We make use of the following claim:

Lemma 2.3 ([15])**.** *Let I be interpretation of τ in σ. Then for all $L[\tau]$-formulas ϕ and all σ-structures A*

$$A \models \phi^I \iff A^I \models \phi.$$

$\square$

A great part of our paper deals with MSO logic of rooted trees, which are structures with a single parent-child binary relation. For general graphs, however, there are two established but inequivalent views of them as relational structures. In the one-sorted adjacency model of graphs, MSO specifically reads as follows:

Definition 2.4 (MSO$_1$ logic of graphs)**.** The language of MSO$_1$ contains the expressions built from the following elements:

- variables $x, y, \ldots$ for vertices, and $X, Y, \ldots$ for sets of vertices,
- the predicates $x \in X$ and $edge(x, y)$ with the standard meaning,
- equality for variables, the connectives $\wedge, \vee, \neg, \to$, and the quantifiers $\forall, \exists$ over vertex and vertex-set variables.

MSO$_2$ logic of graphs extends MSO$_1$ by allowing quantification over edge sets. Formally, one can consider graphs as two-sorted structures (the two sorts being vertex-set and edge-set of a graph) with adjacency and incidence predicates.

Although our results are concerned also with MSO$_2$, we refrain from giving full definition of MSO$_2$ logic of graphs, since by the following theorem we can avoid using MSO$_2$ explicitly.

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[1]The name *one-dimensional (noncopying) transduction* is also used in an algorithmic context, see, e.g., [4]. Transductions mean, however, usually a more general concept than what we define and use here.

Theorem 2.5 (Courcelle [3], also [4])**.** *Let* $\mathcal{C}$ *be the class of finite graphs of tree-width at most* k *(for any fixed* k*). A property of graphs in* $\mathcal{C}$ *is* CMSO_2*-expressible if, and only if, it is* CMSO_1*-expressible. Moreover, for any fixed* k*, if* ϕ *is an* CMSO_2 *sentence, the size of an equivalent* CMSO_1 *sentence* ϕ' *can be bounded by an elementary function of* $|\phi|$*.*[2] $\qquad\square$

We often use labelled graphs, i.e. graphs where each vertex can have labels from some finite set *Lab*. Labels are modelled by unary predicates. Sometimes we refer to colours instead of labels; the colour of a vertex is the combination of its labels (each vertex has exactly one colour, the number of colours is $\sim 2^{|Lab|}$).

For an introduction to parameterized complexity we suggest [9]. Here we just recall that a problem $\mathcal{P}$ parameterized by k, i.e., with an input $\langle x, k \rangle \in \Sigma^* \times \mathbb{N}$, is *fixed parameter tractable*, or *fpt*, if it admits an algorithm in time $\mathcal{O}\big(f(k) \cdot |x|^{\mathcal{O}(1)}\big)$ where f is an arbitrary computable function.

3. Trees of Bounded Height and MSO

The primary purpose of this section is to prove Theorem 3.2; that for any m-coloured tree T of constant height h there exists an efficiently computable subtree $T_0 \subseteq T$ (a *kernel*) such that, for any MSO sentence ϕ of fixed quantifier rank r, it is $T \models \phi \iff T_0 \models \phi$, and the size of T_0 is bounded by an elementary function of r and m (the dependence on h being non-elementary, though). Particularly, since checking of an MSO property ϕ can be easily solved in time $\mathcal{O}^*\big(2^{c|\phi|}\big)$ on a graph with c vertices (in this case T_0) by recursive exhaustive expansion of all quantifiers of ϕ, this gives a kernelization-based elementary fpt algorithm for MSO model checking of rooted m-coloured trees of constant height h (Corollary 3.4).

We need a bit more formal notation. The *height*[3] h of a rooted tree T is the farthest distance from its root, and a node is at the *level* ℓ if its distance from the root is $h - \ell$. For a node v of a rooted tree T, we call a *limb of* v a subtree of T rooted at some child node of v. Our rooted trees are unordered, and they "grow top-down", i.e. we depict the root on the top. We switch from considering m-coloured trees to more convenient t-labelled ones, the difference being that one vertex may have several labels at once (and so $m \sim 2^t$). We say that two such rooted labelled trees are *l-isomorphic* if there is an isomorphism between them preserving the root and all the labels.

For obtaining the desired kernel, we shall use a concept of reducing a (rooted t-labelled) tree T as follows.

Definition 3.1 (f-reduction, f-reduced)**.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function, called a *threshold function*. Assuming a node $v \in V(T)$ at level $i + 1 > 0$ and a limb B of v such that there exist at least $f(i)$ other limbs of v in T which are all l-isomorphic to B; we say that T *f-reduces in one step* to $T - V(B)$. A tree T *f-reduces to* $T_0 \subseteq T$ if there is a sequence of one-step f-reductions from T to T_0. The tree T_0 is *f-reduced* if no further f-reduction step is possible.

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[2]We note that the theorem is explicitly stated only for MSO_1 and MSO_2 in [3, 4] (Theorems 1.44 and 5.22 of [4]). To see that the same holds for CMSO_1 and CMSO_2 see Section 5.2.6 of [4]. The second part of the theorem concerning the size of an equivalent CMSO_1 formula can be obtained from the proofs. Therefore, we cannot cite a precise bound of $|\phi'|$ in terms of $|\phi|$, but one can read from the proofs that the bound is actually polynomial and even linear for a fixed k.

[3] There is a conflict in the literature about whether the height of a rooted tree should be measured by the "root-to-leaves distance" or by the "number of levels" (a difference of 1 on finite trees). We adopt the convention that the height of a single-node tree is 0 (i.e., the former view).

A straightforward induction shows that any f-reduced t-labelled rooted tree of fixed height has size bounded with respect to f and t.

3.1. **Reduction to kernel.** Considering MSO sentences with q element quantifiers and s set quantifiers, we use the following threshold function R to define our reduced kernel (where k is an arbitrary integer parameter that will count all the labels used in our proof):

$$R_{q,s,k}(i) \;=\; q \cdot N_{q,s,k}(i)^s, \qquad \text{where} \tag{3.1}$$

$$N_{q,s,k}(0) \;=\; 2^k + 1 \geq 2 \quad \text{and}$$

$$N_{q,s,k}(i+1) \;=\; 2^k \cdot \left(R_{q,s,k}(i)+1\right)^{N_{q,s,k}(i)} \leq 2^k \cdot \left(2q \cdot N_{q,s,k}(i)^s\right)^{N_{q,s,k}(i)} \tag{3.2}$$

Note also that all these values are non-decreasing in the parameters q, s, k.

The idea behind our approach can be simplified as follows. Fix $i \geq 0$ and any $a \geq R_{q,s,k}(i)$ where $k = t + 3q + s$, and choose an arbitrary $R_{q,s,k}$-reduced rooted t-labelled tree U of height i. Then no MSO sentence with q element variables and s set variables can distinguish between a disjoint copies and $a+1$ disjoint copies of U. Our full result then reads:

Theorem 3.2. *Let T be a rooted t-labelled tree of height h, and let ϕ be an MSO sentence with q element quantifiers and s set quantifiers. Suppose that $u \in V(T)$ is a node at level $i+1$ where $i < h$.*

  a) *If, among all the limbs of u in T, there are more than $R_{q,s,t+3q+s}(i)$ pairwise l-isomorphic ones, then let $T' \subseteq T$ be obtained by deleting one of the latter limbs from T. Then, $T \models \phi \iff T' \models \phi$.*

  b) *Consequently, the tree T $R_{q,s,t+3q+s}$-reduces to $T_0 \subseteq T$ such that T_0 itself is $R_{q,s,t+3q+s}$-reduced, and $T \models \phi \iff T_0 \models \phi$.*

  c) *The tree T_0 can be computed in linear time from T and ϕ, and the size of T_0 is bounded by*

$$|V(T_0)| \leq exp^{(h)} \left[(2^{h+5} - 12) \cdot (t+q+s)(q+s)\right].$$

In the latter expression, $exp^{(i)}(x)$ denotes the i-fold exponential function defined inductively as follows: $exp^{(0)}(x) = x$ and $exp^{(i+1)}(x) = 2^{exp^{(i)}(x)}$. Then $exp^{(h)}(x)$ is an elementary function of x for each particular height h. Note that the runtime bound in c) is within classical complexity—not parameterized.

In the case of FO logic, a statement analogous to Theorem 3.2 can be obtained using folklore arguments of finite model theory (even full recursive expansion of all q vertex quantifiers in ϕ could "hit" only bounded number of limbs of u and the rest would not matter). However, in order to obtain suitable explicit bounds, in the case of MSO logic there are additional nontrivial complications which require careful considerations (in addition to standard tools) during the proof. Briefly saying, one has to recursively consider the internal structure of the limbs of u, and show that even an expansion of a vertex-set quantifier in ϕ does not effectively distinguish too many of them (and hence some of them remain irrelevant for the decision whether $T \models \phi$).

Before proceeding with formal proof of Theorem 3.2, we need to clarify the meaning of the values N:

Lemma 3.3. *For any natural $i, q, s,$ and k, there are at most $N_{q,s,k}(i)$ pairwise non-l-isomorphic $R_{q,s,k}$-reduced rooted k-labelled trees of height $\leq i$.*

Proof. This claim readily follows from Equation (3.1) and (3.2) by induction on i. The base case $i = 0$ is trivial, and the count includes also the empty tree. A rooted k-labelled tree T of height $\leq i + 1$ can be described by the label of its root r (2^k possibilities), and the set of its limbs, each one of height $\leq i$. This set of limbs can be fully described by the numbers of limbs (between 0 and $R_{q,s,k}(i)$) in every of $\leq N_{q,s,k}(i)$ possible l-isomorphism classes. Hence by (3.2) we have got at most $N_{q,s,k}(i+1)$ possible distinct descriptions of T. □

Proof of Theorem 3.2. For clarity, we start with a proof sketch.
  (I) We are going to use a so called "quantifier elimination" approach.[4] That means, assuming $T \models \phi \not\Longleftrightarrow T' \models \phi$, we look at the "distinguishing choice" of the first quantifier in ϕ, and encode it in the labelling of T (e.g., when $\phi \equiv \exists x.\psi$, we give new exclusive labels to the value of x and to its parent/children in T and T'). By an inductive assumption, we then argue that the shorter formula ψ cannot distinguish between these newly labelled T and T', which is a contradiction.
  (II) The traditional quantifier elimination approach—namely of set quantifiers in ϕ, however, might not be directly applicable to even very many pairwise l-isomorphic limbs in T if their size is unbounded. Roughly explaining, the problem is that a single valuation of a set variable on these repeated limbs may potentially pairwise distinguish all of them. Hence additional combinatorial arguments are necessary to bound the size of the limbs in consideration.
  (III) Having resolved technical ((II)), the rest of the proof is a careful composition of inductive arguments using the formula (3.1) $R_{q,s,k}(i) = q \cdot N_{q,s,k}(i)^s$.

a) The whole proof goes through by *means of contradiction*. That is, we assume $T \models \phi$ while $T' \models \neg\phi$ (a counterexample to Theorem 3.2 a, where T' implicitly depends on the choice of u), up to natural symmetry between ϕ and $\neg\phi$ in this context. Let $t' = t + 3q + s$. Let $B_1, \ldots, B_p \subseteq T$ where $p > R_{q,s,t'}(i) \geq 1$ be the pairwise l-isomorphic limbs of u in T, as anticipated in Theorem 3.2 a). Note that the height of B_1 is at most i (but it may possibly be lower than i).

So, say, $T' = T - V(B_1)$. We will apply nested induction, primarily targeting the structure of the sentence ϕ, or simply the value $q + s$. For that we assume ϕ in the prenex form, i.e., with a leading section of all quantifiers. If $q = s = 0$, then ϕ is a propositional formula which evaluates to true or false without respect to T or T'. Hence we further assume $q + s > 0$. Note also the little trick with choice of $t' = t + 3q + s$ which "makes room" for ((I)) adding further labels to T in the course of the proof.

(Minimality setup) To overcome the complication in ((II)), we have to deal with limbs $B_1, \ldots, B_p$ of bounded size. So, among all the assumed counterexamples to Theorem 3.2 a) for this particular ϕ or symmetric $\neg\phi$, choose one (meaning precisely the choice of T and u within it) which minimizes the size of B_1 (same as the sizes of $B_2, \ldots, B_p$). This minimality choice actually represents a secondary induction in our proof.

We would like to show that the l-isomorphic limbs $B_1, \ldots, B_p$ are $R_{q,s,t'}$-reduced. Suppose not, and let $w_k \in V(B_k)$ be a node at level $j + 1 \leq i$ such that among all the limbs of w_k in B_k there are more than $R_{q,s,t'}(j)$ pairwise l-isomorphic ones, hereafter denoted by $D_{k,1}, \ldots, D_{k,r}$ where $r > R_{q,s,t'}(j)$. This choice is made for all $k = 1, \ldots, p$ symmetrically, i.e., all the subtrees $B_k^- = B_k - V(D_{k,1})$ where $k = 1, \ldots, p$ are pairwise l-isomorphic, too.

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[4]This approach has been inspired by [10], though here it is applied in a wider setting of MSO logic.

We define a sequence of trees by $U_0 = T$ and $U_k = U_{k-1} - V(D_{k,1})$ for $k = 1, \ldots, p$. Recall that $U_0 \models \phi$. If it ever happened that $U_{k-1} \models \phi$ but $U_k \models \neg\phi$, then we would consider U_{k-1} and w_k in place of T and u above, and hence contradict the choice minimizing B_1 (which would be replaced with smaller $D_{k,1}$). We may thus say that $U_p \models \phi$. We similarly define $U_1' = T'$ and $U_k' = U_{k-1}' - V(D_{k,1})$ for $k = 2, \ldots, p$ (recall that B_1 has been removed from T'). With an analogous argument we conclude that $U_p' \models \neg\phi$.

Note that, now, $B_1^-, \ldots, B_p^-$ are pairwise l-isomorphic limbs of u in U_p, and they are strictly smaller than B_1. Since $U_p' = U_p - V(B_1^-)$, we may have chosen U_p and u in place of T, u, again contradicting minimality of B_1 in the choice above. Indeed, the (original) limbs $B_1, \ldots, B_p$ are $R_{q,s,t'}$-reduced in T.

(Quantifier elimination: $\exists x$) As the main induction step we now "eliminate" the leading quantifier of ϕ as follows. Suppose first that $\phi \equiv \exists x.\,\psi$. Let $a \in V(T)$ be such that $T[x = a] \models \psi(x)$. Clearly, it can be chosen $a \notin V(B_1)$ since B_1 is l-isomorphic to other $B_2, \ldots, B_p$. On the other hand, $T'[x = b] \not\models \psi(x)$ for all $b \in V(T')$.

We define a $(t+3)$-labelled tree T^a which results from T by adding a new label L_x exclusively to the node a, a new label L_{px} exclusively to the parent node of a, and L_{cx} to the child nodes of a. A tree $T^{a'} = T^a - V(B_1)$ is formed analogously from T'. Then we translate the formula $\psi(x)$ with free x into a closed one ψ^x as defined next: All label predicates $L(x)$ in $\psi(x)$ are simply evaluated as $L(a)$ over T (which is the same as over T'). Any predicate $x = y$ is replaced with $L_x(y)$. Finally, all predicates for edges (x, y) and (y, x) in this parent-child order are replaced with $L_{cx}(y)$ and $L_{px}(y)$, respectively. It is trivial that $T[x = a] \models \psi(x) \iff T^a \models \psi^x$, and $T'[x = a] \not\models \psi(x) \iff T^{a'} \not\models \psi^x$.

All the limbs $B_1, \ldots, B_p$ remain pairwise l-isomorphic in T^a unless, say, $a \in V(B_p)$. Even in the latter case we anyway obtain, using (3.1), at least $p - 1 > R_{q,s,t'}(i) - 1$ pairwise l-isomorphic limbs of u in T^a, including B_1. It is

$$R_{q,s,t'}(i) - 1 = q \cdot N_{q,s,t'}(i)^s - 1 \geq (q-1) \cdot N_{q,s,t'}(i)^s \geq R_{q-1,s,t'}(i).$$

Note also that $q - 1$ is the number of element quantifiers in ψ, and that the combined parameter $t + 3 + 3(q - 1) + s = t + 3q + s = t'$ remains the same. Hence we can apply the inductive assumption to T^a, u, and ψ^x—concluding that $T^a \models \psi^x \iff T^{a'} \models \psi^x$, a contradiction.

(Quantifier elimination: $\exists X$) We are finally getting to the heart of the proof. Suppose now that $\phi \equiv \exists X.\,\psi$. Let $A \subseteq V(T)$ be such that $T[X = A] \models \psi(X)$. On the other hand, $T'[X = A'] \not\models \psi(X)$ for all $A' \subseteq V(T')$. We define a $(t+1)$-labelled tree T^A which results from T by adding a new label L_X precisely to all members of A. Then we translate the formula $\psi(X)$ with free X into a closed one ψ^X by replacing every occurrence of $y \in X$ with $L_X(y)$. Trivially, $T[X = A] \models \psi(X) \iff T^A \models \psi^X$.

Note again that $s - 1$ is the number of set quantifiers in ψ, and that the combined parameter $t + 1 + 3q + (s - 1) = t + 3q + s = t'$ remains the same. A key observation is that "casting" the new label L_X onto the limbs $B_1, \ldots, B_p$ may create at most $N_{q,s,t'}(i)$ l-isomorphism classes among them. This is simply because, for each $k = 1, \ldots, p$, the corresponding B_k^A carries $t + 1 \leq t'$ labels, it is of the same height as B_k and $R_{q,s,t'}$-reduced, too. Hence, altogether, there are at most $N_{q,s,t'}(i)$ pairwise non-l-isomorphic choices for such B_k^A by Lemma 3.3.

So, among all $B_1, \ldots, B_p$, there are at least $p/N_{q,s,t'}(i)$ pairwise l-isomorphic limbs, and using (3.1),

$$\frac{p}{N_{q,s,t'}(i)} > \frac{R_{q,s,t'}(i)}{N_{q,s,t'}(i)} = q \cdot N_{q,s,t'}(i)^{s-1} \geq R_{q,s-1,t'}(i).$$

For simplicity, let the latter limbs be $B_1, \ldots, B_{p'}$ where $p \geq p' > R_{q,s-1,t'}(i)$. Now we apply the inductive assumption to T^A, u, and ψ^X. Up to symmetry between the limbs, we get $(T^A)' = T^A - V(B_1)$ such that $T^A \models \psi^X \iff (T^A)' \models \psi^X$. Now we can define $A' \subseteq V(T')$ as the set of those nodes having label L_X in $(T^A)'$, and hence $(T^A)' \models \psi^X \iff T'[X = A'] \models \psi(X)$, a contradiction to the initial assumption.

(Quantifier elimination: $\forall$) Finally, the cases of universal quantifiers in ϕ are solved analogously ($\neg\exists$ in place of $\forall$).

b) This part readily follows by a recursive bottom-up application of a) to the whole tree T.

c) T_0 is easily constructed from T by a natural adaptation of the classical linear-time tree-isomorphism algorithm. Notice that there are no "hidden huge constants" depending on ϕ in this algorithm; we simply read the parameters t, q, s by linear-time parsing of ϕ and we compute the values of the threshold function $R_{q,s,t+3q+s}$ "on demand" when we encounter a limb at level i with (too) many l-isomorphic siblings. We use the simple fact that $R_{q,s,k}(i) \geq R_{q,s,k}(i-1)$ and this computation is thus negligible compared to the size of T.

Since T_0 is $R_{q,s,t'}$-reduced by b), where $t' = t + 3q + s$, we can consider $T_0 \subseteq U_{h,q,s,t}$ where $U_{h,q,s,t}$ is the "maximal" $R_{q,s,t'}$-reduced rooted t-labelled tree of height h: $U_{h,q,s,t}$ contains (at each level $j + 1$) precisely $R_{q,s,t'}(j)$ limbs of every l-isomorphism class of rooted t'-labelled trees of height $\leq j$.

Therefore, by Lemma 3.3, the number of descendants at each level j of $U_{h,q,s,t}$ is at most $R_{q,s,t'}(j-1) \cdot N_{q,s,t'}(j-1)$. The total number of vertices in $U_{h,q,s,t}$ is at most

$$1 + R_{q,s,t'}(h-1) \cdot N_{q,s,t'}(h-1) \cdot \left(1 + R_{q,s,t'}(h-2) \cdot N_{q,s,t'}(h-2) \cdot (1 + \ldots)\right) \leq$$

$$\leq \prod_{i=0}^{h-1} \left(1 + R_{q,s,t'}(i) \cdot N_{q,s,t'}(i)\right). \tag{3.3}$$

The task is now to estimate, by induction on i, the value $1 + R_{q,s,t'}(i) \cdot N_{q,s,t'}(i)$ from above by $exp^{(i+1)}\left[(6 \cdot 2^i - 2)t'(q + s)\right]$. Note that $t' \geq q + s \geq 1$.

$$1 + N_{q,s,t'}(0) \cdot R_{q,s,t'}(0) = 1 + q \cdot \left(2^{t'} + 1\right)^{s+1} \leq 2^q \cdot 2^{(t'+1)(s+1)} \leq$$

$$\leq 2^{2t'(s+1)+q} \leq 2^{2t'(s+1+q)} \leq 2^{4t'(s+q)}$$

$$
\begin{aligned}
1 + N_{q,s,t'}(i+1) \cdot R_{q,s,t'}(i+1) \;=\; & 1 + q \cdot N_{q,s,t'}(i+1)^{s+1} = & (3.4)\\
=\; & 1 + q \cdot \left[2^{t'} \cdot \left(R_{q,s,t'}(i) + 1 \right)^{N_{q,s,t'}(i)} \right]^{s+1} \leq \\
\leq\; & 1 + q \cdot \left[2^{t'} \cdot \left(2 R_{q,s,t'}(i) \right)^{N_{q,s,t'}(i)} \right]^{s+1} \leq \\
\leq\; & 1 + q \cdot \left[2^{t'} \cdot \left(2 q N_{q,s,t'}(i)^s \right)^{N_{q,s,t'}(i)} \right]^{s+1} \leq \\
\leq\; & 1 + q \cdot \left[\left(N_{q,s,t'}(i)^{t'+q+s} \right)^{N_{q,s,t'}(i)} \right]^{s+1} \leq \\
\leq\; & 1 + q \cdot \left[2^{(t'+q+s) \cdot N_{q,s,t'}(i)^2} \right]^{s+1} \leq \\
\leq\; & exp^{(1)} \left[q + (t'+q+s)(s+1) \cdot N_{q,s,t'}(i)^2 \right] \leq \\
\leq\; & exp^{(1)} \left[2^{2(t'+q+s)} \cdot N_{q,s,t'}(i)^2 \right] \leq \\
\leq\; & exp^{(1)} \left[2^{2(t'+q+s)} \cdot 2^{2\, exp^{(i)}\left((6 \cdot 2^i - 2)t'(q+s) \right)} \right] \leq \\
\leq\; & exp^{(2)} \left[2\, exp^{(i)} \left((6 \cdot 2^i - 2)t'(q+s) \right) + 2(t'+q+s) \right] \leq \\
\leq\; & exp^{(2)} \left[exp^{(i)} \left(2 \cdot (6 \cdot 2^i - 2)t'(q+s) + (t'+q+s) \right) \right] \leq \\
\leq\; & exp^{(2)} \left[exp^{(i)} \left(2 \cdot (6 \cdot 2^i - 2)t'(q+s) + 2t'(q+s) \right) \right] = \\
=\; & exp^{(2)} \left[exp^{(i)} \left((6 \cdot 2^{i+1} - 2)t'(q+s) \right) \right] = \\
=\; & exp^{(i+2)} \left[(6 \cdot 2^{i+1} - 2)t'(q+s) \right]
\end{aligned}
$$

With (3.3), we then get

$$
\begin{aligned}
|V(T_0)| \leq |V(U_{h,q,s,t})| \;\leq\; & \prod_{i=0}^{h-1} \left(1 + R_{q,s,t'}(i) \cdot N_{q,s,t'}(i) \right) \leq \\
\leq\; & \prod_{i=0}^{h-1} exp^{(i+1)} \left[(6 \cdot 2^i - 2)t'(q+s) \right] \leq \\
\leq\; & exp^{(h)} \left[2 \cdot (6 \cdot 2^{h-1} - 2)t'(q+s) \right] \leq \\
\leq\; & exp^{(h)} \left[(2^{h+3} - 4) \cdot (t + 3q + s)(q+s) \right] \leq \\
\leq\; & exp^{(h)} \left[(2^{h+5} - 12) \cdot (t + q + s)(q+s) \right]. \qquad \square
\end{aligned}
$$

Corollary 3.4. *Let T be a rooted t-labelled tree of constant height $h \geq 1$, and let ϕ be an MSO sentence with r quantifiers. Then $T \models \phi$ can be decided by an fpt algorithm running in elementary time*

$$
\mathcal{O}\left(exp^{(h+1)} \left[2^{h+5} \cdot r(t+r) \right] + |V(T)| \right) = exp^{(h+1)}\left(\mathcal{O}(|\phi|)^2 \right) + \mathcal{O}\left(|V(T)| \right).
$$

Proof. Let T_0 be the kernel obtained in linear time by Theorem 3.2. We (by brute force) exhaustively expand all the quantifiers of ϕ into all possible valuations in T_0, having at most $2^{|V(T_0)|}$ possibilities for each. By searching this "full valuation tree" in time $\mathcal{O}\left(2^{|V(T_0)| \cdot (r+1)} \right)$ we decide whether $T_0 \models \phi$. Using the size bound on T_0 given by Theorem 3.2, where

$r = q + s$, it is

$$
\begin{aligned}
2^{|V(T_0)| \cdot (q+s+1)} &\leq& 2^{exp^{(h)} \left[(2^{h+5}-12) \cdot (t+q+s)(q+s) \right] \cdot (q+s+1)} \leq \\
&\leq& exp^{(h+1)} \left[(2^{h+5} - 12) \cdot (t+q+s)(q+s) + (q+s+1) \right] \\
&\leq& exp^{(h+1)} \left[2^{h+5} \cdot r(t+r) \right]. \qquad\qquad \square
\end{aligned}
$$

3.2. **Counting** MSO **logic.** Theorem 3.2 can be further strengthened by considering CMSO logic. Although relatively easy, this is not a simple corollary, and the additional issues of different sort require us to repeat the overall structure of the previous proof as follows.

Theorem 3.5. *Let T be a rooted t-labelled tree of height h, let ϕ be a CMSO sentence with q element quantifiers and s set quantifiers, and let M be the least common multiple of the b values of all $mod_{a,b}$ predicates occurring in ϕ.*

a) Suppose that $u \in V(T)$ is a node at level $i+1$ where $i < h$. If, among all the limbs of u in T, there are more than $R_{M+q,s,t+3q+s}(i)$ pairwise l-isomorphic ones, then let $T' \subseteq T$ be obtained by deleting exactly M of the latter limbs from T. Then, $T \models \phi \iff T' \models \phi$.

b) Consequently, there is a subtree $T_0 \subseteq T$ computable in linear time (non-parameterized), such that $T \models \phi \iff T_0 \models \phi$ and the size of T_0 is bounded by

$$
|V(T_0)| \leq exp^{(h)} \left[(2^{h+5} - 12) \cdot (t+M+q+s)(M+q+s) \right].
$$

Proof. We closely follow the structure of the proof of Theorem 3.2, and implicitly refer to its assumptions and notation. In particular, let $t' = t + 3q + s$. Due to the effects of (future) quantifier elimination onto the $mod_{a,b}(X)$ predicates we have to deal in ϕ also with special model constants $mod_{a,b}^L$ (where L is a label of the model): The semantics of $mod_{a,b}^L$ in $U \models \psi$ is that the respective model U contains a modulo b nodes holding label L.

a) As in the previous proof, we have a node u in T such that u has many, $p > R_{M+q,s,t'}(i) \geq M$, pairwise l-isomorphic limbs $B_1, \ldots, B_p \subseteq T$. We claim that for $T' = T - V(B_1 \cup \cdots \cup B_M)$ it holds $T \models \phi \iff T' \models \phi$. This is again proved by induction on $q + s$:

For the base of induction, when $q = s = 0$, ϕ is a propositional formula and the outcomes of $T \models \phi$ and $T' \models \phi$ might differ only in the constants $mod_{a,b}^L$. However, M is a multiple of b by definition and the number of (deleted) nodes holding label L in $B_1 \cup \cdots \cup B_M$ is a multiple of M, and hence all the involved model constants $mod_{a,b}^L$ in ϕ indeed do have the same value over T as over T'.

For the induction step with $q + s \geq 1$, we show that the threshold value $p > R_{q,s,t'}^{*M}(i) := R_{M+q,s,t'}(i)$ is sufficient in this proof. Note that $R_{0,0,t'}^{*M}(i) = M$. We again proceed by means of contradiction; assuming $T \models \phi$ while $T' \models \neg\phi$ up to symmetry between ϕ and $\neg\phi$. Let us consider a counterexample which minimizes the size of B_1. Then, analogously to the proof of Theorem 3.2, the l-isomorphic limbs $B_1, \ldots, B_p$ are $R_{q,s,t'}^{*M}$-reduced or a smaller counterexample exists (for this ϕ or $\neg\phi$). The only difference in the argument is that we are now always removing M-tuples of l-isomorphic limbs instead of single ones.

Then a quantifier elimination argument, essentially same as previous, finishes the induction step. The argument from the proof of Theorem 3.2 can be simply repeated word by word, only replacing q with $q' := M + q$ as in $R_{q',s,t'}(i)$.

b) By a recursive bottom-up application of a) to the whole tree T we obtain a tree T_0 which is $R_{q',s,t'}$-reduced, and this T_0 is computable in linear time, too. The size bound then follows from Theorem 3.2c) for $q' = M + q$ in place of q. □

Corollary 3.6. *Let T be a rooted t-labelled tree of constant height $h \geq 1$, and let ϕ be an* CMSO *sentence. Let M be the least common multiple of the b values of all $\mathrm{mod}_{a,b}$ predicates occurring in ϕ. Then $T \models \phi$ can be decided by an fpt algorithm running in elementary time*

$$exp^{(h+1)}\left(\mathcal{O}(M + |\phi|)^2\right) + \mathcal{O}\left(|V(T)|\right).$$ □

4. Algorithmic Consequences for Graphs

If one considers extending algorithmic scope of the previous section to richer classes of structures such as general graphs, the first natural choice would be to employ efficient interpretation of the structures in coloured trees of fixed height together with Corollaries 3.4 and 3.6. This has been the course taken in the conference version of this paper (covering Lampis [17] and Ganian [13] as special cases) and in a greater generality in [14]. Here we use another, slightly more complicated, approach which has the advantage of being able to smoothly incorporate also some wider problem frameworks, such as MSO enumeration and the LinEMSO optimization framework, and others.

4.1. **Smaller tree automata for** MSO. Our approach uses tree automata for MSO properties (cf. Rabin [21], Doner [8] and Thatcher and Wright [22]), and its core idea is to give a stricter bound on the number of states of such an automaton by showing that each reachable state is represented by some of our reduced kernels. Recall that the number of states of the automaton related to, e.g., Courcelle's MSO_2 theorem [2] grows non-elementarily with the quantifier alternation depth of the formula, and that this is generally unavoidable by Frick and Grohe [12] already for MSO properties on trees.

However, consider the following situation; we apply the algorithm of, say, Courcelle's MSO_2 theorem to a class $\mathcal{C}$ of graphs which, in an addition to having bounded tree-width, has an interpretation J in a class of trees of bounded height. Although the related automaton $\mathcal{A}$ has a non-elementary number of states in general, our Theorem 4.1 will show that the number of states of $\mathcal{A}$ reachable by the graphs from $\mathcal{C}$ is indeed elementary (in the input formula). The important part of the formulation of Theorem 4.1 is that we can blindly bound the number of reachable states of $\mathcal{A}$, without knowing or changing the kind of "tree-structured" decomposition (cf. the arbitrary interpretation I versus our J in the statement of Theorem 4.1) used in the original algorithmic metatheorems.

For a class of relational structures $\mathcal{S}$, we say that an MSO interpretation I of $\mathcal{S}$ in a class of rooted trees $\mathcal{T}$ is *hereditary* if, roughly saying, subtrees interpret respective induced substructures. Formally, for any $S \in \mathcal{S}$ which is interpreted in $T \in \mathcal{T}$ with the domain $D \subseteq V(T)$, every rooted subtree $T' \subseteq T$ interprets by I a structure S' that is the restriction of S onto $D \cap V(T')$. Our core claim is now formulated as follows.

Theorem 4.1. *Let $\mathcal{S}$ be a hereditary class of binary relational structures, ψ a* CMSO *property over $\mathcal{S}$, and I a hereditary* CMSO *interpretation of $\mathcal{S}$ in a class $\mathcal{T}$ of rooted (labelled) binary trees. If there exists a hereditary* CMSO *interpretation J of $\mathcal{S}$ in the class $\mathcal{U}_d$ of rooted labelled trees of fixed height d, then the following holds: There is a finite deterministic tree*

automaton $\mathcal{A}_{\psi,I}$ accepting the language of those $T \in \mathfrak{T}$ such that $T \models \psi^I$ (ψ under the interpretation I); and the number of states of $\mathcal{A}_{\psi,I}$ is at most

$$exp^{(d+1)}\left(\mathcal{O}(M + |\psi^J| + \iota)^2\right), \tag{4.1}$$

where (i) ψ^J is ψ under the interpretation J, (ii) M is the least common multiple of the b values of all $mod_{a,b}$ predicates occurring in ψ^J, and (iii) ι is a constant depending on $I, \mathfrak{T}$ but not on ψ.

Recall, again, that sole existence of the automaton $\mathcal{A}_{\psi,I}$ follows from Rabin's [21], while the selling point of this claim is an elementary size bound on $\mathcal{A}_{\psi,I}$ for each fixed d. Note also that we intentionally formulate the theorem with *two* interpretations I and J which are not mutually related in any way (even though an existence of I can be easily derived from the existence of J). The purpose of this "separation of I from J" has been explained at the beginning of this section.

Proof. Our approach builds upon the classical Myhill–Nerode regularity tool in automata theory. We actually apply its tree-automata version [16] – we show that the number of classes of the congruence relation on trees defined with respect to ψ^I is bounded as in the statement of the theorem. The existence of the automaton $\mathcal{A}_{\psi,I}$ with the same number of states then follows.

For a CMSO sentence ϕ we define an equivalence relation $\sim_\phi$ on the universe of rooted (labelled) trees as follows. If T and B_2 are rooted trees, and B_1 is a limb of a node $v \in V(T)$, then let $T[B_1 \to B_2]$ denote the tree obtained from $T - V(B_1)$ by attaching the root of B_2 as a child of v. For this proof, we additionally treat also the case of an "improper limb" $B_1 = T$ and then, specially, $T[B_1 \to B_2] = B_2$. It is

$$B_1 \sim_\phi B_2 \qquad \text{if, and only if,} \qquad T \models \phi \iff T[B_1 \to B_2] \models \phi$$

over all rooted trees T such that B_1 is a limb of T or $B_1 = T$.

We are interested in the equivalence classes of $\sim_{\psi^I}$ when restricted to rooted subtrees of the members of $\mathfrak{T}$. Let $S \in \mathcal{S}$ be interpreted by I in a tree $T \in \mathfrak{T}$ (i.e., $S \simeq T^I$), and let B be a limb in T. Let S_1 and S_2 be the induced substructures of S interpreted in B and $T - V(B)$, respectively. Let $U \in \mathcal{U}_d$ be an interpretation of $S \simeq U^J$ under J. Let U_1, U_2 be two disjoint copies of U, and let U_3 be the rooted tree (of height $d+1$) obtained from $U_1 \cup U_2$ by adding a new root as the parent of the former roots of U_1, U_2.

Our first claim is that there exists an assignment of additional labels to U_3 and an interpretation J_3 of S in U_3 such that; (a) J_3 depends only on J and on $I, \mathfrak{T}$ (this will define our ι), (b) $S_i \simeq U_i^{J_3}$ for $i = 1, 2$, and (c) the additional labelling of U_1 within U_3 is independent of $T - V(B)$.

To prove this claim, consider any of the binary relational symbols R over $\mathcal{S}$ and the CMSO formula $\varrho(x, y) \equiv R(x, y)^I$ which interprets R into $\mathfrak{T}$. Already by Rabin's theorem [21]—while simply amended by finite-state $mod_{a,b}$ predicates in ϱ—this $\varrho(x, y)$ has a finite number of equivalence classes with the interpretation of x in $V(B) \cap dom(S)$ and y into $(V(T) \cap dom(S)) \setminus V(B)$ (and the same applies also to $\varrho(y, x)$ interpreting the inverse R^{-1} in case of a non-symmetric relation). Hence there exists an integer m_R depending only on $I, \mathfrak{T}$ such that $V(S_1)$ has a partition $\mathcal{P}_R$ with $|\mathcal{P}_R| \leq m_R$ parts, and $S \models R(x_1, y) \iff S \models R(x_2, y)$ for any two x_1, x_2 from the same part of $\mathcal{P}_R$ and any $y \in V(S_2)$. Importantly, the partition $\mathcal{P}_R$ does not depend at all on S_2 and $T - V(B)$.

We choose $\iota = 2 + \sum_{R \text{ over } \mathcal{S}} (m_R + m_{R^{-1}})$ and assign ι new labels to the nodes of U_3 as follows: For $i = 1, 2$, a separate new label L_i is issued to the whole domain of S_i in U_i. Furthermore, on the domain of S_1 in U_1, every vertex gets one of m_R new labels identifying which part of $\mathcal{P}_R$ it belongs to. This is repeated for all relational symbols R and their inverses R^{-1} over $\mathcal{S}$. On the domain of S_2 in U_2, the label identifying a part $P \in \mathcal{P}_R$ is simply given to all the vertices which are adjacent to P via R. The interpretation J_3 now follows naturally; $U_3 \models R(x,y)^{J_3}$ if, and only if,

- x, y in the domain of S are from the same one of U_1, U_2 —formally, $L_1(x) \wedge L_1(y)$ or $L_2(x) \wedge L_2(y)$ —and $U_3 \models R(x,y)^J$ (since J is hereditary), or
- x, y in the domain of S are from different ones of U_1, U_2 —formally, $L_1(x) \wedge L_2(y)$ or vice versa—and the additional labels in U_3 "encode" $R(x,y)$ as defined previously.

This finishes the proof of the first claim; the existence of J_3. Note, moreover, that $T \models \psi^I \iff S \models \psi \iff U_3 \models \psi^{J_3}$.

Second, we claim that every equivalence class of $\sim_{\psi^I}$ contains a special small representative (not necessarily unique), and so the index of $\sim_{\psi^I}$ cannot be too large. We apply Theorem 3.5 a) to ψ^{J_3} and the tree U_3, precisely to all nodes of U_1 within it. Let U_1 reduce to U_0 by this application(s), and let $U_{30} = U_3[U_1 \to U_0]$. Then $U_3 \models \psi^{J_3} \iff U_{30} \models \psi^{J_3}$. Since J is hereditary, $U_0^{J_3}$ as a restriction of U_0^J is isomorphic to an induced substructure $S_0 \subseteq S_1$. Let $B_0 \subseteq B$ be the subtree giving $dom(S_0)$ in the interpretation I, and let $S_{30} = S - (V(S_1) \setminus V(S_0)) \simeq U_{30}^{J_3}$. Then, again, $B_0^I \simeq S_0$ and $T[B \to B_0]^I \simeq S_{30}$ since I is hereditary. This B_0 will be a representative of the $\sim_{\psi^I}$-class of B.

By the previous, we have got $T \models \psi^I \iff U_{30} \models \psi^{J_3} \iff S_{30} \models \psi \iff T[B \to B_0] \models \psi^I$. This has been so far verified for one particular tree $T \in \mathcal{T}$ having B as its limb. Consider now arbitrary $T' \in \mathcal{T}$ having $B \subseteq T'$ as its limb or $B = T'$, and correspondingly define $S' \in \mathcal{S}$, $S' \simeq T'^I$ and $U' \in \mathcal{U}_d$ such that $S' \simeq U'^J$ by the assumptions. If $S_1', S_2' \subseteq S'$ are the induced substructures interpreted in B and $T' - V(B)$, then $S_1' = S_1$. We take $U_1' := U_1$ and U_2' a disjoint copy of U', and analogously construct U_3' from $U_1' \cup U_2'$ by adding a new root. As in the first claim, and emphasizing the condition (c), we get a labelling of U_3' such that $S' \simeq U_3'^{J_3}$ and $S_i' \simeq U_i'^{J_3}$ for $i = 1, 2$. Again by Theorem 3.5 a), for $U_{30}' = U_3'[U_1' \to U_0]$ it holds $T' \models \psi^I \iff S' \models \psi \iff U_3' \models \psi^{J_3} \iff U_{30}' \models \psi^{J_3} \iff T'[B \to B_0] \models \psi^I$. Consequently, $B \sim_{\psi^I} B_0$ as desired.

Therefore, the number of equivalence classes of $\sim_{\psi^I}$ is at most as large as the number of pairwise non-l-isomorphic $R_{M+q,s,t+3q+s}$-reduced rooted k-labelled trees (B_0) of height $\leq d$, where q, s are the numbers of element and set quantifiers in ψ^{J_3} and t is the number of labels addressed in ψ^{J_3}. This quantity is at most $N_{M+q,s,t+3q+s}(d)$ by Lemma 3.3. Under a very rough estimate, $q, s \leq t + 3q + s \leq \mathcal{O}(|\psi^J| + \iota)$ by the construction of J_3. Using the calculation of (3.4) (for $i + 1 = d$, and again with a broad margin) we get that $N_{M+q,s,t+3q+s}(d) < exp^{(d+1)}\left(\mathcal{O}(M + |\psi^J| + \iota)^2 \right)$. This is an upper bound on the number of states of desired minimal $\mathcal{A}_{\psi,I}$ by the Myhill–Nerode theorem. □

4.2. **Solving extended** MSO **properties.** Unfortunately, direct algorithmic applicability of Corollaries 3.4 and 3.6 is limited to pure decision problems (such as, e.g., 3-colourability), but many practical problems are formulated as optimization ones. The usual way of transforming optimization problems into decision ones does not work for us since the MSO language cannot handle arbitrary numbers.
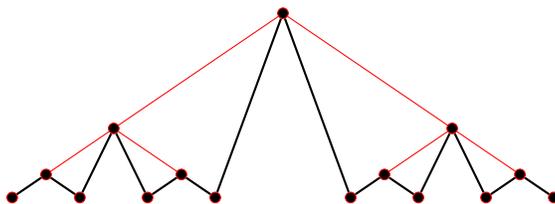
Figure 1: The path of length 14 has tree-depth $3 + 1 = 4$ since it is contained in the closure of the depicted (red) tree of height 3. It can be proved that this is optimal.

Nevertheless, there is a known solution. Arnborg, Lagergren, and Seese [1] (while studying graphs of bounded tree-width), and later Courcelle, Makowsky, and Rotics [5] (for graphs of bounded clique-width), specifically extended the expressive power of MSO logic to define so-called LinEMSO optimization problems. Briefly saying, the LinEMSO language allows, in addition to ordinary MSO expressions, to compare between and optimize over linear evaluational terms.

We follow, for an illustration, a simpler definition of LinEMSO$_1$ given in [5]. Consider any MSO$_1$ formula $\psi(X_1, \ldots, X_p)$ with free set variables, and state the following problem on an input graph G:

$$opt \left\{ f_{lin}(Z_1, \ldots, Z_p) : Z_1, \ldots, Z_p \subseteq V(G), \ G \models \psi(Z_1, \ldots, Z_p) \right\},$$

where opt can be min or max, and f_{lin} is a linear evaluational function. It is

$$f_{lin}(Z_1, \ldots, Z_p) = \sum_{i=1}^{p} \sum_{j=1}^{m} \left(a_{i,j} \cdot \sum_{x \in Z_i} f_j(x) \right)$$

where m and $a_{i,j}$ are (integer) constants and f_j are (integer) weight functions on the vertices of G. Typically f_{lin} is just a cardinality function. Such as,

$$\psi(X) \equiv \forall v, w \left(v \notin X \lor w \notin X \lor \neg \, edge(v, w) \right) \text{ and } \text{``max } |X| \text{''}$$

describes the maximum independent set problem, or

$$\psi(X) \equiv \forall v \exists w \left[v \in X \lor \left(w \in X \land edge(v, w) \right) \right] \text{ and } \text{``min } |X| \text{''}$$

is the minimum dominating set problem.

The algorithms given in [1, 5] for solving such LinEMSO optimization (and enumeration as well) problems are implicitly based on a finite tree automaton associated with the formula ψ in the problem description. Now, Theorem 4.1 can immediately be used to tighten runtime analysis of each of the mentioned algorithms, when the input is restricted to graph classes having not only bounded tree-width or clique-width, respectively, but at the same time being interpretable in a class of trees of fixed height.

This goal first gets us to the following definition:

Definition 4.2 (Tree-depth [19])**.** The *closure* $cl(F)$ of a rooted forest F is the graph obtained from F by adding from each node all edges to its descendants. The *tree-depth* $td(G)$ of a graph G is one more than the smallest height (distance from the root to all leaves) of a rooted forest F such that $G \subseteq cl(F)$.

Note that tree-depth is always an upper bound for tree-width. Some useful properties of it can be derived from the following asymptotic characterization: If L is the length of a

longest path in a graph G, then $\lceil \log_2(L+2) \rceil \le td(G) \le L+1$. See Figure 1. For a simple proof of this, as well as for a more extensive study of tree-depth, we refer the reader to [20, Chapter 6]. Here we need the following:

Lemma 4.3. *Let d be an integer and $\mathcal{R}_d$ denote the class of all rooted d-labelled trees of height d. The class of all graphs of tree-depth at most d has a hereditary MSO_1 interpretation into $\mathcal{R}_{d+1}$.*

Proof. Let $td(G) \le d$ and W be a rooted forest of height d such that $G \subseteq cl(W)$, and let $T \in \mathcal{R}_{d+1}$ be obtained from W by adding a new common root. The intended interpretation identically maps $V(G)$ into $V(T)$. In particular, each vertex quantifier $\exists x \ldots$ is simply replaced with $\exists x. \neg L_0(x) \wedge \ldots$, where L_0 is a special label given to the root of T.

Every vertex v of G at distance $i \ge 1$ from the root of T is given the label L_i. Every vertex v of G, such that there exists an ancestor u of v in T and $uv \in E(G)$, is also given the label L_j where $1 \le j < i$ and j is the distance of u from the root in T. Note that $L_i(x) \wedge \bigwedge_{j>i} \neg L_j(x)$ is true iff x is at the distance i from the root. It is a routine to express the edge relation η of G as follows: $\eta(x,y) \equiv x \ne y \wedge (\alpha(x,y) \vee \alpha(y,x))$ where

$$\alpha(x,y) \equiv \mathrm{ancest}(x,y) \wedge \bigvee_{i=1,\ldots,d} \big[L_i(x) \wedge L_i(y) \wedge \bigwedge_{j>i} \neg L_j(x) \big].$$

Here $\mathrm{ancest}(x,y)$ means that x is an ancestor of y (the transitive closure of the parental relation in T). This is clearly MSO_1-expressible (in fact, even FO-expressible on trees of bounded height). $\square$

Corollary 4.4. *Let $\mathcal{C}$ be a class of graphs of bounded tree-depth. Then every $\mathrm{LinEMSO}_2$ problem $\mathcal{P}$ can be solved on $\mathcal{C}$ by a linear-time fpt algorithm with an elementary runtime dependence on $\mathcal{P}$. Then same holds also if CMSO_2 is allowed in the description of $\mathcal{P}$.*

Proof. Let the definition of $\mathcal{P}$ be based on an CMSO formula ψ (for simplicity, ψ can be thought of an CMSO_1 formula thanks to Theorem 2.5). Then, by Lemma 4.3, Theorem 4.1 applies here and the runtime analysis of [1] can be tightened to an elementary bound (4.1) of Theorem 4.1. $\square$

In exactly the same way we can claim an analogous statement for $\mathrm{LinEMSO}_1$:

Corollary 4.5. *Let $\mathcal{C}$ be a class of graphs of bounded clique-width, and assume that $\mathcal{C}$ has a hereditary MSO_1 interpretation in a class of trees of fixed height. Then every $\mathrm{LinEMSO}_1$ problem $\mathcal{P}$ can be solved on $\mathcal{C}$ by a linear-time fpt algorithm with an elementary runtime dependence on $\mathcal{P}$, provided a clique-width expression for the input graph is given. Then same holds also if CMSO_1 is allowed in the description of $\mathcal{P}$.* $\square$

In relation to Corollary 4.5 it becomes interesting to ask what graph classes have an MSO_1 interpretation in a class of trees of fixed height. The answer [14] is mentioned in another context later in Section 5.2 (Definition 5.4).

5. Consequences for Expressive Power of FO

A non-algorithmic straightforward corollary of Theorem 3.2 is the fact that FO and MSO logic can express the same collection of properties on classes of trees of bounded height; they have equal *expressive power*. Formally:

Proposition 5.1. *Let h, t be integers. If ϕ is an MSO sentence, then there is an FO sentence $\psi_{h,t}$ such that, for any rooted t-labelled tree T of height at most h, it is $T \models \phi \iff T \models \psi_{h,t}$.*

Proof. Let ϕ be an MSO sentence with q element quantifiers and s set quantifiers. By Theorem 3.2, there is a finite set $\mathcal{U}_\phi$ of pairwise non-l-isomorphic $R_{q,s,t+3q+s}$-reduced trees W such that $W \models \phi$, and $T \models \phi$ if and only if the $R_{q,s,t+3q+s}$-reduction of T is l-isomorphic to a member of $\mathcal{U}_\phi$.

We write an FO sentence $\psi_{h,t} \equiv \exists x.\, root(x) \wedge \bigvee_{W \in \mathcal{U}_\phi} \tau_W(x)$. The intended meaning of τ_W is that $T \models \tau_W(r)$ where $r \in V(T)$ if, and only if, the subtree $T_r \subseteq T$ induced on r and all of its descendants reduces, up to l-isomorphism, to W. Assuming existence of τ_W for a moment, we see that $T \models \phi \iff T \models \psi_{h,t}$.

We build τ_W recursively by induction on the height of W. For height zero, i.e. when W is a single vertex, $\tau_W(x)$ simply tests the correct label of x and that x has no children. Now let W be of height $h > 0$, with the root w and its limbs $W_{i,j}$ where $i = 1, \ldots, a$ and $j = 1, \ldots, b_i$, such that all $W_{i,j}$ for $j = 1, \ldots, b_i$ are l-isomorphic to the same U_i, and U_i for $i = 1, \ldots, a$ are pairwise non-l-isomorphic. Let $\mathcal{S}$ denote the set of those U_i for which $b_i = R_{q,s,t+3q+s}(h-1)$ (the threshold in Theorem 3.2).

To conclude the proof, we set

$$
\begin{aligned}
\tau_W(x) \quad \equiv \quad & \exists\, (y_{i,j} : i = 1, \ldots, a,\ j = 1, \ldots, b_i) \, \Big[\bigwedge_{i,j} parent(x, y_{i,j}) \wedge \\
& \wedge \bigwedge_{i,j,i',j'} y_{i,j} \neq y_{i',j'} \wedge \bigwedge_{i,j} \tau_{U_i}(y_{i,j}) \wedge \\
& \wedge \Big(\forall z.\ parent(x, z) \rightarrow \Big(\bigvee_{i,j} z = y_{i,j} \vee \bigvee_{U_i \in \mathcal{S}} \tau_{U_i}(z) \Big) \Big) \Big] \,,
\end{aligned}
$$

meaning that; (1) among the limbs of x in T there exist pairwise distinct ones such that, when recursively reduced, they are in a one-to-one l-isomorphism correspondence to the limbs of w in W; and (2) all the other limbs of x in T reduce to ones l-isomorphic to some U_i reaching the reduction threshold above. $\qquad\square$

The purpose of this section is to investigate generalizations of Proposition 5.1 to richer graph classes.

5.1. **Case of bounded tree-depth.**

Elberfeld, Grohe, and Tantau [11] proved that FO and MSO_2 have equal expressive power on the graphs of bounded tree-depth—Theorem 5.2. Having Theorem 3.2 at hand, we can provide a relatively simple alternative proof of this result along the construction from Proposition 5.1. Though, in this section we take a different route, which might look unnecessarily complicated at the first sight, but which allows for a smooth extension to a new result about the expressive power of FO and MSO_1 in Section 5.3. Along this route we introduce a sophisticated combinatorial tool, namely well-quasi-ordering, in the logical context.

Theorem 5.2 (Elberfeld, Grohe, and Tantau [11]). *Let $\mathcal{D}$ denote a class of graphs of bounded tree-depth (Definition 4.2). Then FO and MSO_2 have the same expressive power on $\mathcal{D}$.*

Proof. Let ψ_2 be an MSO_2 sentence. Since MSO_1 has the same expressive power as MSO_2 on graphs of bounded tree-width (by Theorem 2.5), we may as well consider an equivalent

MSO$_1$ sentence ψ. Our alternative proof can be then outlined in three steps (the first two of which are analogous to Proposition 5.1):

(I) By Lemma 4.3, there is a hereditary MSO$_1$ interpretation J of $\mathcal{D}$ in the class $\mathcal{U} = \mathcal{U}_d$ of rooted labelled trees of height at most d, for some integer constant d. By Theorem 3.2, there is a finite set $\mathcal{U}_0 \subseteq \mathcal{U}$ of (f-reduced) kernels; every $T \in \mathcal{U}$ f-reduces to an "easily definable" $T_0 \in \mathcal{U}_0$ such that $T \models \psi^J \iff T_0 \models \psi^J$ (where the reduction threshold f depends only on ψ, J, d and we will simply say "*reduces*" in this proof).

(II) For $G \in \mathcal{D}$, hence, $G \models \psi$ is equivalent to saying that $G \simeq T_G^J$ for a tree $T_G \in \mathcal{U}$ such that T_G reduces to a tree in $\mathcal{U}_0^{\psi,J} = \{ U \in \mathcal{U}_0 : U \models \psi^J \}$ (a disjunction of finitely many cases over the members of $\mathcal{U}_0^{\psi,J}$).

(III) A problem is that T_G is only implicit and we cannot directly address T_G (and reducibility of it to $U \in \mathcal{U}_0^{\psi,J}$) from within the resulting FO formula over G. To resolve this problem, we consider a related hereditary property over G which is characterized by finitely many obstacles by Theorem 2.2. We use it to build the desired FO sentence expressing over G that (some) implicit T_G reduces to a particular reduced tree $U \in \mathcal{U}_0^{\psi,J}$.

The rest of the proof will give the details of crucial step (III).

Let $T \in \mathcal{U}$, and let $\ell(T)$ denote the domain (vertex set) of T^J. We say that a graph $H \in \mathcal{D}$ is T-*coloured* (with respect to implicit J) if H is associated with an injective mapping from $\ell(T)$ into $V(H)$. When dealing with such a coloured graph H we automatically consider a subgraph relation preserving these colours. The key definition is now that of consistency with a T-colouring:

Assume a T-coloured graph $G \in \mathcal{D}$. We say that G is *consistent with its T-colouring* (shortly *consistent*) if the following holds:

- there exists an induced supergraph $H \supseteq_i G$, $H \in \mathcal{D}$ and a tree $T_H \in \mathcal{U}$ such that $H \simeq T_H^J$, the isomorphism mapping of $T_H^J \simeq H$ restricted to $\ell(T)$ is the given T-colouring of G, and
- $T \subseteq T_H$ (sharing the root with T) and T_H reduces to T.

The definition automatically gives that the property of consistency with T is hereditary (closed under induced colour-preserving subgraphs) on the universe of T-coloured graphs from $\mathcal{D}$. Therefore, by Theorem 2.2 and Proposition 2.1, there is a finite set $Obst(T)$ of T-coloured graphs such that T-coloured G is consistent if and only if G has no induced subgraph isomorphic to a member of $Obst(T)$. Consequently, this property can be expressed by an FO formula $consistent_T$.

We also use one special property of the interpretation J from Lemma 4.3, that J is "tree-ordered", meaning that an edge uv in the graph T^J may exist only if the interpretation of u in T is an ancestor of that of v or vice versa. This property, informally, will allow us to simply identify limbs of T within T^J using connectivity: if T_1, T_2 are two disjoint limbs in T, then there is no edge between $\ell(T_1)$ and $\ell(T_2)$ in the graph T^J.

Now we use the previous to describe the property that $G \in \mathcal{D}$ *reduces to* $U \in \mathcal{U}_0$, i.e., that there exists $T_G \in \mathcal{U}$ such that $G \simeq T_G^J$ and T_G reduces to U. (Recall that if G reduces to U, then $G \models \psi \iff U \models \psi^J$.) Considering the restriction of the mapping $T_G^J \simeq G$ to $\ell(U)$ we say that G reduces to U *respecting this U-colouring* of G. We introduce the following shorthand notation:

- For a tree $T \in \mathcal{U}$, let $\widehat{x_T}$ denote a collection of variables indexed by the elements of $\ell(T)$, i.e., $\widehat{x_T} = (x_v : v \in \ell(T))$. Let $\widehat{x_T} \cup \widehat{x_{T'}}$ stand for a union of two such collections and $z \in \widehat{x_T}$ mean that $z = x_v$ for some member x_v of $\widehat{x_T}$.
- Let $\mathcal{L}(T)$ be the set of all non-l-isomorphic limbs $B \subsetneq T$ that reach the considered reduction threshold, i.e., if B is of node $v \in V(T)$ and of height i, then at least $f(i) - 1$ other limbs of v in T are l-isomorphic to B. For $B \in \mathcal{L}(T)$ we denote by $T \odot B$ a tree such that $T \odot B$ reduces in one step to T, i.e., $T \odot B$ results by adding a disjoint copy of B as a sibling of B into T.
- We write $G \models consistent_T(\widehat{x_T})$ in FO to mean that G is consistent with the T-colouring given by an assignment of $\widehat{x_T}$.
- We write $G \models connected(\widehat{x_T}, y, z)$ to mean that $y, z \notin \widehat{x_T}$ and y and z are connected in G by a path avoiding all $\widehat{x_T}$. Although connectivity is not in FO in general, our formula $connected$ is FO over $\mathcal{D}$ since graphs of bounded tree-depth are hereditary and have bounded diameter as well [20].

Let $T_1 \in \mathcal{U}$, and let T_0 be a limb of T_1 or $T_0 = T_1$. We write in FO that $G \models reduce_{T_1,T_0}(\widehat{x_{T_1}}, y)$ with the following intended (and so far informal) meaning: There exists a graph H and a tree $T_H \in \mathcal{U}$ such that $T_H \supseteq T_1$, and

- $G \subseteq_i H \simeq T_H^J$, the elements of $\ell(T_1)$ coincide with the assignment of $\widehat{x_{T_1}}$ in G, and T_H reduces to T_1,
- the connected component of $G - \widehat{x_{T_1}}$ containing y recursively reduces (within the reduction of T_H onto T_1) to T_0.

This is formally written down as follows

$$reduce_{T_1,T_0}(\widehat{x_{T_1}}, y) \equiv \bigvee_{T_2 \in \mathcal{L}(T_0)} \exists \widehat{z_{T_2}}. \, \mathbf{\Psi}_{T_1,T_0,T_2} \tag{5.1}$$

where

$$\mathbf{\Psi}_{T_1,T_0,T_2} \equiv \bigwedge_{v \in V(T_2)} connected(\widehat{x_{T_1}}, y, z_v) \wedge \tag{5.2}$$

$$consistent_{T_1 \odot T_2}(\widehat{x_{T_1}} \cup \widehat{z_{T_2}}) \wedge$$

$$\forall y' \left[(connected(\widehat{x_{T_1}}, y, y') \wedge y' \notin \widehat{z_{T_2}}) \rightarrow \right.$$

$$\left. reduce_{T_1 \odot T_2, T_2}(\widehat{x_{T_1}} \cup \widehat{z_{T_2}}, y') \right], \tag{5.3}$$

except that for T_2 of height 0 (i.e., T_2 a single node, hence singleton $\widehat{z_{T_2}}$!) it is

$$\mathbf{\Psi}_{T_1,T_0,T_2} \equiv y \in \widehat{z_{T_2}} \wedge \forall y'(connected(\widehat{x_{T_1}}, y, y') \rightarrow y' = y) \wedge$$

$$consistent_{T_1 \odot T_2}(\widehat{x_{T_1}} \cup \widehat{z_{T_2}}). \tag{5.4}$$

Note that in (5.1) the only role of y is to be a representative of a connected component of $G - \widehat{x_{T_1}}$; the outcome of $G \models reduce_{T_1,T_0}(\widehat{x_{T_1}}, y)$ is invariant upon the choice of y from the same component. Task (III) is now finished with

$$reduce\text{-}to_U \equiv \exists \widehat{x_U} \left\{ consistent_U(\widehat{x_U}) \wedge \forall z \, (z \in \widehat{x_U} \vee reduce_{U,U}(\widehat{x_U}; z)) \right\}. \tag{5.5}$$

Assume that $G \in \mathcal{D}$ reduces to $U \in \mathcal{U}_0$. Then, following the steps of this reduction, it is a simple routine to verify that $G \models reduce\text{-}to_U$ (one always chooses $H = G$ when speaking about consistency). On the other hand, assume (5.5) $G \models reduce\text{-}to_U$. Fix a satisfying assignment of $\widehat{x_U}$ into $V(G)$ as the U-colouring of G. If $|\ell(U)| = |V(G)|$, then $G \simeq U^J$ by $G \models consistent_U(\widehat{x_U})$. Otherwise, for every connected component P_0 of $G - \widehat{x_U}$ choose

$p \in P_0$; hence (5.5) $G \models reduce_{U,U}(\widehat{x_U}; p)$. We now analyze the recursive definition (5.1) of $reduce_{U,U}$, aiming to show the following claim: the induced subgraph $G_1 = G[\widehat{x_U} \cup P_0]$ reduces to U respecting the U-colouring induced by $\widehat{x_U}$. If this is true for each component P_0 then, clearly, G itself reduces to U.

Let initially $U_0 = W_0 = U$, and set $W_1 = T_2$ where $T_2 \in \mathcal{L}(W_0)$ of height > 0 is a satisfying branch (any one) of the "big" disjunction in (5.1) and fix a satisfying assignment of $\widehat{z_{W_1}}$ into P_0 thereafter. Let $U_1 = U_0 \odot W_1$ and (with a negligible abuse of notation) $\widehat{x_{U_1}} = \widehat{x_{U_0}} \cup \widehat{z_{W_1}}$. If $|\ell(W_1)| = |P_0|$, then we are immediately done with the claim by $consistent_{U_1}(\widehat{x_{U_1}})$ in (5.2): according to the hereditary property of consistency it is $G_1 \simeq U_1^J$ and '$U_1 \to U_0$' is a valid reduction step. Otherwise, recursively for every connected component P_1 of $G_1 - \widehat{x_{U_1}}$ (which is one of components of $G - \widehat{x_{U_1}}$, too) we choose $q \in P_1$; hence (5.3) $G \models reduce_{U_1,W_1}(\widehat{x_{U_1}}; q)$.

In each recursion branch (say, by DFS on the recursion tree) we continue with the previous argument at "depth" $i \geq 1$: we analyze (5.1) $G \models reduce_{U_i,W_i}(\widehat{x_{U_i}}; q)$ over every connected component $P_i \ni q$ of $G_i - \widehat{x_{U_i}}$, denote by $G_{i+1} = G[\widehat{x_{U_i}} \cup P_i]$, and analogously obtain W_{i+1} ($= T_2 \in \mathcal{L}(W_i)$ from a satisfying branch) and $U_{i+1} = U_i \odot W_{i+1}$, with an assignment of $\widehat{z_{W_{i+1}}}$ into P_i. This recursion is finite since the height of W_{i+1} strictly decreases and then generally terminates at (5.4) for W_{i+1} of height 0. In this terminal case we get by (5.4) for $\boldsymbol{\Psi}_{U_i,W_i,W_{i+1}}$ that the assignment of $\widehat{z_{W_{i+1}}}$ is q and $\{q\} = P_i$, and so $U_{i+1}^J \simeq G_{i+1}$ by $consistent_{U_{i+1}}$. This finishes the proof since U_{i+1}, by the construction, reduces to U. $\qquad\square$

Notice how much of the previous proof is very general, using just the fact that the graph class $\mathcal{D}$ has *some* hereditary MSO_1 interpretation J in the class $\mathcal{U}$. Interestingly, well-quasi-ordering of $\mathcal{D}$ under induced subgraphs can be derived already from this assumption and Theorem 2.2. There is, however, one technical point in the proof which heavily depends on special properties of the interpretation J—it is the use of predicate $connected(\widehat{x_T}, y, z)$ in the proof of Theorem 5.2. This is based on a quite unique property of tree-depth, and before attempting to give a similar result for other graph classes interpretable in $\mathcal{U}$, we have to find a suitable replacement property in the next section.

5.2. **Tree-models and their properties.** Graph classes of bounded shrub-depth have been introduced recently in [14] as those having MSO_1 interpretations in the class(es) of rooted labelled trees of fixed height. Equivalently, they are defined by a very special kind of a hereditary interpretation:

Definition 5.3 (Tree-model [14]). We say that a graph G has a *tree-model of m colours and depth d* if there exists a rooted tree T (of height d) such that
  i. the set of leaves of T is exactly $V(G)$,
 ii. the length of each root-to-leaf path in T is exactly d,
iii. each leaf of T is assigned one of m colours (T is *m-coloured*),
 iv. and the existence of a G-edge between $u, v \in V(G)$ depends solely on the colours of u, v and the distance between u, v in T.

The class of all graphs having such a tree-model is denoted by $\mathcal{TM}_m(d)$.

Notice that, in (iv..) of the definition, the existence of an edge uv depends on finite information. For example, $K_n \in \mathcal{TM}_1(1)$ or $K_{n,n} \in \mathcal{TM}_2(1)$. Definition 5.3 is further
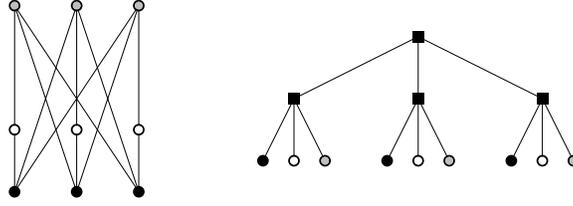
Figure 2: The graph obtained from $K_{3,3}$ by subdividing a matching belongs to $\mathcal{TM}_3(2)$.

illustrated in Figure 2. It is easy to see that each class $\mathcal{TM}_m(d)$ is closed under complements and induced subgraphs, but neither under disjoint unions, nor under subgraphs.

Definition 5.4 (Shrub-depth [14]). A class of graphs $\mathcal{S}$ has *shrub-depth* d if there exists m such that $\mathcal{S} \subseteq \mathcal{TM}_m(d)$, while for all natural m it is $\mathcal{S} \not\subseteq \mathcal{TM}_m(d-1)$.

Note that Definition 5.4 is asymptotic as it makes sense only for infinite graph classes; the shrub-depth of a single finite graph is always at most one (0 for empty or one-vertex graphs). For instance, the class of all cliques has shrub-depth 1. Similarly, although Definition 5.3 does not explicitly specify rules for the existence of edges (iv..), Definition 5.4 suggests a natural associated FO interpretation J of the class $\mathcal{S}$:

Definition 5.5 (Shrub interpretation). For a graph class $\mathcal{S}$ of shrub-depth d, a *shrub interpretation* of $\mathcal{S}$ in the class $\mathcal{U}$ of rooted labelled trees of height d is a hereditary interpretation J satisfying the following: for $G \in \mathcal{S}$ with a tree-model T of depth d, it is $G \simeq T_1^J$ where T_1 inherits T with all the leaf colours, and each leaf v of T_1 is additionally equipped with all labels of the form (i, c) where c is the T-colour of a vertex u adjacent to v in G such that the distance between u, v in T is $2i$.

Lemma 5.6. *A shrub interpretation is* FO *definable for each fixed* d. $\square$

For more relations of shrub-depth to other established concepts such as cographs or clique-width we refer the reader to [14]. Here we just summarize:

Proposition 5.7 ([14]). *Let* $\mathcal{G}$ *be a graph class and* d *an integer. Then:*
a) *If* $\mathcal{G}$ *is of tree-depth* $\leq d$, *then* $\mathcal{G}$ *is of shrub-depth* $\leq d$.
b) *If* $\mathcal{G}$ *is of bounded shrub-depth, then* $\mathcal{G}$ *is of bounded clique-width.* $\square$

Proposition 5.8. *(See also [14] for a more general statement.) Let* $\mathcal{S}$ *be a graph class of bounded shrub-depth. Assume the graphs of* $\mathcal{S}$ *are arbitrarily coloured from a finite set of colours. Then* $\mathcal{S}$ *is well-quasi-ordered under the colour-preserving induced subgraph order.*

Proof. Consider an infinite sequence $(G_1, G_2, \dots) \subseteq \mathcal{S}$, and the corresponding tree-models $(T_1, T_2, \dots)$. Let T_i^+, $i = 1, 2, \dots$, denote the rooted tree with leaf labels composed of the colours of T_i and the colours in G_i. By Theorem 2.2, $T_1^+, T_2^+, \dots$ of bounded diameter are WQO under rooted coloured subtree relation, and, consequently, so are the coloured graphs $G_1, G_2, \dots$, as desired. $\square$

For the rest of this section we will focus on the shrub interpretation J associated with a graph class $\mathcal{S}$ of bounded shrub-depth. This allows us to smoothly adapt the notions introduced during the proof of Theorem 5.2. In particular, $\ell(T)$ (the domain of T^J) is now the set of leaves of T and the notion of a T-colouring corresponds to that. We also

literally adopt the definition of $G \in \mathcal{S}$ being *consistent with its T-colouring*. Thanks to Proposition 5.8, this property can be expressed by an FO formula *consistent*$_T$ (depending on $\mathcal{S}$), too.

Though, we need a bit stronger and crucial property of separability for T. It deals with a $(T, red, blue)$-*coloured* graph which is a T-coloured graph in which two other (non-T-coloured) vertices are assigned colours *red* and *blue*. When the *red* vertex is x and the *blue* one is y in a graph H, then we call this also a (T, x, y)-*colouring* of H.

Definition 5.9 (Separability for $(T, red, blue)$-coloured)**.** Let $\mathcal{S}$ be a class of bounded shrub-depth with a shrub interpretation J in $\mathcal{U}$ (cf. Definition 5.5). Assume a $(T, red, blue)$-coloured graph G where r, b denote the vertices of colours *red* and *blue*, respectively. We say that G is *separating* (r from b, implicitly) *for T* if there exists an induced supergraph $H \supseteq_i G$ and a tree $T_H \in \mathcal{U}$, $H \simeq T_H^J$, such that the following hold:

- $T \subseteq T_H$ (sharing the root with T), the isomorphism mapping of $T_H^J \simeq H$ restricted to $\ell(T)$ is the given T-colouring of G, and T_H reduces to T;
- the least common ancestor of the nodes interpreting r, b within T_H belongs to $V(T)$.

Note; separability clearly implies consistency with T. And again by Proposition 5.8, the separability property can be expressed with an FO formula *separable*$_T(\widehat{x_T}, r, b)$ over the universe of (T, r, b)-coloured graphs from $\mathcal{S}$.

To make practical use of the (generally vague) separability property, we have to restrict our domain to so called unsplittable tree-models, as follows.

Assume T is a tree-model of a graph G, and B is a limb of $v \in V(T)$, such that W is the set of leaves of B. We say that a tree-model T' is obtained from T by *splitting B along $X \subseteq W$* if a disjoint copy B' of B with the same parent v is added into T, and then B is restricted to all its root-to-leaf paths ending in $W \setminus X$ while B' is restricted to all such paths ending in X' (the corresponding copy of X). A tree-model T is *splittable* if some limb in T can be split along some subset X, making a tree-model T' which represents the same graph G as T does. A tree-model is *unsplittable* if it is not splittable. Notice that any tree-model can be turned into an unsplittable one; simply since the splitting process *must end* eventually.

An *unsplittable shrub interpretation J* is an interpretation satisfying Definition 5.5 with the target class $\mathcal{U}$ containing only trees of unsplittable tree-models. We also implicitly assume that the threshold function f in the definition of 'f-reduce' is always at least 2. Then we claim:

Lemma 5.10. *Assume an* unsplittable *shrub interpretation J in $\mathcal{U}$. For a graph $G \simeq T_G^J$ let $T_G \in \mathcal{U}$ reduce to T_0, and associate G with this T_0-colouring. Let $L_1, \ldots, L_k$ denote the leaf sets of all k subtrees in $T_G - V(T_0)$. If $\sim$ is the binary relation on $V(G) \setminus \ell(T_0)$ defined as $x \sim y$ iff the corresponding (T_0, x, y)-colouring of G is* not *separating, then $\sim$ is a relation of equivalence whose classes coincide with the sets $L_1, \ldots, L_k$.*

Before giving a proof, we add the following immediate corollary which may be interesting on its own (though not used here).

Corollary 5.11. *If J is unsplittable, then the partition $L_1, \ldots, L_k$ from Lemma 5.10 is unique for given G and T_0 (regardless of T_G).*   □
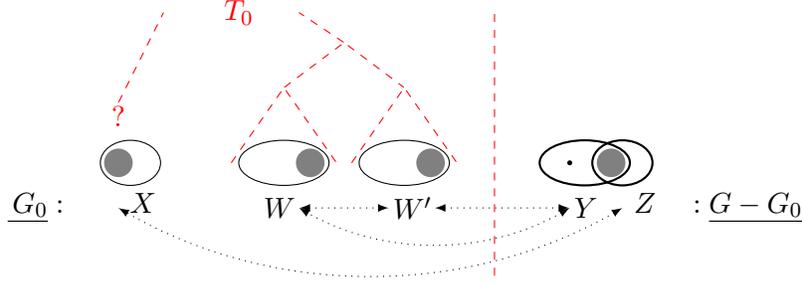
Figure 3: A situation which cannot happen, in a graph G with an unsplittable tree-model T_0 of an induced subgraph $G_0 \subseteq G$, and with the sets W, W', Y and X, Z as in Lemma 5.13.

Remark 5.12. Note that it is *not possible* to simply relate which equivalence class of $\sim$ corresponds to which subtree of $T_G - V(T_0)$. This is since the definition of separability deals with a supergraph $H \supseteq_i G$ and then a tree $T_H \supsetneq T_G$ may reduce to T_0 in a different way than T_G does.

Proof of Lemma 5.10. Suppose that x, y belong to distinct ones of the sets $L_1, \ldots, L_k$. Then already $H = G$ in Definition 5.9 witnesses that the (T_0, x, y)-coloured graph G is separating, and so $x \not\sim y$.

In the other direction, we aim for a contradiction. Two vertices p, q are called *twins* with respect to a set R if the neighbourhood of p in R equals that of q in R. Moreover, $P, Q \subseteq V(G)$ are *twin sets* in G if there is a bijection $h : P \to Q$ such that h is an isomorphism of $G[P]$ onto $G[Q]$, and $p, h(p)$ are twins wrt. $V(G) \setminus (P \cup Q)$ for each $p \in P$.

We assume (up to symmetry) that $x, y \in L_1$, but the (T_0, x, y)-coloured graph G is separating for T_0. Hence there is a graph $H \supseteq_i G$ and a tree $T_H \supseteq T_0$, $H \simeq T_H^J$, such that the least common ancestor of x, y within T_H belongs to $V(T_0)$. If $K_1, \ldots, K_{k'}$ denote the sets of leaves of all k' subtrees in $T_H - V(T_0)$ then, say, $x \in K_1$ but $y \notin K_1$. We summarize what this means according to our definitions:

i. Since T_G reduces to T_0, the set L_1 is the leaf set of a subtree U_1 in $T_G - V(T_0)$, and this U_1 is reducible to U_1' such that U_1' is isomorphic to (at least) two disjoint sibling limbs B, B' in T_0. Clearly, we may assume that x, y are in U_1'. We consider the induced subgraph $G' = G - (V(U_1) \setminus V(U_1'))$, and $L_1', M, M' \subseteq V(G')$ the leaf sets of U_1', B, B', respectively. Then L_1', M, M' are pairwise twin sets in G' by Definition 5.3.

ii. The same as in (i..) can be claimed for H, T_H, and K_1 in place of G, T_H, and L_1; giving us sets (pairwise twin) $K_1' \subseteq K_1$ and $N, N' \subseteq \ell(T_0)$ in a suitable induced subgraph of H. We may similarly assume $x \in K_1'$. Moreover, since N, N' are the leaf sets of some limbs in T_0, too, each of N, N' may either be disjoint or in an inclusion with each of M, M'. Consequently, up to symmetry, N can be assumed disjoint from M'. In the graph G' we, in particular, have that for every vertex $u \in K_1' \cap V(G')$ there is $w \in N$ such that u, w are twins wrt. $(L_1' \cup M \cup M') \setminus (K_1' \cup N)$.

Now, we get a contradiction (to being unsplittable) by applying next Lemma 5.13 with $G := G'$, $W := M$, $W' := M'$, $Y := L_1'$, $X := N$, $Z := K_1' \cap V(G')$. $\square$

Lemma 5.13. *Let G be a graph, and T_0 be a tree-model of an induced subgraph $G_0 \subseteq_i G$. Let T_0 contain two (disjoint) isomorphic limbs B, B' of a node v, and $W, W' \subseteq V(G_0)$ be the sets of leaves of B, B', respectively. Let $X \subseteq V(G_0)$ be a set disjoint from W'. If there exist sets $Y, Z \subseteq V(G) \setminus V(G_0)$ such that*

a) *W, W', Y are pairwise twin sets in G_0,*
b) *for each $z \in Z$ there is some $h(z) \in X$ such that $z, h(z)$ are twins with respect to $(W \cup W' \cup Y) \setminus (Z \cup X)$,*
c) *$Y \neq Y \cap Z \neq \emptyset$,*

then the tree model T_0 is splittable.

The statement is illustrated in Figure 3.

Proof. Our aim is to prove that, assuming a)–c) to be true, the limb B in T_0 can be split along the set $W_Z \subseteq W$ which in the twin set relation (a) corresponds to $Y \cap Z$ (and analogously for B'). We take arbitrary $z \in Y \cap Z$ and $y \in Y \setminus Z$, and denote by $w, t \in W$ and $w', t' \in W'$ the vertices such that w, w' correspond to y and t, t' to z in the twin set relation (in particular, $t \in W_Z$). By Definition 5.3, only the edges between W_Z and $W \setminus W_Z$ in G_0 are potentially affected by the splitting operation on B, and so it is enough to show that $wt \in E(G_0)$ iff $w't \in E(G_0)$ to get the desired conclusion.

Assume $wt \in E(G_0)$. Then $yz \in E(G)$ by (a) for the twin pair W, Y. Since $y \notin Z \cup X$, from (b) and $yz \in E(G)$ we get $yh(z) \in E(G)$. Then, $h(z) \in X$ implies $h(z) \notin W'$ which is disjoint from X, and so from (a) for the twin pair W', Y it follows $w'h(z) \in E(G)$. Furthermore, $w' \notin Z \cup X$, and hence $w'z \in E(G)$ from (b). Finally, $w't \in E(G)$ by (a) for the twin pair W, Y, and $w't \in E(G_0)$. In the exactly same way, $wt \notin E(G_0)$ implies $w't \notin E(G_0)$. $\square$

5.3. **Case of bounded shrub-depth.** We get to the main new result of Section 5:

Theorem 5.14. *Let $\mathcal{S}$ denote any class of graphs of bounded shrub-depth (Definition 5.4). Then* FO *and* MSO$_1$ *have the same expressive power on $\mathcal{S}$.*

Proof. Our proof follows the same steps as that of Theorem 5.2.

(I) Let J be a shrub interpretation (Lemma 5.6) of $\mathcal{S}$ in the class $\mathcal{U}$ of rooted labelled trees of height $\leq d$, for some integer constant d. By implicitly restricting $\mathcal{U}$ we may assume it is unsplittable. By Theorem 3.2, there is a finite set $\mathcal{U}_0 \subseteq \mathcal{U}$ of reduced kernels; every $T \in \mathcal{U}$ reduces to an "easily definable" $T_0 \in \mathcal{U}_0$ such that $T \models \psi^J \iff T_0 \models \psi^J$.

(II) For $G \in \mathcal{S}$, hence, $G \models \psi$ is equivalent to saying that $G \simeq T_G^J$ for a tree $T_G \in \mathcal{U}$ such that T_G reduces to a tree in $\mathcal{U}_0^{\psi, J} = \{U \in \mathcal{U}_0 : U \models \psi^J\}$.

(III) It remains to build a desired FO sentence expressing over G that (some) implicit unsplittable T_G reduces to a particular unsplittable tree $U \in \mathcal{U}_0^{\psi, J}$.

The rest of the proof will again give the details of crucial step (III). Using the tools developed in Section 5.2 this is now a relatively easy task:

$$reduce_{T_1, T_0}(\widehat{x_{T_1}}, y) \equiv sep\text{-}is\text{-}equivalence_{T_1}(\widehat{x_{T_1}}) \wedge \bigvee_{T_2 \in \mathcal{L}(T_0)} \exists \widehat{z_{T_2}}. \; \boldsymbol{\Psi}_{T_1, T_0, T_2}$$

where

$$\boldsymbol{\Psi}_{T_1,T_0,T_2} \equiv \bigwedge_{v \in V(T_2)} \neg\, separable_{T_1}(\widehat{x_{T_1}}, y, z_v) \land$$

$$consistent_{T_1 \odot T_2}(\widehat{x_{T_1}} \cup \widehat{z_{T_2}}) \land$$

$$\forall y' \left[(\neg\, separable_{T_1}(\widehat{x_{T_1}}, y, y') \land y' \notin \widehat{z_{T_2}}) \rightarrow \right.$$

$$\left. reduce_{T_1 \odot T_2, T_2}(\widehat{x_{T_1}} \cup \widehat{z_{T_2}}, y') \right],$$

except that for T_2 of height 0 (i.e., T_2 a single node, hence singleton $\widehat{z_{T_2}}$) it is

$$\boldsymbol{\Psi}_{T_1,T_0,T_2} \equiv y \in \widehat{z_{T_2}} \land \forall y'(\neg\, separable_{T_1}(\widehat{x_{T_1}}, y, y') \rightarrow y' = y) \land$$

$$consistent_{T_1 \odot T_2}\left(\widehat{x_{T_1}} \cup \widehat{z_{T_2}}\right).$$

Here '$sep\text{-}is\text{-}equivalence_T(\widehat{x_T})$' asserts that the binary relation defined by '$separable_T(\widehat{x_T}, \cdot, \cdot)$' is reflexive, symmetric, and transitive on $V(G) \setminus \widehat{x_T}$, which has a routine FO expression. Task (III) is now finished with

$$reduce\text{-}to_U \equiv \exists \widehat{x_U} \left\{ consistent_U(\widehat{x_U}) \land \forall z \, (z \in \widehat{x_U} \lor reduce_{U,U}(\widehat{x_U}; z)) \right\}.$$

We finally claim that $G \in \mathcal{S}$ reduces to $U \in \mathcal{U}_0$ if, and only if, $G \models reduce\text{-}to_U$. Under Lemma 5.10 this is just a plain repetition of the arguments in the proof of Theorem 5.2. $\square$

6. Conclusions

Even though our prime motivation was to provide an algorithmic improvement over Courcelle's theorem on "low-depth" variants of tree-width and clique-width, the main importance of our results probably lies in the fact that they provide deeper understanding of MSO logic on graph classes interpretable in trees of bounded height. This understanding already played crucial role in the proofs establishing that FO logic has equal expressive power as MSO on these graph classes.

We believe that our results and techniques for obtaining them (especially the use of well-quasi-ordering) could lead to new results in the future. Above all we suggest that there is likely no major obstacle to suitable extensions of the results of Sections 4 and 5 to classes of general relational structures. In particular, the notion of shrub-depth extends easily there.

We conclude with a conjecture stating the converse of our result on expressive power or FO and MSO_1 on graph classes of bounded shrub-depth.

Conjecture 6.1. Consider a hereditary (i.e., closed under induced subgraphs) graph class $\mathcal{G}$. If the expressive powers of FO and MSO_1 are equal on $\mathcal{G}$, then the shrub-depth of $\mathcal{G}$ is bounded (by a suitable constant).

Resolving this conjecture would probably require an asymptotic characterization of shrub-depth in terms of forbidden induced subgraphs. Such characterization would probably also allow us to answer the following question: Are there hereditary graph classes of unbounded shrub-depth having an MSO_1 model-checking algorithm with elementary runtime dependence on the formula? For tree-depth and MSO_2 the answer is no (unless EXP=NEXP), because unbounded tree-depth implies the existence of long paths and by the result of Lampis [18] this in turn implies non-existence of such an algorithm.

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