

PREDICATIVE THEORIES OF CONTINUOUS LATTICES

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ABSTRACT. We introduce a notion of strong proximity join-semilattice, a predicative notion of continuous lattice which arises as the Karoubi envelop of the category of algebraic lattices. Strong proximity join-semilattices can be characterised by the coalgebras of the lower powerlocale on the wider category of proximity posets (also known as abstract bases or R-structures). Moreover, locally compact locales can be characterised in terms of strong proximity join-semilattices by the coalgebras of the double powerlocale on the category of proximity posets. We also provide more logical characterisation of a strong proximity join-semilattice, called a strong continuous finitary cover, which uses an entailment relation to present the underlying join-semilattice. We show that this structure naturally corresponds to the notion of continuous lattice in the predicative point-free topology. Our result makes the predicative and finitary aspect of the notion of continuous lattice in point-free topology more explicit.

1. INTRODUCTION

A continuous lattice [GHK⁺03] is a complete lattice L in which every element $x \in L$ is expressed as a directed join of elements way-below x , where y is *way-below* x if

$$x \leq \bigvee U \rightarrow \exists z \in U (y \leq z)$$

for every directed subset $U \subseteq L$. The reference to an arbitrary directed subset in the definition of the way-below relation, however, makes the theory of continuous lattice difficult to develop in predicative mathematics such as those of Bishop [Bis67] or Martin-Löf [ML84], where powers of sets are not accepted as legitimate objects. Negri [Neg98, Neg02] has addressed this problem in the predicative point-free topology (known as formal topology [Sam87]). Her notion of continuous lattice assumes only the structure on the basis of a continuous lattice together with a primitive way-below relation on the basis (see Definition 4.30). However, the condition on the primitive way-below relation refers to an arbitrary element of the continuous lattice represented by such a structure. Hence, the structure is not a predicative presentation of a continuous lattice; rather, it should be considered as specifying the condition such a predicative presentation, if it exists, should satisfy. The problem of how to present continuous lattices in a uniform way using only predicative data remains open.

Key words and phrases: continuous lattice; locally compact locale; proximity lattice; entailment relation; point-free topology.

The aim of this paper is to address the above predicativity problem by introducing several finitary presentations of continuous lattices in terms of the basis of a domain. These presentations are *finitary* in that they do not make any reference to an arbitrary element of the continuous lattices they represent but only to finite joins of their bases.

Our idea is built on a well-known fact in domain theory: a continuous domain can be completely recovered from a certain structure on its basis, which can be simply described using an idempotent relation on the basis. Such a structure on the basis is called an abstract basis by Abramsky and Jung [AJ94, Section 2.2.6] and an R-structure by Smyth [Smy77]. In this paper, we introduce corresponding structures for continuous lattices, called *proximity \vee -semilattice* and *strong proximity \vee -semilattice*. Both proximity \vee -semilattice and its strong variant are abstract bases enriched with a structure of join semilattice (*\vee -semilattice* for short). Proximity \vee -semilattices naturally arise as the Karoubi envelop of the category of algebraic lattices and Scott continuous functions, while strong proximity \vee -semilattices arise as the Karoubi envelop of the category of algebraic lattices and suplattice homomorphisms. The latter also arise as the coalgebras of the lower powerlocale on the category of proximity posets, the dual of the category of abstract basis. Moreover, locally compact locales admit natural characterisation in terms of strong proximity \vee -semilattice as the coalgebras of the double powerlocale on the category of proximity posets.

We also introduce more logical characterisation of a (strong) proximity \vee -semilattice, called *continuous finitary cover* and *strong continuous finitary cover*. These structures use entailment relations enriched with idempotent relations to present (strong) proximity \vee -semilattices, and they are naturally found in the Karoubi envelop of the category of free suplattices and Scott continuous functions. In particular, a strong continuous finitary cover can be regarded as a finitary presentation of a continuous lattice in formal topology by Negri [Neg98]. We make this observation explicit by establishing an equivalence of the category of strong continuous finitary covers and that of continuous lattices in formal topology. The result clarifies the predicative and finitary aspect of the notions introduced in [Neg98].

Related works. In the context of stably compact locales, the structures analogous to those of proximity \vee -semilattice and continuous finitary cover have already appeared. Smyth [Smy92] and Jung and Sünderhauf [JS96] gave predicative characterisation of stably compact locales using proximity lattices and their strong variants, respectively. These notions were studied further by van Gool [van12], who gave an explicit characterisation of the category of stably compact locales by the Karoubi envelop of the category of spectral locales (the ideal completions of distributive lattices). On the other hand, Vickers [Vic04b] characterised the category of stably compact frames and preframe homomorphisms as the Karoubi envelop of its subcategory of free frames. Building on the earlier work by Jung, Kegelmann, and Moshier [JKM99], he also gave a logical presentation of a proximity lattice using an entailment relation with interpolation property. The strong variant of this notion was investigated by Coquand and Zhang [CZ03] and Kawai [Kaw20]. These structures for stably compact locales are analogous to proximity \vee -semilattices and continuous finitary covers and their strong variants. The difference between the previous works stated above and this paper is as follows: the previous works deal with the Karoubi envelop of the category of spectral locales or that of free frames and preframe homomorphisms; on the other hand, this paper deals with the Karoubi envelop of the category of algebraic lattices (ideal completions of \vee -semilattices) or that of free suplattices and Scott continuous functions.

Notations. We fix some notations for sets and relations which are used throughout the paper. If S is a set, then $\text{Sin}(S)$ denotes the set of singleton subsets of S ; $\text{Fin}(S)$ denotes the set of finitely enumerable subsets of S . Here, a set X is *finitely enumerable* [AR10, Section 7] (or *Kuratowski finite* [Joh02, D5.4]) if there exists a surjection $f: \{0, \dots, n-1\} \rightarrow X$ for some $n \in \mathbb{N}$. We also write $\text{Pow}(S)$ for the collection of subsets of S (usually called the powerset of S), which does not form a set predicativity unless $S = \emptyset$.

Let $r \subseteq X \times Y$ be a relation between sets X and Y . For subsets $U \subseteq X$ and $V \subseteq Y$, the forward image $rU \subseteq Y$ and the inverse image $r^{-1}V \subseteq X$ are defined by

$$\begin{aligned} rU &\stackrel{\text{def}}{=} \{y \in Y \mid \exists x \in U (x r y)\}, \\ r^{-1}V &\stackrel{\text{def}}{=} \{x \in X \mid \exists y \in V (x r y)\}. \end{aligned}$$

For $x \in X$ and $y \in Y$, we write rx for $r\{x\}$ and $r^{-1}y$ for $r^{-1}\{y\}$. The opposite of r is denoted by r^{-} . The relational composition of $r \subseteq X \times Y$ and $s \subseteq Y \times Z$ is denoted by $s \circ r$.

2. SPLITTING OF IDEMPOTENTS

We recall the following fact [Joh82, Chapter VII, Section 2.3], which underlies the development of the later sections:

- (1) Continuous domains are the Scott continuous retracts of algebraic domains.
- (2) Continuous lattices are the suplattice retracts of algebraic lattices.

These allow us to characterise continuous domains (or continuous lattices) in terms of the Karoubi envelop of the category of algebraic domains (resp. algebraic lattices) with suitable notions of morphisms.

We recall some standard notions in domain theory [GHK⁺03]. A *dcpo* is a poset (D, \leq) in which every directed subset has a least upper bound, where a subset $U \subseteq D$ is *directed* if it is inhabited and any two elements of U have an upper bound in U . A function between dcpos D and D' is *Scott continuous* if it preserves directed joins. For elements x, y of a dcpo D , we say that y is *way-below* x , denoted $y \ll x$, if

$$x \leq \bigvee U \rightarrow \exists z \in U (y \leq z)$$

for every directed subset $U \subseteq D$. A *continuous domain* is a dcpo in which every element is a directed join of elements way-below it. A *continuous lattice* is a continuous domain with finite joins. Thus, every continuous lattice is a *suplattice*, a poset with all joins. A *suplattice homomorphism* between continuous lattices is a function which preserves all joins.

Proposition 2.1. *Let D be a continuous domain and $f: D \rightarrow D$ be an idempotent Scott continuous function. Let $D_f = \{f(a) \mid a \in D\}$.*

- (1) D_f is a continuous domain.
- (2) If D is a continuous lattice, then so is D_f .
- (3) If D is a continuous lattice and f is a suplattice homomorphism, then D_f is a sub-semilattice of D .

Proof. 1. We recall the proof from Johnstone [Joh82, Chapter VII, Lemma 2.3] (see also Vickers [Vic04b, Theorem 3]). First, the way-below relation \ll_f of D_f can be characterised in terms of the way-below relation \ll of D as follows:

$$f(a) \ll_f f(b) \leftrightarrow \exists c \in D (f(a) \leq f(c) \ \& \ c \ll f(b)). \quad (2.1)$$

It is easy to see that the set $\downarrow_{\ll_f} f(a) = \{f(b) \in D_f \mid f(b) \ll_f f(a)\}$ is directed. Then

$$\bigvee \downarrow_{\ll_f} f(a) = \bigvee \{f(b) \in D_f \mid b \ll f(a)\} = f\left(\bigvee \{b \in D \mid b \ll f(a)\}\right) = f(a),$$

where the second and the third equations follow from the Scott continuity and the idempotency of f , respectively.

2. If D is a continuous lattice with a \vee -semilattice structure $(D, 0, \vee)$, then D_f admits a \vee -semilattice structure as follows:

$$0_f \stackrel{\text{def}}{=} f(0), \quad f(a) \vee_f f(b) \stackrel{\text{def}}{=} f(f(a) \vee f(b)). \quad (2.2)$$

3. Immediate from (2.2), noting that f is a \vee -semilattice homomorphism. \square

Let (S, \leq) be a poset. An *ideal completion* of a poset (S, \leq) is the collection $\text{Idl}(S)$ of ideals of S ordered by inclusion, where an *ideal* is a downward closed and upward directed subset of S . The poset $(\text{Idl}(S), \subseteq)$ is a continuous domain: directed joins of ideals are unions; the way-below relation is characterised by

$$I \ll J \leftrightarrow \exists a \in J (I \subseteq \downarrow_{\leq} a), \quad (2.3)$$

where $\downarrow_{\leq} a \stackrel{\text{def}}{=} \{b \in S \mid b \leq a\}$. An *algebraic domain* is a continuous domain which can be expressed as the ideal completion of a poset. An *algebraic lattice* is a continuous lattice which can be expressed as the ideal completion of a \vee -semilattice. For an algebraic lattice which is expressed as $\text{Idl}(S)$ for some \vee -semilattice $(S, 0, \vee)$, its finite joins can be characterised by

$$0 \stackrel{\text{def}}{=} \{0\}, \quad I \vee J \stackrel{\text{def}}{=} \bigcup_{a \in I, b \in J} \downarrow_{\leq} (a \vee b). \quad (2.4)$$

Proposition 2.2.

- (1) *Every continuous domain is a Scott continuous retract of an algebraic domain.*
- (2) *Every continuous lattice is a suplattice retract of an algebraic lattice.*

Proof. Every continuous domain (or continuous lattice) is a Scott continuous (resp. suplattice) retract of its ideal completion. See Johnstone [Joh82, Chapter VII, Theorem 2.3]. \square

The following construction plays a fundamental role in this paper.

Definition 2.3 [Fre64, Chapter 2, Exercise B]. An *idempotent* in a category \mathbb{C} is a morphism $f: A \rightarrow A$ such that $f \circ f = f$. The *Karoubi envelop* (or *splitting of idempotents*) of \mathbb{C} is a category $\mathbf{Split}(\mathbb{C})$ where objects are idempotents in \mathbb{C} and morphisms $h: (f: A \rightarrow A) \rightarrow (g: B \rightarrow B)$ are morphisms $h: A \rightarrow B$ in \mathbb{C} such that $g \circ h = h = h \circ f$.

An idempotent $f: A \rightarrow A$ in \mathbb{C} *splits* if there exists a pair of morphisms $r: A \rightarrow B$ and $s: B \rightarrow A$ such that $s \circ r = f$ and $r \circ s = \text{id}_B$. If \mathbb{C} is a full subcategory of \mathbb{D} where every idempotent splits in \mathbb{D} and if every object in \mathbb{D} is a retract of an object of \mathbb{C} , then \mathbb{D} is equivalent to $\mathbf{Split}(\mathbb{C})$.

Theorem 2.4. *Consider the subcategories of continuous domains in Table 1.*

- (1) ContDom is equivalent to $\mathbf{Split}(\text{AlgDom})$.
- (2) $\text{ContLat}_{\text{Scott}}$ is equivalent to $\mathbf{Split}(\text{AlgLat}_{\text{Scott}})$.
- (3) ContLat is equivalent to $\mathbf{Split}(\text{AlgLat})$.

Proof. By Proposition 2.1, every idempotent in ContDom , $\text{ContLat}_{\text{Scott}}$, and ContLat splits. The desired conclusion follows from Proposition 2.2. \square

ContDom	the category of continuous domains and Scott continuous functions
AlgDom	the full subcategory of ContDom consisting of algebraic domains
ContLat_{Scott}	the full subcategory of ContDom consisting of continuous lattices
AlgLat_{Scott}	the full subcategory of ContDom consisting of algebraic lattices
ContLat	the category of continuous lattices and suplattice homomorphisms
AlgLat	the full subcategory of ContLat consisting of algebraic lattices

Table 1: Subcategories of continuous domains.

Predicative characterisation	Karoubi envelop	Domain theoretic dual
proximity posets + approximable relations	AlgDom	ContDom
proximity \vee -semilattices + approximable relations	AlgLat_{Scott}	ContLat_{Scott}
strong proximity \vee -semilattices + join-approximable relations	AlgLat	ContLat
localized strong prox. \vee -semilat. + proximity relations	—	LKFrm

Table 2: Main structures in Section 3.

3. PROXIMITY \vee -SEMILATTICES

We introduce predicative characterisations of the Karoubi envelopes of the category of algebraic domains and its subcategories of algebraic lattices. The left column of Table 2 shows some of the major structures introduced in this section. The categories of these structures provide predicative characterisations of the duals of the Karoubi envelopes of the categories on the middle column as well as the duals of the categories on the right column (cf. Theorem 2.4).¹

3.1. Proximity posets and proximity \vee -semilattices. Our predicative characterisation of continuous domains rests on an elementary characterisation of Scott continuous functions between algebraic domains.

First, we recall that the ideal completion of a poset is the free dcpo on that poset.

Lemma 3.1. *For each poset (S, \leq) , there is a monotone (i.e. order preserving) function $i_S: S \rightarrow \text{Idl}(S)$ defined by*

$$i_S(a) \stackrel{\text{def}}{=} \downarrow_{\leq} a \tag{3.1}$$

with the following universal property: for any monotone function $f: S \rightarrow D$ to a dcpo D , there exists a unique Scott continuous function $\bar{f}: \text{Idl}(S) \rightarrow D$ such that $\bar{f} \circ i_S = f$.

Proof. See Vickers [Vic89, Proposition 9.1.2 (v)]. The unique extension $\bar{f}: \text{Idl}(S) \rightarrow D$ is defined by

$$\bar{f}(I) \stackrel{\text{def}}{=} \bigvee_{a \in I} f(a). \tag{3.2} \quad \square$$

¹LKFrm in Table 2 denotes the category of locally compact frames, i.e., the dual of the category of locally compact locales (cf. Section 3.5).

For posets (S, \leq) and (S', \leq') , a monotone function $f: S' \rightarrow \text{Idl}(S)$ corresponds to a relation between S and S' as characterised below.

Definition 3.2. Let (S, \leq) and (S', \leq') be posets. A relation $r \subseteq S \times S'$ is *approximable* if (AppI) r^{-b} is an ideal for each $b \in S'$, (AppU) ra is an upward closed subset for each $a \in S$.

Proposition 3.3. Let (S, \leq) and (S', \leq') be posets. There exists a bijective correspondence between approximable relations from S to S' and Scott continuous functions from $\text{Idl}(S')$ to $\text{Idl}(S)$. Through this correspondence, the identity function on $\text{Idl}(S)$ corresponds to the order \leq on S and the composition of two Scott continuous functions contravariantly corresponds to the relational composition of the approximable relations.

Proof. An approximable relation $r \subseteq S \times S'$ bijectively corresponds to a monotone function $f_r: S' \rightarrow \text{Idl}(S)$ defined by

$$f_r(b) \stackrel{\text{def}}{=} r^{-b}, \quad (3.3)$$

and hence to a Scott continuous function $\bar{f}_r: \text{Idl}(S') \rightarrow \text{Idl}(S)$ defined in (3.2). Conversely, each Scott continuous function $f: \text{Idl}(S') \rightarrow \text{Idl}(S)$ determines an approximable relation $r_f \subseteq S \times S'$ by

$$a r_f b \stackrel{\text{def}}{\iff} a \in f(i_{S'}(b)). \quad (3.4)$$

It is straightforward to check that the above correspondence is bijective. The second statement is also straightforward to check. \square

Let Pos_{App} be the category in which objects are posets and morphisms are approximable relations between posets: the identity on a poset is its underlying order; the composition of two approximable relations is the relational composition.

Proposition 3.4. Pos_{App} is dually equivalent to AlgDom .

Proof. Immediate from Proposition 3.3. \square

By Theorem 2.4, we have the following characterisation of the category of continuous domains.

Theorem 3.5. $\text{Split}(\text{Pos}_{\text{App}})$ is dually equivalent to ContDom .

The objects and morphisms of $\text{Split}(\text{Pos}_{\text{App}})$ can be explicitly described as follows.

Definition 3.6. A *proximity poset* is a structure (S, \leq, \prec) where (S, \leq) is a poset and \prec is a relation on S satisfying

- (1) $\downarrow_{\prec} a \stackrel{\text{def}}{=} \{b \in S \mid b \prec a\}$ is a rounded ideal of S ,
- (2) $\uparrow_{\prec} a \stackrel{\text{def}}{=} \{b \in S \mid b \succ a\}$ is a rounded upward closed subset S .

Here, an ideal I is *rounded* if

$$a \in I \leftrightarrow \exists b \succ a (b \in I).$$

Similarly, an upward closed subset $U \subseteq S$ is *rounded* if

$$a \in U \leftrightarrow \exists b \prec a (b \in U).$$

We write (S, \prec) or simply S for a proximity poset, leaving the underlying order of S implicit.

Definition 3.7. Let (S, \prec) and (S', \prec') be proximity posets. A relation $r \subseteq S \times S'$ is *approximable* if r is an approximable relation between the underlying posets of S and S' such that $r \circ \prec = r = \prec' \circ r$. Equivalently, a relation $r \subseteq S \times S'$ is approximable if

- (1) r^-b is a rounded ideal for each $b \in S'$,
- (2) ra is a rounded upward closed subset for each $a \in S$.

Henceforth, we write \mathbf{PxPos} for $\mathbf{Split}(\mathbf{Pos}_{\text{App}})$: objects of \mathbf{PxPos} are proximity posets and morphisms between them are approximable relations. The identity on a proximity poset (S, \prec) is \prec ; the composition of two approximable relations is the relational composition. Note that $\mathbf{Pos}_{\text{App}}$ is a full subcategory of \mathbf{PxPos} , where each poset (S, \leq) is identified with a proximity poset with \leq as its idempotent relation. Thus, the terminology *approximable relation* for morphisms of \mathbf{PxPos} is consistent with that of $\mathbf{Pos}_{\text{App}}$.

Remark 3.8.

- (1) The notion of proximity poset is essentially equivalent to that of *abstract basis* [AJ94, Definition 2.2.20] or *R-structure* [Smy77], where the morphisms between abstract bases are also characterised as approximable relations [AJ94, Definition 2.2.27]. An abstract basis lacks the underlying poset structure on S . In this paper, we include the poset structure to stress the fact that every continuous domain is a retract of an algebraic domain, which is represented by that poset structure.
- (2) Another closely related notion is that of *infosys* and approximable mapping by Vickers [Vic93, Definition 2.1 and Definition 2.18]. Constructively, however, the notion of infosys seems to be more general than that of continuous domain, and hence does not seem to be equivalent to proximity poset. Nevertheless, some of the conjectures Vickers has made in the context of infosys are proved to be correct for proximity posets; see Section 3.4 and Section 3.6.

To obtain a similar characterisation of continuous lattices, consider a full subcategory $\mathbf{JLat}_{\text{App}}$ of $\mathbf{Pos}_{\text{App}}$ consisting of \vee -semilattices. Then, the dual equivalence in Proposition 3.4 restricts to a dual equivalence between $\mathbf{JLat}_{\text{App}}$ and $\mathbf{AlgLat}_{\text{Scott}}$. Hence, by Theorem 2.4, we have the following characterisation of the category of continuous lattices and Scott continuous functions (cf. Table 2).

Theorem 3.9. $\mathbf{Split}(\mathbf{JLat}_{\text{App}})$ is dually equivalent to $\mathbf{ContLat}_{\text{Scott}}$.

The objects of $\mathbf{Split}(\mathbf{JLat}_{\text{App}})$ can be explicitly described as follows.

Definition 3.10. A *proximity \vee -semilattice* is a proximity poset whose underlying poset is a \vee -semilattice. We write $(S, 0, \vee, \prec)$ or simply S for a proximity \vee -semilattice, where $(S, 0, \vee)$ is a \vee -semilattice and (S, \prec) is a proximity poset.

Henceforth, we write \mathbf{PxJLat} for $\mathbf{Split}(\mathbf{JLat}_{\text{App}})$, which forms a full subcategory of \mathbf{PxPos} consisting of proximity \vee -semilattices.

Remark 3.11. It is well known that $\mathbf{ContLat}_{\text{Scott}}$ is equivalent to the category of injective T_0 -spaces (cf. Scott [Sco72]). Thus, \mathbf{PxJLat} can be seen as a predicative characterisation of the latter category.

3.2. Representation of continuous domains. By Theorem 3.5, each proximity poset (S, \prec) represents a continuous domain. In the case of continuous domain, the equivalence in Theorem 2.4 is mediated by the canonical splitting of idempotents of AlgDom by the image factorisation described in Proposition 2.1. This suggests that the continuous domain represented by (S, \prec) arises as the image of $\text{Idl}(S)$ under the idempotent Scott continuous function on $\text{Idl}(S)$ induced by \prec (cf. Proposition 3.3). This image, which we denote by $\text{RIdl}(S)$, is the collection of ideals of the form

$$\downarrow_{\prec} I \stackrel{\text{def}}{=} \prec I = \{a \in S \mid \exists b \in I (a \prec b)\}$$

for some $I \in \text{Idl}(S)$. By the idempotency of \prec , these ideals are exactly the rounded ideals of S (hence the notation $\text{RIdl}(S)$). By (2.1) and (2.3), the way-below relation in $\text{RIdl}(S)$ can be characterised by

$$I \ll J \leftrightarrow \exists a \in J (I \subseteq \downarrow_{\prec} a).$$

Furthermore, if $(S, 0, \vee, \prec)$ is a proximity \vee -semilattice, then $\text{RIdl}(S)$ has finite joins by Proposition 3.9. By (2.2) and (2.4), these joins can be characterised by

$$0 \stackrel{\text{def}}{=} \downarrow_{\prec} 0, \quad I \vee J \stackrel{\text{def}}{=} \bigcup_{a \in I, b \in J} \downarrow_{\prec} (a \vee b). \quad (3.5)$$

By Theorem 3.5, each approximable relation between proximity posets S and S' represents a Scott continuous function between $\text{RIdl}(S')$ and $\text{RIdl}(S)$. This is analogous to the case of posets; Proposition 3.3 and Lemma 3.1, which are stated in terms of posets and their ideals, can be restated in terms of proximity posets and their rounded ideals as follows: each approximable relation $r: S \rightarrow S'$ bijectively corresponds to a Scott continuous function $\overline{f}_r: \text{RIdl}(S') \rightarrow \text{RIdl}(S)$ given by

$$\overline{f}_r(I) \stackrel{\text{def}}{=} r^{-1}I, \quad (3.6)$$

and this correspondence preserves identity and composition of PxPos . This can be established as in Proposition 3.3 using the Lemma 3.12 below in place of Lemma 3.1. Here, a *dcpo interpretation* of a proximity poset (S, \prec) in a dcpo D is a monotone function $f: S \rightarrow D$ such that $f(a) = \bigvee_{b \prec a} f(b)$. Note that $\downarrow_{\prec} a$ is directed so that $\bigvee_{b \prec a} f(b)$ exists.

Lemma 3.12. *For each proximity poset (S, \prec) , there is a dcpo interpretation $i_S: S \rightarrow \text{RIdl}(S)$ defined by*

$$i_S(a) \stackrel{\text{def}}{=} \downarrow_{\prec} a$$

with the following universal property: for any dcpo interpretation $f: S \rightarrow D$ in a dcpo D , there exists a unique Scott continuous function $\overline{f}: \text{RIdl}(S) \rightarrow D$ such that $\overline{f} \circ i_S = f$.

Proof. Similar to that of Lemma 3.1. □

Then, each approximable relation $r: (S, \prec) \rightarrow (S', \prec')$ bijectively corresponds to a dcpo interpretation $f_r: S' \rightarrow \text{RIdl}(S)$ given by (3.3), and hence to a Scott continuous function $\overline{f}_r: \text{RIdl}(S') \rightarrow \text{RIdl}(S)$ given by (3.6).

3.3. Strong proximity \vee -semilattices. We introduce another characterisation of a continuous lattice, which is obtained by splitting idempotents of the category of algebraic lattices and suplattice homomorphisms. This naturally leads to a predicative characterisation of the category of continuous lattices and suplattice homomorphisms.

First, we give a predicative characterisation of suplattice homomorphisms between algebraic lattices. The following is analogous to Lemma 3.1.

Lemma 3.13. *For each \vee -semilattice $(S, 0, \vee)$, there is a join-preserving function $i_S: S \rightarrow \text{Idl}(S)$ defined (3.1) with the following universal property: for any join-preserving function $f: S \rightarrow L$ to a suplattice L , there exists a unique suplattice homomorphism $\bar{f}: \text{Idl}(S) \rightarrow L$ such that $\bar{f} \circ i_S = f$.*

Proof. See Vickers [Vic89, Proposition 9.1.5 (ii)(iv)]. The unique extension $\bar{f}: \text{Idl}(S) \rightarrow L$ is again given by (3.2). \square

For \vee -semilattices $(S, 0, \vee)$ and $(S', 0', \vee')$, \vee -semilattice homomorphisms $f: S' \rightarrow \text{Idl}(S)$ correspond to the class of approximable relations from S to S' as defined below.

Definition 3.14. Let $(S, 0, \vee)$ and $(S', 0', \vee')$ be \vee -semilattices. An approximable relation $r: S \rightarrow S'$ is *join-preserving* if

$$(\text{App}0) \quad a \ r \ 0' \rightarrow a = 0,$$

$$(\text{App}\vee) \quad a \ r \ (b \vee' c) \rightarrow \exists b', c' \in S (a \leq b' \vee c' \ \& \ b' \ r \ b \ \& \ c' \ r \ c).$$

We call join-preserving approximable relations simply as *join-approximable relations*.

Remark 3.15. The two conditions in Definition 3.14 correspond to the condition $(G\text{-}\vee)$ in Jung and Sünderhauf [JS96, Definition 25], which is also called *join-approximable* by van Gool [van12, Definition 1.9 (4)].

Proposition 3.3 restricts to a bijective correspondence between join-approximable relations and suplattice homomorphisms.

Proposition 3.16. *Let $(S, 0, \vee)$ and $(S', 0', \vee')$ be \vee -semilattices. There exists a bijective correspondence between join-approximable relations from S to S' and suplattice homomorphisms from $\text{Idl}(S')$ to $\text{Idl}(S)$. Through this correspondence, the identity function on $\text{Idl}(S)$ corresponds to the order \leq on S and the composition of suplattice homomorphisms contravariantly corresponds to the relational composition of the join-approximable relations.*

Proof. We show that the correspondence described in the proof of Proposition 3.3 restricts to join-approximable relations and suplattice homomorphisms. First, if $r: (S, 0, \vee) \rightarrow (S', 0', \vee')$ is a join-approximable relation, the function $f_r: S' \rightarrow \text{Idl}(S)$ given by (3.3) preserves finite joins by the characterisation of finite joins of ideals in (2.4). Thus, the resulting Scott continuous function $\bar{f}_r: \text{Idl}(S') \rightarrow \text{Idl}(S)$ is a suplattice homomorphism by Lemma 3.13.

Conversely, if $f: \text{Idl}(S') \rightarrow \text{Idl}(S)$ is a suplattice homomorphism, then the approximable relation $r_f: S \rightarrow S'$ given by (3.4) is join-preserving again by the characterisation (2.4). \square

Let $\mathbf{JLat}_{\text{JApp}}$ be a subcategory of $\mathbf{Pos}_{\text{App}}$ consisting of \vee -semilattices and join-approximable relations between them. By Proposition 3.16, we have the following.

Proposition 3.17. $\mathbf{JLat}_{\text{JApp}}$ is dually equivalent to \mathbf{AlgLat} .

Thus, by Theorem 2.4, we have the following characterisation of the category of continuous lattices.

Theorem 3.18. $\mathbf{Split}(\mathbf{JLat}_{\mathbf{JApp}})$ is dually equivalent to $\mathbf{ContLat}$.

The objects of $\mathbf{Split}(\mathbf{JLat}_{\mathbf{JApp}})$ can be explicitly described as follows.

Definition 3.19. A *strong proximity \vee -semilattice* is a proximity \vee -semilattice $(S, 0, \vee, \prec)$ satisfying

- (1) $a \prec 0 \rightarrow a = 0$,
- (2) $a \prec (b \vee c) \rightarrow \exists b', c' \in S (a \leq b' \vee c' \ \& \ b' \prec b \ \& \ c' \prec c)$.

Henceforth, we write $\mathbf{SPxJLat}$ for $\mathbf{Split}(\mathbf{JLat}_{\mathbf{JApp}})$, which forms a subcategory of \mathbf{PxJLat} consisting of strong proximity \vee -semilattices and join-approximable relations (in the sense of Definition 3.7 and Definition 3.14).

Note that the objects of $\mathbf{SPxJLat}$ and \mathbf{PxJLat} are essentially the same as they both represent continuous lattices. Proposition 3.21 below makes this explicit (cf. Kawai [Kaw20, Theorem 5.8]). We need the following construction for its proof.

Definition 3.20. For a relation $r \subseteq S \times S'$, its *lower extension* $r_L \subseteq \mathbf{Fin}(S) \times \mathbf{Fin}(S')$ is defined by

$$A r_L B \stackrel{\text{def}}{\iff} \forall a \in A \exists b \in B (a r b).$$

Proposition 3.21. For each proximity \vee -semilattice S , there exists a strong proximity \vee -semilattice S' which is isomorphic to S in \mathbf{PxJLat} .

Proof. Given a proximity \vee -semilattice $S = (S, 0, \vee, \prec)$, define a preorder \leq^\vee on $\mathbf{Fin}(S)$ by

$$A \leq^\vee B \stackrel{\text{def}}{\iff} \forall C \prec_L A \exists D \prec_L B (\bigvee C \prec \bigvee D).$$

Let $S' = (\mathbf{Fin}(S), \leq^\vee)$ be the poset reflection of \leq^\vee , and define a \vee -semilattice structure $(S', 0^\vee, \vee^\vee)$ on S' by

$$0^\vee \stackrel{\text{def}}{=} \emptyset, \quad A \vee^\vee B \stackrel{\text{def}}{=} A \cup B.$$

It is easy to see that the above operations are well-defined with respect to the equality on S' . Moreover, define a relation \prec^\vee on S' by

$$A \prec^\vee B \stackrel{\text{def}}{\iff} \exists C \prec_L B (A \leq^\vee C).$$

Then, $S' = (S', 0^\vee, \vee^\vee, \prec^\vee)$ is a strong proximity \vee -semilattice. Indeed, since \prec_L is idempotent and $\prec_L \subseteq \prec^\vee \subseteq \leq^\vee$, the relation \prec^\vee is idempotent, which clearly satisfies (AppI). As for the property (AppU), suppose $A \prec^\vee B$ and $B \leq^\vee C$. By the former, there exists $B' \prec_L B$ such that $A \leq^\vee B'$, so by the latter, there exists $C' \prec_L C$ such that $\bigvee B' \prec \bigvee C'$. Thus, there exists D such that $C' \prec_L D \prec_L C$. Then, one can easily see that $A \leq^\vee D$, so that $A \prec^\vee C$. To see that S' is strong, we have $A \prec^\vee \emptyset$ implies $A \leq^\vee \emptyset$, so S' satisfies (App0). Next, suppose that $A \prec^\vee B \vee^\vee C$. Then, there exists $D \prec_L B \cup C$ such that $A \leq^\vee D$. Split D into D_B and D_C such that $D = D_B \cup D_C$, $D_B \prec_L B$, and $D_C \prec_L C$. Then, $A \leq^\vee D_B \vee^\vee D_C$, $D_B \prec^\vee B$, and $D_C \prec^\vee C$. Hence S' satisfies (App \vee).

Now, define relations $r \subseteq \mathbf{Fin}(S) \times S$ and $s \subseteq S \times \mathbf{Fin}(S)$ by

$$A r a \stackrel{\text{def}}{\iff} A \prec^\vee \{a\}, \quad a s A \stackrel{\text{def}}{\iff} \exists B \prec_L A (a \prec \bigvee B).$$

We show that these are approximable relations between S' and S which are inverse to each other. The property (AppI) for s follows from the fact that $0 \prec 0$ and $a \prec b \ \& \ a' \prec b' \rightarrow a \vee a' \prec b \vee b'$. From the definition of \leq^\vee , it is easy to see that s also satisfies (AppU). Moreover, we have $s \circ \prec = s$ by the idempotency of \prec , and $s \subseteq \prec_L \circ s \subseteq \prec^\vee \circ s \subseteq \leq^\vee \circ s = s$. Thus,

s is an approximable relation from S and S' . The fact that r is approximable is also easy to check. To see that r and s are inverse to each other, first, we have $s \circ r \subseteq \prec^\vee$, noting that $a s A$ implies $\{a\} \leq^\vee A$. Conversely, if $A \prec^\vee B$, then there exists $C \prec_L B$ such that $A \prec^\vee C$. Then, $A r \bigvee C s B$, and so $A (s \circ r) B$. Thus, $s \circ r = \prec^\vee$. Next, we have $\prec \subseteq r \circ s$ by the idempotency of \prec . Conversely, if $a s A r b$, there exists $B \prec_L A$ such that $a \prec \bigvee B$. Since $A r b$ implies $A \leq^\vee \{b\}$, there exists $C \prec_L \{b\}$ such that $\bigvee B \prec \bigvee C$. Then, $a \prec b$, and thus $r \circ s = \prec$. \square

3.4. Algebraic theory of continuous lattices. We give yet another predicative characterisation of continuous lattices in terms of algebras of the lower powerlocale on the category of continuous domains. Specifically, from the localic point of view, the category of continuous domains and continuous functions can be regarded as a subcategory of locales whose frames are Scott topologies of continuous domains. This is the view of the category **Infosys** of infosys [Vic93], a classical dual of **PxPos**. In this view, the construction in Definition 3.22 below corresponds to that of the lower powerlocale on the Scott topologies of continuous domains [Vic93, Theorem 4.3 (ii)]. In the case of infosys, the construction forms a monad on **Infosys**, and Vickers conjectured that the category of algebras for the monad is equivalent to **ContLat** [Vic93, Section 5.1]. In classical domain theory, the corresponding characterisation of continuous lattices has been given by Schalk [Sch93, Section 6.2.2].

In what follows, we work in the dual context of **PxPos**, where the construction in Definition 3.22 (also called the lower powerlocale in this paper) forms a comonad on **PxPos**. Then, the main result of this subsection says that the category of strong proximity \vee -semilattices is equivalent to that of the coalgebras of the lower powerlocale.

Definition 3.22 [Vic93, Definition 4.1]. Let (S, \prec) be a proximity poset. The *lower powerlocale* $P_L(S)$ of (S, \prec) is a proximity poset $((\text{Fin}(S), \leq_L), \prec_L)$ where $(\text{Fin}(S), \leq_L)$ denotes the poset reflection of the preorder \leq_L .

Proposition 3.23. *Let (S, \prec) be a proximity poset.*

- (1) $P_L(S)$ is a strong proximity \vee -semilattice.
- (2) A relation $\varepsilon_S^L \subseteq \text{Fin}(S) \times S$ defined by

$$A \varepsilon_S^L a \stackrel{\text{def}}{\iff} A \prec_L \{a\} \tag{3.7}$$

is an approximable relation from $P_L(S)$ to (S, \prec) with the following universal property: for any strong proximity \vee -semilattice $(S', 0', \vee', \prec')$ and an approximable relation $r: S' \rightarrow S$, there exists a unique join-approximable relation $\bar{r}: S' \rightarrow P_L(S)$ such that $\varepsilon_S^L \circ \bar{r} = r$.

Proof. 1. The poset $(\text{Fin}(S), \leq_L)$ has finite joins characterised by

$$0 \stackrel{\text{def}}{=} \emptyset, \quad A \vee B \stackrel{\text{def}}{=} A \cup B.$$

It is straightforward to show that \prec_L is an idempotent approximable relation on $(\text{Fin}(S), \leq_L)$ which satisfies (App0) and (App \vee) with respect to the above operations.

2. We clearly have $\varepsilon_S^L \circ \prec_L = \varepsilon_S^L$. By (AppI) for \prec , we also have $\prec \circ \varepsilon_S^L = \varepsilon_S^L$. Since $(\text{Fin}(S), \leq_L)$ has finite joins computed by unions, ε_S^L satisfies (AppI). Moreover, the property (AppU) follows from the corresponding property of \prec . Thus, ε_S^L is an approximable relation.

Next, given an approximable relation $r: S' \rightarrow S$ from a strong proximity \vee -semilattice $(S', 0', \vee', \prec')$, define a relation $\bar{r} \subseteq S' \times \text{Fin}(S)$ by

$$b \bar{r} A \stackrel{\text{def}}{\iff} \exists B \in \text{Fin}(S') (b \leq' \bigvee B \ \& \ B \ r_L A).$$

It is straightforward to check that \bar{r} is a join-approximable relation from S' to $\text{P}_L(S)$ such that $\varepsilon_S^L \circ \bar{r} = r$. For the uniqueness of \bar{r} , suppose that $u: S' \rightarrow \text{P}_L(S)$ is a join-approximable relation such that $\varepsilon_S^L \circ u = r$. Suppose $b \ u \ A$. Since $\prec_L \circ u = u$, there exists $A' \in \text{Fin}(S)$ such that $b \ u \ A' \prec_L A$. Since $A' = \bigcup \{\{a'\} \mid a' \in A'\}$ and u is join-preserving, there exists $B \in \text{Fin}(S')$ such that $b \leq' \bigvee B$ and $B \ u_L \{\{a'\} \mid a' \in A'\}$. Then, $B \ r_L A$ by $\varepsilon_S^L \circ u \subseteq r$. Hence $b \bar{r} A$. Conversely, suppose $b \bar{r} A$. Then, there exists $B \in \text{Fin}(S')$ such that $b \leq' \bigvee B$ and $B \ r_L A$. Thus, for each $b' \in B$, there exists $a \in A$ such that $b' (\varepsilon_S^L \circ u) a$. Then, $b' \ u \ \{a\}$, and since $\{a\} \leq_L A$, we have $b' \ u \ A$ by (AppU). Hence, $b \ u \ A$ by (AppI). \square

By Proposition 3.23, the construction $\text{P}_L(S)$ determines a right adjoint to the forgetful functor from SPxJLat to PxPos . The functor $\text{P}_L: \text{PxPos} \rightarrow \text{SPxJLat}$ sends an approximable relation $r: S \rightarrow S'$ to a join-approximable relation $\text{P}_L(r): \text{P}_L(S) \rightarrow \text{P}_L(S')$ defined by

$$\text{P}_L(r) \stackrel{\text{def}}{=} r_L.$$

The counit $\varepsilon_S^L: \text{P}_L(S) \rightarrow S$ of the adjunction is given by (3.7), and the unit $\eta_S^L: S \rightarrow \text{P}_L(S)$ at a strong proximity \vee -semilattice $(S, 0, \vee, \prec)$ is given by

$$a \eta_S^L A \stackrel{\text{def}}{\iff} a \prec \bigvee A. \quad (3.8)$$

The adjunction induces a comonad $\langle \text{P}_L, \varepsilon^L, \nu^L \rangle$ on PxPos with a co-multiplication $\nu_S^L \stackrel{\text{def}}{=} \eta_{\text{P}_L(S)}^L$, which by definition satisfies

$$A \nu_S^L \mathcal{U} \leftrightarrow A \prec_L \bigcup \mathcal{U}.$$

Let $\text{coAlg}(\text{P}_L)$ denote the category of P_L -coalgebras and coalgebra homomorphisms, and let $K: \text{SPxJLat} \rightarrow \text{coAlg}(\text{P}_L)$ be the comparison functor. Note that K sends each strong proximity \vee -semilattice S to a P_L -coalgebra $\eta_S^L: S \rightarrow \text{P}_L(S)$ and it is an identity map on morphisms (cf. [Mac98, Chapter VI, Section 3]).

Lemma 3.24. *The functor $K: \text{SPxJLat} \rightarrow \text{coAlg}(\text{P}_L)$ is full and faithful.*

Proof. K is obviously faithful. To see that K is full, let $r: S \rightarrow S'$ be a P_L -coalgebra homomorphism between strong proximity \vee -semilattices $(S, 0, \vee, \prec)$ and $(S', 0', \vee', \prec')$. Then, for any $a \in S$ and $B \in \text{Fin}(S')$, we have

$$\begin{aligned} a \ r \ \bigvee' B &\iff \exists b \in S' (a \ r \ b \prec' \bigvee' B) && \text{(by } r = \prec' \circ r) \\ &\iff a (\eta_{S'}^L \circ r) B \\ &\iff a (\text{P}_L(r) \circ \eta_S^L) B && \text{(} r \text{ is a homomorphism)} \\ &\iff \exists A \in \text{Fin}(S) (a \prec \bigvee A \ \& \ A \ r_L B) \\ &\iff \exists A \in \text{Fin}(S) (a \leq \bigvee A \ \& \ A (r_L \circ \prec_L) B) && \text{(} S \text{ is strong)} \\ &\iff \exists A \in \text{Fin}(S) (a \leq \bigvee A \ \& \ A \ r_L B). && \text{(by } r = r \circ \prec) \end{aligned}$$

Thus, r is join-preserving. Hence K is full. \square

It remains to show that K is essentially surjective. This will become clear after we recall the fact that the comonad P_L is of Kock–Zöberlein type (see Escardó [Esc98, Section 4.1]). The dual notion is that of KZ-monad [Koc95].

Definition 3.25. Let $\langle T, \varepsilon, \nu \rangle$ be a comonad on a poset enriched category \mathbb{C} , where T preserves the order on homsets. Then, T is called a *KZ-comonad* (*coKZ-comonad*) if $\varepsilon_{TX} \leq T\varepsilon_X$ (resp. $T\varepsilon_X \leq \varepsilon_{TX}$) for each object X of \mathbb{C} .

Proposition 3.26. *Let $\langle T, \varepsilon, \nu \rangle$ be a comonad on a poset enriched category \mathbb{C} , where T preserves the order on homsets. Then, the following are equivalent:*

- (1) T is a KZ-comonad.
- (2) $\alpha: X \rightarrow TX$ is a T -coalgebra if and only if $\alpha \dashv \varepsilon_X$ and $\varepsilon_X \circ \alpha = \text{id}_X$, where

$$\alpha \dashv \varepsilon_X \stackrel{\text{def}}{\iff} \text{id}_X \leq \varepsilon_X \circ \alpha \ \& \ \alpha \circ \varepsilon_X \leq \text{id}_{TX}.$$
²

Proof. See Escardó [Esc98, Lemma 4.1.1] for a proof for KZ-monads. □

We have a similar characterisation for coKZ-comonads, which is obtained by replacing item 2 of Proposition 3.26 with the following:

$$\alpha: X \rightarrow TX \text{ is a } T\text{-coalgebra if and only if } \varepsilon_X \dashv \alpha \text{ and } \varepsilon_X \circ \alpha = \text{id}_X. \quad (3.9)$$

By the uniqueness of left and right adjoints, for a (co)KZ-comonad, each object admits at most one coalgebra structure. The following characterisation of coalgebras and isomorphisms between them is useful.

Corollary 3.27 [Esc98, Corollary 4.2.3]. *Let $\langle T, \varepsilon, \nu \rangle$ be a (co)KZ-comonad on a poset enriched category \mathbb{C} .*

- (1) *The following are equivalent for an object X :*
 - (a) X admits a T -coalgebra structure.
 - (b) X is a retract of TX .
 - (c) There exists $\alpha: X \rightarrow TX$ such that $\varepsilon_X \circ \alpha = \text{id}_X$.
- (2) *Any isomorphism in \mathbb{C} between the underlying objects of T -coalgebras is an isomorphism of the coalgebras.*

Proof. 1. See Escardó [Esc98, Corollary 4.2.3].

2. We only give a proof for KZ-comonads. Let $\alpha: X \rightarrow TX$ and $\beta: Y \rightarrow TY$ be T -coalgebras, and let $f: X \rightarrow Y$ be an isomorphism in \mathbb{C} with an inverse $g: Y \rightarrow X$. It suffices to show that f is a coalgebra homomorphism. Since

$$\varepsilon_X \circ Tg \circ \beta \circ f = g \circ \varepsilon_Y \circ \beta \circ f = g \circ f = \text{id}_X,$$

$Tg \circ \beta \circ f$ is a T -coalgebra on X by 1c. Since a coalgebra structure on X is unique, we must have $Tg \circ \beta \circ f = \alpha$. Thus, $\beta \circ f = Tf \circ \alpha$. □

Each homset of PxPos is ordered by the inclusion of graphs of approximable relations, and the functor P_L clearly preserves this order.

Proposition 3.28. *The comonad $\langle P_L, \varepsilon^L, \nu^L \rangle$ is a coKZ-comonad on PxPos .*

²As usual in the context of poset enriched categories, the situation $\alpha \dashv \varepsilon_X$ is called an *adjunction*, where α and ε_X are the *left adjoint* and the *right adjoint*, respectively.

Proof. Let (S, \prec) be a proximity poset. For each $\mathcal{U} \in \text{Fin}(\text{Fin}(S))$ and $A \in \text{Fin}(S)$, we have

$$\begin{aligned} \mathcal{U} P_L(\varepsilon_S^L) A &\iff \mathcal{U} (\prec_L)_L \{\{a\} \mid a \in A\} \\ &\implies \mathcal{U} (\prec_L)_L \{A\} \\ &\iff \mathcal{U} \varepsilon_{P_L(S)}^L A. \end{aligned}$$

Thus $P_L(\varepsilon_S^L) \leq \varepsilon_{P_L(S)}^L$. \square

By Proposition 3.28 and Corollary 3.27 (1), P_L -coalgebras are precisely the retracts of free P_L -coalgebras in the category of proximity posets. In view of Proposition 3.23 (1), Theorem 3.18, and Proposition 2.1 (2), each P_L -coalgebra represents a continuous lattice. Hence, the functor $K: \text{SPxJLat} \rightarrow \text{coAlg}(P_L)$ must be essentially surjective, which we now make explicit.

Lemma 3.29. *For each P_L -coalgebra $\alpha: S \rightarrow P_L(S)$, there exists a strong proximity \vee -semilattice S' which is isomorphic to S in PxPos .*

Proof. Let $\alpha: S \rightarrow P_L(S)$ be a P_L -coalgebra on a proximity poset (S, \prec) . Let $\prec_\alpha \stackrel{\text{def}}{=} \alpha \circ \varepsilon_S^L$, and put $S' = (\text{Fin}(S), \prec_\alpha)$ where $\text{Fin}(S)$ is regarded as a free \vee -semilattice over S . Then, it is easy to see that S' is a proximity \vee -semilattice. Moreover, $\varepsilon_{S'}^L$ and α are approximable relations from S' to S and S to S' , respectively, and they are inverse to each other. On the other hand, by Proposition 3.21, there is a strong proximity \vee -semilattice S'' which is isomorphic to S' in PxJLat . Then, S'' is isomorphic to S in PxPos . \square

By Lemma 3.29 and Corollary 3.27 (2), the comparison functor $K: \text{SPxJLat} \rightarrow \text{coAlg}(P_L)$ is essentially surjective. Since K is full and faithful (cf. Lemma 3.24), K determines an equivalence of the categories.³

Theorem 3.30. *SPxJLat is equivalent to $\text{coAlg}(P_L)$.*

By Theorem 3.18 and Theorem 3.30, we have the following characterisation of continuous lattices.

Theorem 3.31. *$\text{coAlg}(P_L)$ is dually equivalent to ContLat .*

3.5. Localized strong proximity \vee -semilattices. We characterise locally compact locales in terms of strong proximity \vee -semilattices. Recall that a *frame* is a poset (X, \wedge, \vee) with finite meets \wedge and joins \vee for all subsets of X where finite meets distribute over all joins. A homomorphism between frames X and Y is a function $f: X \rightarrow Y$ which preserves finite meets and all joins. The *category of locales* is the opposite of the category of frames and frame homomorphisms. A locale is *locally compact* if it is a continuous lattice (see Johnstone [Joh82, Chapter VII, Section 4]).

The following structure characterises locally compact locales.

Definition 3.32. A strong proximity \vee -semilattice $(S, 0, \vee, \prec)$ is *localized* if

$$a \prec b \leq c \vee d \rightarrow \exists a_1 \in (b \downarrow_\prec c) \exists a_2 \in (b \downarrow_\prec d) (a \prec a_1 \vee a_2), \quad (3.10)$$

where $b \downarrow_\prec c \stackrel{\text{def}}{=} \downarrow_\prec b \cap \downarrow_\prec c$.

³This equivalence does not require the axiom of choice because an explicit description of a quasi-inverse of K can be obtained from the proof of Lemma 3.29.

Since S is assumed to be strong, the antecedent of (3.10) can be equivalently stated as $a \prec b \prec c \vee d$.

Lemma 3.33. *For strong proximity \vee -semilattices, the condition (3.10) is equivalent to*

$$a \prec a' \leq \bigvee A \ \& \ a' \leq \bigvee B \ \rightarrow \ \exists C \in \text{Fin}(A \downarrow_{\prec} B) (a \prec \bigvee C), \quad (3.11)$$

where $A \downarrow_{\prec} B \stackrel{\text{def}}{=} \downarrow_{\prec} A \cap \downarrow_{\prec} B$.

Proof. First, (3.10) implies

$$a \prec a' \leq \bigvee A \ \rightarrow \ \exists C \in \text{Fin}(a' \downarrow_{\prec} A) (a \prec \bigvee C), \quad (3.12)$$

where $a' \downarrow_{\prec} A \stackrel{\text{def}}{=} \{a'\} \downarrow_{\prec} A$. This can be proved by induction on the size of A . Note that the antecedent of (3.12) can be equivalently stated as $a \prec a' \prec \bigvee A$. Now, suppose $a \prec a' \leq \bigvee A$ and $a' \leq \bigvee B$. By (3.12), there exists $C \in \text{Fin}(a' \downarrow_{\prec} A) \subseteq \text{Fin}(\bigvee B \downarrow_{\prec} A)$ such that $a \prec \bigvee C$, and since S is strong, there exists $D \prec_L C$ such that $a \leq \bigvee D$. For each $d \in D$, there exist $c \in \downarrow_{\prec} A$ and $E_d \in \text{Fin}(c \downarrow_{\prec} B)$ such that $d \prec \bigvee E_d$ by (3.12). Thus, $E_d \in \text{Fin}(A \downarrow_{\prec} B)$ for each $d \in D$. Then, by putting $E = \bigcup_{d \in D} E_d$, we have $E \in \text{Fin}(A \downarrow_{\prec} B)$ and $\bigvee D \prec \bigvee E$, and so $a \prec \bigvee E$.

Conversely, assume that (3.11) holds, and let $a \prec b \leq c \vee d$. By letting $a' = b$, $A = \{b\}$, and $B = \{c, d\}$ in (3.11), we find $C \in \text{Fin}(b \downarrow_{\prec} \{c, d\})$ such that $a \prec \bigvee C$. Split C into C_c and C_d such that $C = C_c \cup C_d$, $C_c \in \text{Fin}(b \downarrow_{\prec} c)$, and $C_d \in \text{Fin}(b \downarrow_{\prec} d)$. By putting $a_1 = \bigvee C_c$ and $a_2 = \bigvee C_d$, we have $a \prec a_1 \vee a_2$, $a_1 \in (b \downarrow_{\prec} c)$, and $a_2 \in (b \downarrow_{\prec} d)$. \square

As in the case of (3.10), the antecedent of (3.11) can be equivalently stated as $a \prec a' \prec \bigvee A$ and $a' \prec \bigvee B$. By simple induction, one can show that (3.11) is further equivalent to

$$a \prec a' \ \& \ \forall i < n (a' \leq \bigvee A_i) \ \rightarrow \ \exists C \in \text{Fin}\left(\bigcap_{i < n} \downarrow_{\prec} A_i\right) (a \prec \bigvee C) \quad (3.13)$$

for finitely many $A_0, \dots, A_{n-1} \in \text{Fin}(S)$.⁴

The following proposition says that localized strong proximity \vee -semilattices capture the notion of locally compact locale.

Proposition 3.34. *A strong proximity \vee -semilattice S is localized if and only if the collection $\text{RIIdl}(S)$ of rounded ideals of S is a frame.*

Proof. Let $(S, 0, \vee, \prec)$ be a strong proximity \vee -semilattice. Since $\text{RIIdl}(S)$ has all joins, $\text{RIIdl}(S)$ has finite meets characterised by

$$1 \stackrel{\text{def}}{=} \downarrow_{\prec} S, \quad I \wedge J \stackrel{\text{def}}{=} \downarrow_{\prec} (I \cap J). \quad (3.14)$$

Since $\text{RIIdl}(S)$ is a continuous lattice, finite meets distribute over directed joins (cf. Johnstone [Joh82, Chapter VII, Lemma 4.1]). Thus, it suffices to show that the condition (3.10) is equivalent to the distributivity of finite meets over finite joins, i.e.,

$$I \wedge (J \vee K) = (I \wedge J) \vee (I \wedge K) \quad (3.15)$$

for all $I, J, K \in \text{RIIdl}(S)$.

First, suppose that S is localized. Let $a \in I \wedge (J \vee K)$. Since S is strong and J and K are rounded, we have $J \vee K = \bigcup_{c \in J, d \in K} \downarrow_{\prec} (c \vee d) = \bigcup_{c \in J, d \in K} \downarrow_{\leq} (c \vee d)$, and so $I \wedge (J \vee K) = \downarrow_{\prec} (I \cap \bigcup_{c \in J, d \in K} \downarrow_{\leq} (c \vee d))$. Thus, there exist $b \in I$, $c \in J$, and $d \in K$ such

⁴In the case $n = 0$, we assume $\bigcap_{i < n} \downarrow_{\prec} A_i = S$.

that $a \prec b \leq c \vee d$. Then, by (3.10), there exist $a_1 \in b \downarrow_{\prec} c$ and $a_2 \in b \downarrow_{\prec} d$ such that $a \prec a_1 \vee a_2$, and since S is strong, there exist a'_1 and a'_2 such that $a'_1 \prec a_1$, $a'_2 \prec a_2$ and $a \leq a'_1 \vee a'_2$. Then $a \in (I \wedge J) \vee (I \wedge K)$.

Conversely, assume that (3.15) holds for all $I, J, K \in \text{RIIdl}(S)$, and let $a \prec b \leq c \vee d$. Choose $a' \in S$ such that $a \prec a' \prec b$. Then, $a' \in \downarrow_{\prec} b \cap (\downarrow_{\prec} c \vee \downarrow_{\prec} d)$, and so $a \in \downarrow_{\prec} b \wedge (\downarrow_{\prec} c \vee \downarrow_{\prec} d)$. Thus $a \in (\downarrow_{\prec} b \wedge \downarrow_{\prec} c) \vee (\downarrow_{\prec} b \wedge \downarrow_{\prec} d)$ by (3.15). Then, there exist $a_1 \in b \downarrow_{\prec} c$ and $a_2 \in b \downarrow_{\prec} d$ such that $a \prec a_1 \vee a_2$. \square

A continuous lattice has finite meets, so it is a continuous meet semilattice, i.e., continuous domain which has finite meets. In classical domain theory, the category of continuous meet semilattices and Scott continuous meet semilattice homomorphisms is equivalent to the category of algebras of the upper powerdomains of continuous domains (cf. Schalk [Sch93, Section 7.2.5]). In the point-free setting, Vickers [Vic93, Section 5.1] conjectured that this domain theoretic characterisation should hold for the upper powerlocale on the category of infosys. In what follows, we confirm his conjecture in the dual context of PxPos by showing that the category of coalgebras of the upper powerlocale on PxPos is dually equivalent to that of continuous meet semilattices and Scott continuous meet semilattice homomorphisms.

In the context of infosys [Vic93], the construction of the upper powerlocale is given as in Definition 3.36 below, which corresponds to the upper powerlocale on the Scott topologies of continuous domains [Vic93, Theorem 4.3 (iii)].

Definition 3.35. For a relation $r \subseteq S \times S'$, its *upper extension* $r_U \subseteq \text{Fin}(S) \times \text{Fin}(S')$ is defined by

$$A r_U B \stackrel{\text{def}}{\iff} \forall b \in B \exists a \in A (a r b).$$

Definition 3.36 [Vic93, Definition 4.1]. Let (S, \prec) be a proximity poset. The *upper powerlocale* $\text{P}_U(S)$ of (S, \prec) is a proximity poset $(\text{Fin}(S), \leq_U, \prec_U)$ where $(\text{Fin}(S), \leq_U)$ denotes the poset reflection of the preorder \leq_U .

Note that $\text{P}_U(S)$ is indeed a proximity poset: the only non-trivial property to be checked is that $\downarrow_{\prec_U} A$ is directed for each $A \in \text{Fin}(S)$. To see this, let $B, C \in \downarrow_{\prec_U} A$. For each $a \in A$, there exist $b_a \in B$ and $c_a \in C$ such that $b_a \prec a$ and $c_a \prec a$. Thus, there exists $d_a \prec a$ such that $b_a \leq d_a$ and $c_a \leq d_a$. Put $D = \{d_a \mid a \in A\}$. Then, $D \prec_U A$, $B \leq_U D$, and $C \leq_U D$. Similarly, one can show that $\downarrow_{\prec_U} A$ is inhabited.

The construction $\text{P}_U(S)$ gives rise to a functor $\text{P}_U: \text{PxPos} \rightarrow \text{PxPos}$, which is defined on morphisms as follows:

$$\text{P}_U(r) \stackrel{\text{def}}{=} r_U.$$

As in the previous paragraph, one can show that P_U is well-defined on morphisms, i.e., if $r: (S, \prec) \rightarrow (S', \prec')$ is an approximable relation, then r_U is an approximable relation from $\text{P}_U(S)$ to $\text{P}_U(S')$.

There are approximable relations $\varepsilon_S^U: \text{P}_U(S) \rightarrow S$ and $\nu_S^U: \text{P}_U(S) \rightarrow \text{P}_U(\text{P}_U(S))$ defined by

$$\begin{aligned} A \varepsilon_S^U a &\stackrel{\text{def}}{\iff} A \prec_U \{a\}, \\ A \nu_S^U \mathcal{U} &\stackrel{\text{def}}{\iff} A \prec_U \bigcup \mathcal{U}. \end{aligned}$$

It is routine to show that $\langle \text{P}_U, \varepsilon^U, \nu^U \rangle$ is a KZ-comonad on PxPos .

Proposition 3.37. *A proximity poset (S, \prec) is a P_U -coalgebra if and only if $\text{RI}dl(S)$ has finite meets.*

Proof. Suppose that S has a P_U -coalgebra structure $\alpha: S \rightarrow P_U(S)$. Put $\top = \alpha^{-}\emptyset$. Since $\varepsilon_S^U \circ \alpha = \text{id}_S$, for any $a, b \in S$ such that $b \prec a$, there exists $A \in \text{Fin}(S)$ such that $b \alpha A \prec_U \{a\}$. Since $A \leq_U \emptyset$ and α is approximable, we have $b \in \top$. Thus, $\downarrow_{\prec} a \subseteq \top$ for all $a \in S$, which implies that \top is the greatest element of $\text{RI}dl(S)$. Next, we show that $\text{RI}dl(S)$ has binary meets. To this end, it suffices to show that $\downarrow_{\prec} a$ and $\downarrow_{\prec} b$ have a meet for all $a, b \in S$; for then we have $I \wedge J = \bigcup_{a \in I, b \in J} \downarrow_{\prec} a \wedge \downarrow_{\prec} b$ for all $I, J \in \text{RI}dl(S)$. So let $a, b \in S$, and put $a \wedge b = \alpha^{-}\{a, b\}$, which is in $\text{RI}dl(S)$ because α is approximable. For each $c \in a \wedge b$, there exists $C \in \text{Fin}(S)$ such that $c \alpha C \prec_U \{a, b\}$. Then, $C \prec_U \{a\}$ and $C \prec_U \{b\}$, so $C \varepsilon_S^U a$ and $C \varepsilon_S^U b$. Since $\varepsilon_S^U \circ \alpha = \text{id}_S$, we have $c \in \downarrow_{\prec} a \cap \downarrow_{\prec} b$. Thus, $a \wedge b$ is a lower bound of $\downarrow_{\prec} a$ and $\downarrow_{\prec} b$. Let $I \in \text{RI}dl(S)$ such that $I \subseteq \downarrow_{\prec} a \cap \downarrow_{\prec} b$, and let $c \in I$ and $c' \prec c$. Then $\{c\} \prec_U \{a, b\}$, and since $\varepsilon_S^U \circ \alpha = \text{id}_S$, there exists $C \prec_U \{c\}$ such that $c' \alpha C$. Then $C \prec_U \{a, b\}$, and since α is approximable, we have $c' \in a \wedge b$. Thus, $I = \bigvee_{c \in I} \downarrow_{\prec} c \subseteq a \wedge b$. Therefore, $a \wedge b$ is a meet of $\downarrow_{\prec} a$ and $\downarrow_{\prec} b$.

Conversely, suppose that $\text{RI}dl(S)$ has finite meets. Define a relation $\alpha \subseteq S \times \text{Fin}(S)$ by

$$a \alpha A \stackrel{\text{def}}{\iff} \exists b \in S (a \prec b \ \& \ \{b\} \prec_U A). \quad (3.16)$$

Note that $\alpha^{-}A = \bigwedge_{a \in A} \downarrow_{\prec} a$, which shows that α satisfies (AppI). From (3.16), it is also easy to see that α satisfies (AppU) and that $\alpha \circ \prec = \alpha = \prec_U \circ \alpha$. Thus, α is an approximable relation from S to $P_U(S)$. Then, we clearly have $\varepsilon_S^U \circ \alpha = \text{id}_S$. Hence, α is a P_U -coalgebra by Corollary 3.27 (1). \square

Remark 3.38. Since P_U is a KZ-comonad, a P_U -coalgebra structure on a proximity poset, if it exists, is unique. Thus, it is always characterised by (3.16).

Next, we give an intrinsic characterisation of homomorphisms between P_U -coalgebras.

Definition 3.39 [Vic93, Definition 3.6]. An approximable relation $r: (S, \prec) \rightarrow (S', \prec')$ between proximity posets is *Lawson approximable* if

- (1) $a \prec a' \rightarrow \exists b \in S' (a r b)$,
- (2) $a \prec a' r b \ \& \ a' r c \rightarrow \exists d \in b \downarrow_{\prec'} c (a r d)$.

In Vickers [Vic93, Definition 3.6], Lawson approximable relation is defined by

$$a \prec a' \ \& \ \{a'\} r_U B \rightarrow \exists b \in S' (a r b \ \& \ \{b\} \prec'_U B) \quad (3.17)$$

for each $a, a' \in S$ and $B \in \text{Fin}(S')$. By induction on the size of B , one can show that (3.17) is equivalent to the two conditions in Definition 3.39.

The following is noted by Vickers [Vic93, Section 5.1]. We give a proof for the sake of completeness.

Proposition 3.40. *Let $\alpha: S \rightarrow P_U(S)$ and $\beta: S' \rightarrow P_U(S')$ be P_U -coalgebras on proximity posets (S, \prec) and (S', \prec') . For any approximable relation $r: S \rightarrow S'$, the following are equivalent:*

- (1) r is a P_U -coalgebra homomorphism.
- (2) r is Lawson approximable.
- (3) The Scott continuous function $\overline{f_r}: \text{RI}dl(S') \rightarrow \text{RI}dl(S)$ given by (3.6) preserves finite meets.

Proof. Before getting down to the proof, note that α and β are given by (3.16) (cf. Remark 3.38).

(1 \rightarrow 2) Suppose that r is a P_U -coalgebra homomorphism. Let $a \prec a'$. Then $a \alpha \emptyset$. Since $\emptyset P_U(r) \emptyset$, there exists $b \in S'$ such that $a r b$ and $b \beta \emptyset$. Next, suppose $a \prec a' r b$ and $a' r c$. Then $a (P_U(r) \circ \alpha) \{b, c\}$. Thus, there exists $d \in S'$ such that $a r d$ and $d \beta \{b, c\}$. Then $d \in b \downarrow_{\prec'} c$.

(2 \rightarrow 1) Suppose that r is Lawson approximable, and let $a (\beta \circ r) B$. Then, there exist $b, b' \in S'$ such that $a r b \prec' b'$ and $\{b'\} \prec'_U B$. Thus, there exists $a' \in S$ such that $a \prec a'$ and $\{a'\} r_U B$. Then $a (P_U(r) \circ \alpha) B$. Conversely, suppose $a (P_U(r) \circ \alpha) B$. Then, there exist $a' \in S$ and $A \in \text{Fin}(S)$ such that $a \prec a'$ and $\{a'\} \prec_U A r_U B$, so that $\{a'\} r_U B$. Since r is Lawson approximable, there exists $b \in S'$ such that $a r b$ and $\{b\} \prec'_U B$ by (3.17). Then, $a (\beta \circ r) B$.

The equivalence (2 \leftrightarrow 3) is also straightforward to check. \square

Theorem 3.41. *The category of P_U -coalgebras over $P \times \text{Pos}$ is dually equivalent to the category of continuous meet semilattices and Scott continuous meet semilattice homomorphisms.*

Proof. By Theorem 3.5, Proposition 3.37, and Proposition 3.40. \square

Remark 3.42. As of this writing, we still lack an intrinsic characterisation of continuous meet semilattices in terms of proximity posets. As far as we know, P_U -coalgebras provide the best predicative characterisation of continuous meet semilattices so far.

Definition 3.43. A join-preserving Lawson approximable relations between localized strong proximity \vee -semilattices is called a *proximity relation*.

Let $\text{LSP} \times \text{JLat}$ be the category of localized strong proximity \vee -semilattices and proximity relations.

Theorem 3.44. *$\text{LSP} \times \text{JLat}$ is equivalent to the category of locally compact locales.*

Proof. By Theorem 3.18 and Proposition 3.34, the class of localized strong proximity \vee -semilattices characterises locally compact locales among continuous lattices. Since every strong proximity \vee -semilattice is a P_U -coalgebra by Proposition 3.37, proximity relations between localized strong proximity \vee -semilattices characterise locale maps between the corresponding locally compact locales by Proposition 3.40. \square

3.6. Algebraic theory of locally compact locales. We give another predicative characterisation of locally compact locales in terms of algebras of the double powerlocale on the category of continuous domains. Specifically, building on the result of Section 3.5, we characterise localized strong proximity \vee -semilattices by the coalgebras of the double powerlocale on $P \times \text{Pos}$. This algebraic characterisation of locally compact locales has been conjectured by Vickers in the dual context of infosys [Vic93, Section 5.1].

3.6.1. *Finite subsets.* We first recall some properties of finitely enumerable subsets from Vickers [Vic04b, Section 4]. Let S be a set. For each $\mathcal{U} \in \text{Fin}(\text{Fin}(S))$, define $\mathcal{U}^* \in \text{Fin}(\text{Fin}(S))$ inductively by

$$\emptyset^* \stackrel{\text{def}}{=} \{\emptyset\}, \quad (\mathcal{U} \cup \{A\})^* \stackrel{\text{def}}{=} \{B \cup C \mid B \in \mathcal{U}^* \ \& \ C \in \text{Fin}^+(A)\},$$

where $\text{Fin}^+(A)$ denotes the set of inhabited finitely enumerable subsets of A .⁵ Note that for each $B \in \mathcal{U}^*$, we have $B \not\subseteq A$ for all $A \in \mathcal{U}$, where

$$U \not\subseteq V \stackrel{\text{def}}{\iff} \exists a \in S (a \in U \cap V)$$

for $U, V \subseteq S$.

Example 3.45. For $a, b, c \in S$, we have

$$\begin{aligned} \{\{a\} \{b\}\}^* &= \{\{a, b\}\}, \\ \{\{a, b\}, \{c\}\}^* &= \{A \cup \{c\} \mid A \in \text{Fin}^+(\{a, b\})\} = \{\{a, b, c\}, \{a, c\}, \{b, c\}\}, \\ \{\{a, b\}, \{c\}\}^{**} &= \{A \cup B \cup C \mid A \in \text{Fin}^+(\{a, c\}), B \in \text{Fin}^+(\{b, c\}), C \in \text{Fin}^+(\{a, b, c\})\} \\ &= \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c\}\}. \end{aligned}$$

Lemma 3.46. *Let S be a set. For each $\mathcal{U} \in \text{Fin}(\text{Fin}(S))$ and $U \subseteq S$, we have*

- (1) $\forall C \in \mathcal{U} (U \not\subseteq C) \rightarrow \exists B \in \mathcal{U}^* (B \subseteq U)$,
- (2) $\forall C \in \mathcal{U}^* (U \not\subseteq C) \rightarrow \exists B \in \mathcal{U} (B \subseteq U)$,
- (3) $\forall A \in \mathcal{U}^{**} \exists B \in \mathcal{U} (B \subseteq A)$ and $\forall B \in \mathcal{U} \exists A \in \mathcal{U}^{**} (A \subseteq B)$.

Proof. Items 1 and 2 correspond to Proposition 15 and Lemma 16 in Vickers [Vic04b], respectively. Item 3 follows from 1 and 2 since $\forall C \in \mathcal{U}^* (C \not\subseteq A)$ for all $A \in \mathcal{U}^{**}$ and $\forall C \in \mathcal{U}^* (C \not\subseteq B)$ for all $B \in \mathcal{U}$. \square

Lemma 3.47. *For any relation $r \subseteq S \times S'$, we have*

$$\mathcal{U} (r_L)_U \mathcal{V} \leftrightarrow \mathcal{U}^* (r_U)_L \mathcal{V}^*$$

for all $\mathcal{U} \in \text{Fin}(\text{Fin}(S))$ and $\mathcal{V} \in \text{Fin}(\text{Fin}(S'))$.

Proof. First, suppose $\mathcal{U} (r_L)_U \mathcal{V}$, and let $A' \in \mathcal{U}^*$. For each $B \in \mathcal{V}$, there exists $A \in \mathcal{U}$ such that $A r_L B$. Since $A \not\subseteq A'$, we have $r A' \not\subseteq B$. Thus, by Lemma 3.46 (1), there exists $B' \in \mathcal{V}^*$ such that $B' \subseteq r A'$, i.e., $A' r_U B'$. Thus, $\mathcal{U}^* (r_U)_L \mathcal{V}^*$.

Conversely, suppose $\mathcal{U}^* (r_U)_L \mathcal{V}^*$. Then, $\mathcal{V}^* ((r^-)_L)_U \mathcal{U}^*$, so by the proof of the converse, we have $\mathcal{V}^{**} ((r^-)_U)_L \mathcal{U}^{**}$, i.e., $\mathcal{U}^{**} (r_L)_U \mathcal{V}^{**}$. Since $\mathcal{U} \subseteq_U \mathcal{U}^{**}$ and $\mathcal{V}^{**} \subseteq_U \mathcal{V}$ by Lemma 3.46 (3), we have $\mathcal{U} (r_L)_U \mathcal{V}$. \square

The following is also useful later.

Lemma 3.48. *Let S be a set. For each $A \in \text{Fin}(S)$ and $\mathcal{U} \in \text{Fin}(\text{Fin}(S))$, we have*

- (1) $\{A\}^* \subseteq_U \text{Sin}(A) \subseteq_U \{A\}^*$,
- (2) $\{\text{Sin}(A) \mid A \in \mathcal{U}\}^* \subseteq_U \{\text{Sin}(B) \mid B \in \mathcal{U}^*\} \subseteq_U \{\text{Sin}(A) \mid A \in \mathcal{U}\}^*$.

⁵In Vickers [Vic04b, Section 4], the set \mathcal{U}^* is equal to $\{\text{Im } \gamma \mid \gamma \in \text{Ch}(\mathcal{U})\}$, where $\text{Ch}(\mathcal{U})$ is the set of choices of \mathcal{U} and $\text{Im } \gamma$ is the image of a choice γ ; see Definition 12 and Definition 13, and the proof of Proposition 14 in [Vic04b].

Proof. 1. This is immediate from $\{A\}^* = \text{Fin}^+(A)$.

2. For the first inclusion, let $B \in \mathcal{U}^*$. Then, $B \checkmark A$ for all $A \in \mathcal{U}$, i.e., $\text{Sin}(B) \checkmark \text{Sin}(A)$ for all $A \in \mathcal{U}$. By Lemma 3.46 (1), there exists $C \in \{\text{Sin}(A) \mid A \in \mathcal{U}\}^*$ such that $C \subseteq \text{Sin}(B)$. Next, let $C \in \{\text{Sin}(A) \mid A \in \mathcal{U}\}^*$. Then, $C \checkmark \text{Sin}(A)$ for each $A \in \mathcal{U}$, i.e., $\{a \in S \mid \{a\} \in C\} \checkmark A$ for each $A \in \mathcal{U}$. By Lemma 3.46 (1), there exists $B \in \mathcal{U}^*$ such that $B \subseteq \{a \in S \mid \{a\} \in C\}$. Then $\text{Sin}(B) \subseteq C$. \square

3.6.2. *Double powerlocale.* As is well known in locale theory [Vic04a], the two compositions $\text{P}_U \circ \text{P}_L$ and $\text{P}_L \circ \text{P}_U$ of the upper and lower powerlocales are naturally isomorphic. Indeed, for each proximity poset (S, \prec) , there is an approximable relation $\sigma_S: \text{P}_U(\text{P}_L(S)) \rightarrow \text{P}_L(\text{P}_U(S))$ defined by

$$\mathcal{U} \sigma_S \mathcal{V} \stackrel{\text{def}}{\iff} \mathcal{U} (\prec_L)_U \mathcal{V}^*. \quad (3.18)$$

For any approximable relation $r: (S, \prec) \rightarrow (S', \prec')$, we have

$$\begin{aligned} & \mathcal{U} (\sigma_{S'} \circ \text{P}_U(\text{P}_L(r))) \mathcal{V} \\ & \iff \mathcal{U} ((\prec' \circ r)_L)_U \mathcal{V}^* \\ & \iff \mathcal{U} ((r \circ \prec)_L)_U \mathcal{V}^* && (r \text{ is approximable}) \\ & \iff \exists \mathcal{W} (\mathcal{U} (\prec_L)_U \mathcal{W}^{**} \& \mathcal{W}^{**} (r_L)_U \mathcal{V}^*) && (\text{by Lemma 3.46 (3)}) \\ & \iff \exists \mathcal{W} (\mathcal{U} (\prec_L)_U \mathcal{W}^{**} \& \mathcal{W}^* (r_U)_L \mathcal{V}) && (\text{by Lemma 3.47}) \\ & \iff \mathcal{U} (\text{P}_L(\text{P}_U(r)) \circ \sigma_S) \mathcal{V}, && (\text{by Lemma 3.46 (3)}) \end{aligned}$$

where \mathcal{W} ranges over $\text{Fin}(\text{Fin}(S))$. Thus, σ is natural in S . Moreover, there is an approximable relation $\tau_S: \text{P}_L(\text{P}_U(S)) \rightarrow \text{P}_U(\text{P}_L(S))$ defined by

$$\mathcal{V} \tau_S \mathcal{U} \stackrel{\text{def}}{\iff} \mathcal{V} (\prec_U)_L \mathcal{U}^*. \quad (3.19)$$

We have

$$\begin{aligned} & \mathcal{U} (\tau_S \circ \sigma_S) \mathcal{V} \\ & \iff \exists \mathcal{W} (\mathcal{U} (\prec_L)_U \mathcal{W}^* \& \mathcal{W} (\prec_U)_L \mathcal{V}^*) \\ & \iff \exists \mathcal{W} (\mathcal{U} (\prec_L)_U \mathcal{W}^* \& \mathcal{W}^* (\prec_L)_U \mathcal{V}^{**}) && (\text{by Lemma 3.47}) \\ & \iff \exists \mathcal{W} (\mathcal{U} (\prec_L)_U \mathcal{W}^* \& \mathcal{W}^* (\prec_L)_U \mathcal{V}) && (\text{by Lemma 3.46 (3)}) \\ & \iff \mathcal{U} (\prec_L)_U \mathcal{V}, && (\text{by Lemma 3.46 (3)}) \end{aligned}$$

where \mathcal{W} ranges over $\text{Fin}(\text{Fin}(S))$. Thus, $\tau_S \circ \sigma_S = \text{id}_{\text{P}_U(\text{P}_L(S))}$. Similarly, we have $\sigma_S \circ \tau_S = \text{id}_{\text{P}_L(\text{P}_U(S))}$. Hence σ and τ are inverse to each other.

We show that $\sigma: \text{P}_U \circ \text{P}_L \rightarrow \text{P}_L \circ \text{P}_U$ satisfies the distributive law of comonad, which is the dual of the distributive law of monad [Bec69].

Definition 3.49. Let $\langle K, \varepsilon^K, \nu^K \rangle$ and $\langle T, \varepsilon^T, \nu^T \rangle$ be comonads on a category \mathbb{C} . A *distributive law* of K over T is a natural transformation $\sigma: T \circ K \rightarrow K \circ T$ which makes the

following diagrams commute:

$$\begin{array}{ccc}
& T \circ K & \\
\varepsilon^T K \swarrow & \downarrow \sigma & \searrow T\varepsilon^K \\
K & K \circ T & T \\
K \xleftarrow{K\varepsilon^T} & & \xrightarrow{\varepsilon^K T} T
\end{array}
\quad (1) \quad
\begin{array}{ccc}
& T \circ K & \\
\sigma \downarrow & \downarrow \sigma & \downarrow T\sigma \\
K \circ T \circ K & K \circ T & T \circ K \circ T \\
K\sigma \downarrow & \downarrow \sigma & \downarrow \sigma T \\
K \circ K \circ T & K \circ T & K \circ T \circ T
\end{array}
\quad (2) \quad
\begin{array}{ccc}
T \circ K \circ K & \xleftarrow{T\nu^K} T \circ K & \xrightarrow{\nu^T K} T \circ T \circ K \\
\sigma \downarrow & \downarrow \sigma & \downarrow T\sigma \\
K \circ T \circ K & & T \circ K \circ T \\
K\sigma \downarrow & \downarrow \sigma & \downarrow \sigma T \\
K \circ K \circ T & \xleftarrow{\nu^K T} K \circ T & \xrightarrow{K\nu^T} K \circ T \circ T
\end{array}$$

In this situation, $T \circ K$ is a comonad with a counit ε and a co-multiplication ν defined by

$$\begin{aligned}
\varepsilon &= T \circ K \xrightarrow{T\varepsilon^K} T \xrightarrow{\varepsilon^T} \text{id}_{\mathbb{C}}, \\
\nu &= T \circ K \xrightarrow{T\nu^K} T \circ K \circ K \xrightarrow{\nu^T K \circ K} T \circ T \circ K \circ K \xrightarrow{T\sigma K} T \circ K \circ T \circ K.
\end{aligned}
\quad (3.20)$$

Proposition 3.50. σ given by (3.18) is a distributive law of P_L over P_U .

Proof. The commutativity of the diagrams (1) and (2) are easy to check. We show that the diagram (3) commutes for $T = P_U$ and $K = P_L$. Fix a proximity poset (S, \prec) . It suffices to show

$$P_L(\tau_S) \circ \nu_{P_U(S)}^L \circ \sigma_S \leq \sigma_{P_L(S)} \circ P_U(\nu_S^L), \quad (3.21)$$

$$\sigma_{P_L(S)} \circ P_U(\nu_S^L) \circ \tau_S \leq P_L(\tau_S) \circ \nu_{P_U(S)}^L. \quad (3.22)$$

In the proof below, we identify each proximity poset with its underlying set.

First, to see that (3.21) holds, let $\mathcal{X} \in P_U(P_L(S))$ and $\mathbb{U} \in P_L(P_U(P_L(S)))$, and suppose $\mathcal{X} (P_L(\tau_S) \circ \nu_{P_U(S)}^L \circ \sigma_S) \mathbb{U}$. Then, there exist $\mathcal{Y} \in P_L(P_U(S))$ and $\mathbb{V} \in P_L(P_L(P_U(S)))$ such that $\mathcal{X} \sigma_S \mathcal{Y} \nu_{P_U(S)}^L \mathbb{V} P_L(\tau_S) \mathbb{U}$. Thus

- (1) $\mathcal{X} (\prec_L)_U \mathcal{Y}^*$,
- (2) $\text{Sin}(\mathcal{Y}) ((\prec_U)_L)_L \mathbb{V}$,
- (3) $\mathbb{V} ((\prec_U)_L)_L \{\mathcal{U}^* \mid \mathcal{U} \in \mathbb{U}\}$.

By 2 and 3, we have $\text{Sin}(\mathcal{Y}) ((\prec_U)_L)_L \{\mathcal{U}^* \mid \mathcal{U} \in \mathbb{U}\}$. Then

$$\begin{aligned}
& \text{Sin}(\mathcal{Y}) ((\prec_U)_L)_L \{\mathcal{U}^* \mid \mathcal{U} \in \mathbb{U}\} \\
& \iff \{\{Y\}^{**} \mid Y \in \mathcal{Y}\} ((\prec_U)_L)_L \{\mathcal{U}^* \mid \mathcal{U} \in \mathbb{U}\} && \text{(by Lemma 3.46 (3))} \\
& \iff \{\{Y\}^* \mid Y \in \mathcal{Y}\} ((\prec_L)_U)_L \mathbb{U} && \text{(by Lemma 3.47)} \\
& \iff \{\text{Sin}(Y) \mid Y \in \mathcal{Y}\} ((\prec_L)_U)_L \mathbb{U} && \text{(by Lemma 3.48 (1))} \\
& \iff \{\text{Sin}(Y) \mid Y \in \mathcal{Y}\}^* ((\prec_L)_L)_U \mathbb{U}^* && \text{(by Lemma 3.47)} \\
& \iff \{\text{Sin}(Y) \mid Y \in \mathcal{Y}^*\} ((\prec_L)_L)_U \mathbb{U}^*. && \text{(by Lemma 3.48 (2))}
\end{aligned}$$

On the other hand, 1 implies $\{\text{Sin}(X) \mid X \in \mathcal{X}\} ((\prec_L)_L)_U \{\text{Sin}(Y) \mid Y \in \mathcal{Y}^*\}$. Thus, by putting $\mathbb{W} = \{\text{Sin}(Y) \mid Y \in \mathcal{Y}^*\}$, we have $\mathcal{X} P_U(\nu_S^L) \mathbb{W}$ and $\mathbb{W} \sigma_{P_L(S)} \mathbb{U}$. Hence $\mathcal{X} (\sigma_{P_L(S)} \circ P_U(\nu_S^L)) \mathbb{U}$.

Next, we verify (3.22). Let $\mathcal{Y} \in P_L(P_U(S))$ and $\mathbb{U} \in P_L(P_U(P_L(S)))$, and suppose $\mathcal{Y} (\sigma_{P_L(S)} \circ P_U(\nu_S^L) \circ \tau_S) \mathbb{U}$. Then, there exist $\mathcal{X} \in P_U(P_L(S))$ and $\mathbb{W} \in P_U(P_L(P_L(S)))$ such that $\mathcal{Y} \tau_S \mathcal{X} P_U(\nu_S^L) \mathbb{W} \sigma_{P_L(S)} \mathbb{U}$. Thus

- (4) $\mathcal{Y} (\prec_U)_L \mathcal{X}^*$,
- (5) $\{\text{Sin}(X) \mid X \in \mathcal{X}\} ((\prec_L)_L)_U \mathbb{W}$,

(6) $\mathbb{W}((\prec_L)_L)_U \mathbb{U}^*$.

By 5 and 6, we have $\{\text{Sin}(X) \mid X \in \mathcal{X}\}((\prec_L)_L)_U \mathbb{U}^*$. Then

$$\begin{aligned}
& \{\text{Sin}(X) \mid X \in \mathcal{X}\}((\prec_L)_L)_U \mathbb{U}^* \\
& \iff \{\text{Sin}(X) \mid X \in \mathcal{X}\}^{**}((\prec_L)_L)_U \mathbb{U}^* && \text{(by Lemma 3.46 (3))} \\
& \iff \{\text{Sin}(X) \mid X \in \mathcal{X}\}^*((\prec_L)_U)_L \mathbb{U} && \text{(by Lemma 3.47)} \\
& \iff \{\text{Sin}(X) \mid X \in \mathcal{X}^*\}((\prec_L)_U)_L \mathbb{U} && \text{(by Lemma 3.48 (2))} \\
& \iff \{\{X\}^* \mid X \in \mathcal{X}^*\}((\prec_L)_U)_L \mathbb{U} && \text{(by Lemma 3.48 (1))} \\
& \iff \{\{X\}^{**} \mid X \in \mathcal{X}^*\}((\prec_U)_L)_L \{\mathcal{U}^* \mid \mathcal{U} \in \mathbb{U}\} && \text{(by Lemma 3.47)} \\
& \iff \text{Sin}(\mathcal{X}^*)((\prec_U)_L)_L \{\mathcal{U}^* \mid \mathcal{U} \in \mathbb{U}\}. && \text{(by Lemma 3.46 (3))}
\end{aligned}$$

On the other hand, 4 implies $\text{Sin}(\mathcal{Y})((\prec_U)_L)_L \text{Sin}(\mathcal{X}^*)$. Thus, $\mathcal{Y} \nu_{\text{P}_U(S)}^L \text{Sin}(\mathcal{X}^*)$ and $\text{Sin}(\mathcal{X}^*) \text{P}_L(\tau_S) \mathbb{U}$. Hence $\mathcal{Y}(\text{P}_L(\tau_S) \circ \nu_{\text{P}_U(S)}^L) \mathbb{U}$.

The commutativity of (4) can be proved similarly. \square

Thus, $\text{P}_U \circ \text{P}_L$ and $\text{P}_L \circ \text{P}_U$ give rise to equivalent comonads on $\text{P}\times\text{Pos}$.

Definition 3.51. The *double powerlocale* on $\text{P}\times\text{Pos}$ is the composition $\text{P}_U \circ \text{P}_L$ (or equivalently the composition $\text{P}_L \circ \text{P}_U$).⁶

Next, we recall some properties of distributive laws to obtain convenient characterisations of coalgebras of double powerlocales and homomorphisms between them. Fix comonads $\langle T, \varepsilon^T, \nu^T \rangle$ and $\langle K, \varepsilon^K, \nu^K \rangle$ on a category \mathbb{C} and a distributive law $\sigma: T \circ K \rightarrow K \circ T$ of K over T . Let $\langle H, \varepsilon^H, \nu^H \rangle$ be the composite comonad $T \circ K$ where ε^H and ν^H are given by (3.20).

Lemma 3.52. If $\alpha: X \rightarrow HX$ is an H -coalgebra, then

$$\alpha_T \stackrel{\text{def}}{=} T\varepsilon_X^K \circ \alpha, \quad \alpha_K \stackrel{\text{def}}{=} \varepsilon_{KX}^T \circ \alpha$$

are T -coalgebra and K -coalgebra, respectively, and make the following diagram commute:

$$\begin{array}{ccc}
TKX & \xrightarrow{\sigma_X} & KTX \\
T\alpha_K \uparrow & & \uparrow K\alpha_T \\
TX & & KX \\
\alpha_T \swarrow & X & \searrow \alpha_K
\end{array}$$

Moreover, $\alpha = T\alpha_K \circ \alpha_T$.

Proof. By direct calculations, one can show $\varepsilon_X^T \circ \alpha_T = \text{id}_X$ and $\nu_X^T \circ \alpha_T = T\alpha_T \circ \alpha_T$, and similarly for α_K . Moreover,

$$\begin{aligned}
T\alpha_K \circ \alpha_T &= T(\varepsilon_{KX}^T \circ \alpha) \circ T\varepsilon_X^K \circ \alpha \\
&= T\varepsilon_{KX}^T \circ T\varepsilon_{TKX}^K \circ TK\alpha \circ \alpha && \text{(by the naturality of } \varepsilon^K \text{)} \\
&= T\varepsilon_{KX}^T \circ T\varepsilon_{TKX}^K \circ (T\sigma_{KX} \circ \nu_{KX}^T \circ T\nu_X^K) \circ \alpha && (\alpha \text{ is an } H\text{-coalgebra)} \\
&= T\varepsilon_{KX}^T \circ TT\varepsilon_{KX}^K \circ \nu_{KX}^T \circ T\nu_X^K \circ \alpha && \text{(by the diagram (2))}
\end{aligned}$$

⁶In this paper, we choose $\text{P}_U \circ \text{P}_L$ as the double powerlocale.

$$\begin{aligned}
&= T\varepsilon_{KX}^T \circ \nu_{KX}^T \circ T\varepsilon_{KX}^K \circ T\nu_X^K \circ \alpha && \text{(by the naturality of } \nu^T \text{)} \\
&= \alpha. && \text{(} K \text{ and } T \text{ are comonads)}
\end{aligned}$$

Similarly, we have $K\alpha_T \circ \alpha_K = \sigma_X \circ \alpha$. \square

Lemma 3.53. *Let $\alpha: X \rightarrow TX$ and $\beta: X \rightarrow KX$ be a T -coalgebra and a K -coalgebra, respectively. Then, $\gamma \stackrel{\text{def}}{=} T\beta \circ \alpha$ is an H -coalgebra if and only if $\sigma_X \circ T\beta \circ \alpha = K\alpha \circ \beta$.*

Proof. (\Rightarrow) This follows from Lemma 3.52 because

$$\begin{aligned}
\gamma_T &= T\varepsilon_X^K \circ T\beta \circ \alpha = \alpha, \\
\gamma_K &= \varepsilon_{KX}^T \circ T\beta \circ \alpha = \beta \circ \varepsilon_X^T \circ \alpha = \beta.
\end{aligned}$$

(\Leftarrow) If $\sigma_X \circ T\beta \circ \alpha = K\alpha \circ \beta$, then

$$\begin{aligned}
\varepsilon_X^H \circ \gamma &= \varepsilon_X^H \circ T\beta \circ \alpha \\
&= \varepsilon_X^T \circ T\varepsilon_X^K \circ T\beta \circ \alpha \\
&= \text{id}_X, && \text{(\alpha and } \beta \text{ are coalgebras)}
\end{aligned}$$

$$\begin{aligned}
\nu_X^H \circ \gamma &= \nu_X^H \circ T\beta \circ \alpha \\
&= T\sigma_{KX} \circ \nu_{KX}^T \circ T\nu_X^K \circ T\beta \circ \alpha && \text{(by (3.20))} \\
&= T\sigma_{KX} \circ \nu_{KX}^T \circ T(K\beta \circ \beta) \circ \alpha && \text{(\beta is a K-coalgebra)} \\
&= T\sigma_{KX} \circ TT(K\beta \circ \beta) \circ \nu_X^T \circ \alpha && \text{(by the naturality of } \nu^T \text{)} \\
&= T(KT\beta \circ \sigma_X) \circ TT\beta \circ \nu_X^T \circ \alpha && \text{(by the naturality of } \sigma \text{)} \\
&= T(KT\beta \circ \sigma_X) \circ TT\beta \circ T\alpha \circ \alpha && \text{(\alpha is a T-coalgebra)} \\
&= TKT\beta \circ TK\alpha \circ T\beta \circ \alpha && \text{(by } \sigma_X \circ T\beta \circ \alpha = K\alpha \circ \beta \text{)} \\
&= H\gamma \circ \gamma.
\end{aligned}$$

Thus, γ is an H -coalgebra. \square

Homomorphisms between H -coalgebras can be characterised by their underlying homomorphisms of T -coalgebras and K -coalgebras. In the following lemma, we use the notation from Lemma 3.52.

Lemma 3.54. *Let $\alpha: X \rightarrow HX$ and $\beta: Y \rightarrow HY$ be H -coalgebras. Then $f: X \rightarrow Y$ is an H -coalgebra homomorphism from α to β if and only if it is a T -coalgebra homomorphism from α_T to β_T and a K -coalgebra homomorphism from α_K to β_K .*

Proof. (\Rightarrow) Immediate from the naturality of ε^K and ε^T .

(\Leftarrow) If f is a homomorphism of the underlying T -coalgebra and K -coalgebra structures of α and β , then

$$\beta \circ f = (T\beta_K \circ \beta_T) \circ f = T\beta_K \circ Tf \circ \alpha_T = T(Kf \circ \alpha_K) \circ \alpha_T = Hf \circ \alpha. \quad \square$$

We come back to the context of the distributive law of P_L over P_U in $P_X\text{Pos}$. Note that by the last part of Lemma 3.52, a double powerlocale structure on a proximity poset, if it exists, is completely determined by the underlying P_L -coalgebra and P_U -coalgebra structures. Since P_L and P_U are (co)KZ-comonads, these coalgebra structures, if they exist, are unique. Thus, each proximity poset admits at most one double powerlocale coalgebra structure.

Proposition 3.55. *A strong proximity \vee -semilattice is localized if and only if it is a coalgebra of the double powerlocale over PxPos .*

Proof. Let $(S, 0, \vee, \prec)$ be a strong proximity \vee -semilattice. Before getting down to the proof, note that by Theorem 3.30 and Proposition 3.37, S is both a P_L -coalgebra and a P_U -coalgebra, and their coalgebra structures $\alpha: S \rightarrow \text{P}_L(S)$ and $\beta: S \rightarrow \text{P}_U(S)$ are given by (3.8) and (3.16), respectively. Then, by Lemma 3.50 and Lemma 3.53, S is a coalgebra of the double powerlocale if and only if

$$\sigma_S \circ \text{P}_U(\alpha) \circ \beta = \text{P}_L(\beta) \circ \alpha. \quad (3.23)$$

(\Rightarrow) Suppose that S is localized. It suffices to show (3.23), or equivalently

$$\tau_S \circ \text{P}_L(\beta) \circ \alpha \leq \text{P}_U(\alpha) \circ \beta, \quad (3.24)$$

$$\sigma_S \circ \text{P}_U(\alpha) \circ \beta \leq \text{P}_L(\beta) \circ \alpha. \quad (3.25)$$

First, to see that (3.24) holds, suppose $a (\tau_S \circ \text{P}_L(\beta) \circ \alpha) \mathcal{U}$. Then, there exist $A \in \text{Fin}(S)$ and $\mathcal{V} \in \text{Fin}(\text{Fin}(S))$ such that

- (1) $a \prec \bigvee A$,
- (2) $A \beta_L \mathcal{V}$,
- (3) $\mathcal{V} (\prec_U)_L \mathcal{U}^*$.

Then, 2 implies $\text{Sin}(A) (\prec_U)_L \mathcal{V}$, so by 3, we have $\text{Sin}(A) (\prec_U)_L \mathcal{U}^*$. Then, by Lemma 3.47 and Lemma 3.48 (1), we have $\{A\} (\prec_L)_U \mathcal{U}$. By putting $B = \{\bigvee A\}$, we have $a \beta B$ and $B \text{P}_U(\alpha) \mathcal{U}$. Thus, $a (\text{P}_U(\alpha) \circ \beta) \mathcal{U}$.

Next, to see that (3.25) holds, suppose $a (\sigma_S \circ \text{P}_U(\alpha) \circ \beta) \mathcal{V}$. Then, there exist $a' \in S$, $B \in \text{Fin}(S)$, and $\mathcal{U} \in \text{Fin}(\text{Fin}(S))$ such that

- (4) $a \prec a'$ and $\{a'\} \prec_U B$,
- (5) $B \alpha_U \mathcal{U}$,
- (6) $\mathcal{U} (\prec_L)_U \mathcal{V}^*$.

By 4 and 5, we have $a' \prec \bigvee C$ for all $C \in \mathcal{U}$. Since S is localized, there exists $A \in \text{Fin}(S)$ such that $a \prec \bigvee A$ and $A \prec_L C$ for all $C \in \mathcal{U}$ by (3.13), i.e., $\{A\} (\prec_L)_U \mathcal{U}$. Then, by Lemma 3.47, we have $\{A\}^* (\prec_U)_L \mathcal{U}^*$, so that $\text{Sin}(A) (\prec_U)_L \mathcal{U}^*$ by Lemma 3.48 (1). On the other hand, 6 implies $\mathcal{U}^* (\prec_U)_L \mathcal{V}$ by Lemma 3.46 (3) and Lemma 3.47. Thus, $\text{Sin}(A) (\prec_U)_L \mathcal{V}$. Moreover, since S is strong, there exists $A' \in \text{Fin}(S)$ such that $a \prec \bigvee A'$ and $A' \prec_L A$. Then, $a \alpha A'$ and $A' \text{P}_L(\beta) \mathcal{V}$, and hence $a (\text{P}_L(\beta) \circ \alpha) \mathcal{V}$.

(\Leftarrow) Suppose that α and β satisfy (3.23), and let $a \prec b \leq c \vee d$. Put $\mathcal{U} = \{\{b\}, \{c, d\}\}$, and choose a' such that $a \prec a' \prec b$. Then $a \beta \{a'\} \text{P}_U(\alpha) \mathcal{U}$, so by the opposite of (3.24), we have $a (\tau_S \circ \text{P}_L(\beta) \circ \alpha) \mathcal{U}$. Thus, there exist $A \in \text{Fin}(S)$ and $\mathcal{V} \in \text{Fin}(\text{Fin}(S))$ such that

- (7) $a \prec \bigvee A$,
- (8) $\text{Sin}(A) (\prec_U)_L \mathcal{V}$,
- (9) $\mathcal{V} (\prec_U)_L \mathcal{U}^*$.

From 8, 9, and Lemma 3.48 (1), we have $\{A\}^* (\prec_U)_L \mathcal{U}^*$. Then $\{A\} (\prec_L)_U \mathcal{U}$ by Lemma 3.47, or equivalently $A \prec_L \{b\}$ and $A \prec_L \{c, d\}$. Split A into C and D so that $A = C \cup D$, $C \subseteq b \downarrow_{\prec} c$ and $D \subseteq b \downarrow_{\prec} d$. By putting $a_1 = \bigvee C$ and $a_2 = \bigvee D$ and using 7, we have $a \prec a_1 \vee a_2$, $a_1 \in b \downarrow_{\prec} c$ and $a_2 \in b \downarrow_{\prec} d$. Hence, S is localized. \square

Proposition 3.55 also implies that the property of being localized is invariant under isomorphisms of strong proximity \vee -semilattices.

We are now ready to prove the main result of this subsection.

Predicative characterisation	Domain theoretic dual	Formal topology
continuous finitary covers + approximable maps	$\text{ContLat}_{\text{Scott}}$	—
strong cont. fin. covers + join-approximable maps	ContLat	continuous basic covers + basic cover maps
localized str. cont. fin. cov. + proximity maps	LKFrm	locally compact formal top. + formal topology maps

Table 3: Main structures in Section 4.

Theorem 3.56. *$\text{LSP}\times\text{JLat}$ is equivalent to the category of coalgebras of the double powerlocale over PxPos .*

Proof. By Lemma 3.24, Lemma 3.40, Lemma 3.54, and Proposition 3.55, the category $\text{LSP}\times\text{JLat}$ embeds into that of coalgebras of the double powerlocale. To see that the embedding is essentially surjective, let S be a double powerlocale coalgebra. By Lemma 3.29, there is a strong proximity \vee -semilattice S' which is isomorphic to S , so by Corollary 3.27 (2), the isomorphism induces an isomorphism of the underlying P_U -coalgebras and P_L -coalgebras of S and S' . Then, it is straightforward to show that the P_U -coalgebra and P_L -coalgebra structures on S' satisfies the condition of Lemma 3.53. Thus, S' is localized by Proposition 3.55, which is isomorphic to S as a double powerlocale coalgebra by Lemma 3.54. \square

By Theorem 3.44 and Theorem 3.56, we have the following characterisation of locally compact locales.

Theorem 3.57. *The category of coalgebras of the double powerlocale over PxPos is equivalent to the category of locally compact locales.*

4. CONTINUOUS FINITARY COVERS

We introduce a notion of continuous finitary cover, which provides a logical characterisation of proximity \vee -semilattice. A continuous finitary cover can be thought of as a presentation of a proximity \vee -semilattice where the underlying \vee -semilattice is presented by generators and relations. The structure provides us with a flexible way of constructing proximity \vee -semilattices by generators and relations, and directly working on the presentation. Moreover, its strong variant provides a predicative and even finitary alternative to the notion of continuous lattice in formal topology [Neg98]. The left column of Table 3 shows some of the major structures introduced in this section, with the corresponding structures in domain theory and formal topology on the middle and the right columns.

The development of this section can be seen as a suplattice analogue of those of entailment systems [Vic04b] and continuous entailment relations [Kaw20]. Similar to [Vic04b, Section 5], the underlying idea of this section is that $\text{ContLat}_{\text{Scott}}$ can be described as the Karoubi envelop of its subcategory of free suplattices (cf. Section 4.1).

4.1. Semi-entailment systems. We begin with an observation that every continuous lattice is a Scott continuous retract of the free suplattice over that lattice. Recall that the *free \vee -semilattice* over a set S can be represented by $\text{Fin}(S)$ ordered by inclusion, where joins are computed by unions. The *free suplattice* over S is the ideal completion of $\text{Fin}(S)$, or equivalently the power set of S (see Johnstone [Joh02, Section C1.1, Lemma 1.1.3]).

Proposition 4.1. *Every continuous lattice is a Scott continuous retract of a free suplattice.*

Proof. Let L be a continuous lattice. Define functions $f: L \rightarrow \text{Idl}(\text{Fin}(L))$ and $g: \text{Idl}(\text{Fin}(L)) \rightarrow L$ by

$$f(a) \stackrel{\text{def}}{=} \{A \in \text{Fin}(L) \mid \bigvee A \ll a\}, \quad g(I) \stackrel{\text{def}}{=} \bigvee_{A \in I} \bigvee A.$$

It is easy to see that f and g are Scott continuous and $g \circ f = \text{id}_L$. \square

Let $\text{FreeSupLat}_{\text{Scott}}$ be the full subcategory of $\text{ContLat}_{\text{Scott}}$ consisting of free suplattices. Since every idempotent splits in $\text{ContLat}_{\text{Scott}}$ (cf. Proposition 2.1 (2)), we have the following.

Theorem 4.2. *$\text{Split}(\text{FreeSupLat}_{\text{Scott}})$ is equivalent to $\text{ContLat}_{\text{Scott}}$.*

The dual of $\text{FreeSupLat}_{\text{Scott}}$ can be identified with a full subcategory of Pos_{App} consisting of free \vee -semilattices. Then, the morphisms of this subcategory can be characterised in terms of generators of free semilattices as follows.

Definition 4.3. Let S and S' be sets. A relation $r \subseteq S \times \text{Fin}(S')$ is *upper* if

$$a r B \rightarrow a r (B \cup B').$$

Given two upper relations $r \subseteq S \times \text{Fin}(S')$ and $s \subseteq S' \times \text{Fin}(S'')$, their *cut composition* $s \cdot r \subseteq S \times \text{Fin}(S'')$ is defined by

$$a (s \cdot r) C \stackrel{\text{def}}{\iff} \exists B \in \text{Fin}(S') (a r B \ \& \ B \tilde{s} C), \quad (4.1)$$

where

$$B \tilde{s} C \stackrel{\text{def}}{\iff} \forall b \in B (b s C). \quad (4.2)$$

The following is analogous to [Vic04b, Proposition 25].

Proposition 4.4. *Let $\text{Fin}(S)$ and $\text{Fin}(S')$ be free \vee -semilattices over sets S and S' . Then, there exists a bijective correspondence between approximable relations from $\text{Fin}(S)$ to $\text{Fin}(S')$ and upper relations from S to $\text{Fin}(S')$. Via this correspondence, the identities and compositions of Pos_{App} correspond to the membership relation \in and cut compositions.*

Proof. The correspondence is as follows. An approximable relation $r \subseteq \text{Fin}(S) \times \text{Fin}(S')$ corresponds to an upper relation $\hat{r} \subseteq S \times \text{Fin}(S')$ defined by

$$a \hat{r} B \stackrel{\text{def}}{\iff} \{a\} r B.$$

Conversely, an upper relation $r \subseteq S \times \text{Fin}(S')$ corresponds to an approximable relation $\tilde{r} \subseteq \text{Fin}(S) \times \text{Fin}(S')$ defined by (4.2). Since finite joins in a free \vee -semilattice are given by unions, the above correspondence is well-defined and bijective. The second part is straightforward to check. \square

Analogous to [Vic04b, Definition 27], let \mathbf{SEnt} denote the category in which objects are sets and morphisms between sets S and S' are upper relations from S to $\text{Fin}(S')$: the identity on a set S is the membership relation \in ; the composition of two upper relations is the cut composition.

By Proposition 4.4 and Proposition 3.4, we have the following.

Proposition 4.5. \mathbf{SEnt} is dually equivalent to $\mathbf{FreeSupLat}_{\text{Scott}}$.

Thus, by Theorem 4.2, we have another characterisation of $\mathbf{ContLat}_{\text{Scott}}$.

Theorem 4.6. $\mathbf{Split}(\mathbf{SEnt})$ is dually equivalent to $\mathbf{ContLat}_{\text{Scott}}$.

The objects and morphisms of $\mathbf{Split}(\mathbf{SEnt})$ are called *semi-entailment systems* and *approximable maps*, respectively. For the record, we put down their explicit descriptions.

Definition 4.7. A *semi-entailment system* is a structure (S, \ll) , where $\ll \subseteq S \times \text{Fin}(S)$ is an upper relation such that $\ll \cdot \ll$.

Let (S, \ll) and (S', \ll') be semi-entailment systems. An *approximable map* from (S, \ll) to (S', \ll') is a relation $r \subseteq S \times \text{Fin}(S')$ such that $\ll \cdot r = r = \ll' \cdot r$.

Note that every approximable map is an upper relation.

In $\mathbf{Split}(\mathbf{SEnt})$, the identity on a semi-entailment system (S, \ll) is \ll ; the composition of two approximable maps is the cut composition.

4.2. Continuous finitary covers. We introduce another description of $\mathbf{Split}(\mathbf{SEnt})$ which is closely related to \mathbf{PxJLat} . First, we characterise a full subcategory of $\mathbf{Split}(\mathbf{SEnt})$ which corresponds to $\mathbf{AlgLat}_{\text{Scott}}$. As in Vickers [Vic04b, Section 6.1], we say that a semi-entailment system (S, \ll) is *reflexive* if

$$a \in A \rightarrow a \ll A.$$

Reflexive semi-entailment systems are also known as finitary covers, or single conclusion entailment relations [RSW18].

Definition 4.8. A *finitary cover* is a pair (S, \blacktriangleleft) where S is a set and \blacktriangleleft is a relation between S and $\text{Fin}(S)$ such that

$$\frac{a \in A}{a \blacktriangleleft A} \qquad \frac{a \blacktriangleleft A}{a \blacktriangleleft A \cup B} \qquad \frac{a \blacktriangleleft A \cup \{b\} \quad b \blacktriangleleft A}{a \blacktriangleleft A}.$$

We write \mathbf{FCov} for the full subcategory of $\mathbf{Split}(\mathbf{SEnt})$ consisting of finitary covers.

Each finitary cover (S, \blacktriangleleft) determines a \vee -semilattice $L(S, \blacktriangleleft)$, which is the poset reflection of $(\text{Fin}(S), \widetilde{\blacktriangleleft})$ with finite joins computed by unions.⁷ Also, each approximable map $r: (S, \blacktriangleleft) \rightarrow (S', \blacktriangleleft')$ determines an approximable relation $\widetilde{r}: L(S, \blacktriangleleft) \rightarrow L(S', \blacktriangleleft')$ between the corresponding \vee -semilattices. Indeed, since $\widetilde{\blacktriangleleft}' \circ \widetilde{r} = \widetilde{\blacktriangleleft}' \cdot r = \widetilde{r}$ and $\widetilde{r} \circ \widetilde{\blacktriangleleft} = \widetilde{r}$, the relation \widetilde{r} is well-defined and satisfies (AppU); since joins of $L(S, \blacktriangleleft)$ are computed by unions, \widetilde{r} also satisfies (AppI). It is easy to see that these assignments determine a full and faithful functor $F: \mathbf{FCov} \rightarrow \mathbf{JLat}_{\text{App}}$.

Proposition 4.9. The functor $F: \mathbf{FCov} \rightarrow \mathbf{JLat}_{\text{App}}$ is essentially surjective. Hence, F determines an equivalence of \mathbf{FCov} and $\mathbf{JLat}_{\text{App}}$.

⁷Specifically, two elements $A, B \in \text{Fin}(S)$ are in the same equivalence class if $A \widetilde{\blacktriangleleft} B$ and $B \widetilde{\blacktriangleleft} A$.

Proof. For each \vee -semilattice $(S, 0, \vee)$, define a finitary cover $(S, \blacktriangleleft_{\vee})$ by

$$a \blacktriangleleft_{\vee} A \stackrel{\text{def}}{\iff} a \leq \bigvee A. \quad (4.3)$$

Define relations $r \subseteq S \times \text{Fin}(S)$ and $s \subseteq \text{Fin}(S) \times S$ by

$$a r A \stackrel{\text{def}}{\iff} a \leq \bigvee A, \quad A s a \stackrel{\text{def}}{\iff} \bigvee A \leq a.$$

It is straightforward to show that r and s are approximable relations between $(S, 0, \vee)$ and $L(S, \blacktriangleleft_{\vee})$ and that they are inverse to each other. \square

Remark 4.10. From the explicit constructions of the finitary cover $(S, \blacktriangleleft_{\vee})$ and isomorphisms r and s in the above proof, we can define a quasi-inverse $G: \mathbf{JLat}_{\text{App}} \rightarrow \mathbf{FCov}$ of F as follows:

$$\begin{aligned} G(S, 0, \vee) &\stackrel{\text{def}}{=} (S, \blacktriangleleft_{\vee}), \\ a G(r) A &\stackrel{\text{def}}{\iff} a r \bigvee A. \end{aligned}$$

Since $\text{PxJLat} = \mathbf{Split}(\mathbf{JLat}_{\text{App}})$ by definition, we have the following.

Corollary 4.11. $\mathbf{Split}(\mathbf{FCov})$ is equivalent to PxJLat .

The objects and morphisms of $\mathbf{Split}(\mathbf{FCov})$ can be explicitly described as follows.

Definition 4.12. A *continuous finitary cover* is a structure $(S, \blacktriangleleft, \ll)$, where (S, \blacktriangleleft) is a finitary cover and $\ll \subseteq S \times \text{Fin}(S)$ is a relation such that $\ll \cdot \ll = \ll$ and $\blacktriangleleft \cdot \ll = \ll = \ll \cdot \blacktriangleleft$.

Let $(S, \blacktriangleleft, \ll)$ and $(S', \blacktriangleleft', \ll')$ be continuous finitary covers. An *approximable map* from $(S, \blacktriangleleft, \ll)$ to $(S', \blacktriangleleft', \ll')$ is a relation $r \subseteq S \times \text{Fin}(S')$ such that $\ll \cdot r = r = \ll' \cdot r$.

Henceforth, we write ContFCov for $\mathbf{Split}(\mathbf{FCov})$, which consists of continuous finitary covers and approximable maps between them: the identity on a continuous finitary cover $(S, \blacktriangleleft, \ll)$ is \ll ; the composition of two approximable maps is the cut composition.

The equivalence of ContFCov and PxJLat in Corollary 4.11 is induced by the functor $F: \mathbf{FCov} \rightarrow \mathbf{JLat}_{\text{App}}$ and its quasi-inverse $G: \mathbf{JLat}_{\text{App}} \rightarrow \mathbf{FCov}$. Specifically, we have a pair of functors $\bar{F}: \text{ContFCov} \rightarrow \text{PxJLat}$ and $\bar{G}: \text{PxJLat} \rightarrow \text{ContFCov}$ which act on morphisms as F and G , respectively, and on objects as follows:

$$\begin{aligned} \bar{F}(S, \blacktriangleleft, \ll) &\stackrel{\text{def}}{=} (L(S, \blacktriangleleft), \widetilde{\ll}), \\ \bar{G}(S, 0, \vee, \prec) &\stackrel{\text{def}}{=} (S, \blacktriangleleft_{\vee}, \ll_{\vee}), \end{aligned} \quad (4.4)$$

where

$$a \ll_{\vee} A \stackrel{\text{def}}{\iff} a \prec \bigvee A. \quad (4.5)$$

Thus, by Theorem 3.9, we have the following.

Theorem 4.13. ContFCov is equivalent to $\text{ContLat}_{\text{Scott}}$.

On the other hand, each continuous finitary cover $(S, \blacktriangleleft, \ll)$ determines a semi-entailment system (S, \ll) , and approximable maps between two continuous finitary covers are precisely the approximable maps between the corresponding semi-entailment systems. Conversely, each semi-entailment system (S, \ll) can be regarded as a continuous finitary cover (S, \in, \ll) .

Proposition 4.14. ContFCov is equivalent to $\mathbf{Split}(\mathbf{SEnt})$.

Proof. As noted above, we have a pair of functors $P: \mathbf{ContFCov} \rightarrow \mathbf{Split}(\mathbf{SEnt})$ and $Q: \mathbf{Split}(\mathbf{SEnt}) \rightarrow \mathbf{ContFCov}$ whose actions on objects are given by

$$\begin{aligned} P(S, \blacktriangleleft, \ll) &\stackrel{\text{def}}{=} (S, \ll), \\ Q(S, \ll) &\stackrel{\text{def}}{=} (S, \in, \ll), \end{aligned}$$

and which are identity on morphisms. Obviously, $P \circ Q$ is an identity. Moreover, we have an approximable map $\ll: QP(S, \blacktriangleleft, \ll) \rightarrow (S, \blacktriangleleft, \ll)$, which is clearly isomorphic and natural in $(S, \blacktriangleleft, \ll)$. \square

In fact, Proposition 4.14 follows from Theorem 4.6 and Theorem 4.13. The point of this proposition, however, is that its proof does not rely on the impredicative notion of $\mathbf{ContLats}_{\text{Scott}}$.

4.3. Strong continuous finitary covers. We characterise a subcategory of $\mathbf{ContFCov}$ which corresponds to $\mathbf{SPxJLat}$.

Definition 4.15. A *strong continuous finitary cover* is a finitary cover (S, \blacktriangleleft) together with an idempotent relation \sqsubset on S such that

$$\exists b \in S (a \sqsubset b \blacktriangleleft A) \leftrightarrow \exists B \in \mathbf{Fin}(S) (a \blacktriangleleft B \sqsubset_L A). \quad (4.6)$$

We write $(S, \blacktriangleleft, \sqsubset)$ or simply S for a strong continuous finitary cover.

The condition (4.6) is equivalent to

$$\blacktriangleleft \cdot \sqsubset_{\exists} = \sqsubset_{\exists} \cdot \blacktriangleleft,$$

where $\sqsubset_{\exists} \subseteq S \times \mathbf{Fin}(S)$ is defined by $a \sqsubset_{\exists} B \stackrel{\text{def}}{\iff} \exists b \in B (a \sqsubset b)$. Put

$$\ll_{\blacktriangleleft} \stackrel{\text{def}}{=} \blacktriangleleft \cdot \sqsubset_{\exists} = \sqsubset_{\exists} \cdot \blacktriangleleft. \quad (4.7)$$

Then, we have

$$\begin{aligned} \ll_{\blacktriangleleft} \cdot \ll_{\blacktriangleleft} &= \blacktriangleleft \cdot \sqsubset_{\exists} \cdot \blacktriangleleft \cdot \sqsubset_{\exists} = \sqsubset_{\exists} \cdot \blacktriangleleft \cdot \blacktriangleleft \cdot \sqsubset_{\exists} \\ &= \sqsubset_{\exists} \cdot \blacktriangleleft \cdot \sqsubset_{\exists} = \sqsubset_{\exists} \cdot \sqsubset_{\exists} \cdot \blacktriangleleft = \sqsubset_{\exists} \cdot \blacktriangleleft = \ll_{\blacktriangleleft}. \end{aligned}$$

Similarly, we have $\ll_{\blacktriangleleft} \cdot \blacktriangleleft = \ll_{\blacktriangleleft} = \ll_{\blacktriangleleft} \cdot \blacktriangleleft$. Thus, $(S, \blacktriangleleft, \ll_{\blacktriangleleft})$ is a continuous finitary cover. With the above identification, strong continuous finitary covers form a full subcategory of $\mathbf{ContFCov}$. In particular, each strong continuous finitary cover $(S, \blacktriangleleft, \sqsubset)$ determines a proximity \vee -semilattice $(L(S, \blacktriangleleft), \widetilde{\ll}_{\blacktriangleleft})$ as in (4.4).

Lemma 4.16. $(L(S, \blacktriangleleft), \widetilde{\ll}_{\blacktriangleleft})$ is a strong proximity \vee -semilattice.

Proof. We show that $\widetilde{\ll}_{\blacktriangleleft}$ satisfies (App0) and (App \vee). For (App0), suppose $A \widetilde{\ll}_{\blacktriangleleft} \emptyset$. Then $A \widetilde{\blacktriangleleft} \emptyset$ by (4.6), and so $A = 0$ in $L(S, \blacktriangleleft)$. For (App \vee), suppose $A \widetilde{\ll}_{\blacktriangleleft} B \cup C$. Then, there exist $B', C' \in \mathbf{Fin}(S)$ such that $B' \sqsubset_{\exists} B$, $C' \sqsubset_{\exists} C$, and $A \widetilde{\blacktriangleleft} B' \cup C'$. Since \blacktriangleleft is reflexive, we have $B' \widetilde{\ll}_{\blacktriangleleft} B$ and $C' \widetilde{\ll}_{\blacktriangleleft} C$. \square

In the opposite direction, each strong proximity \vee -semilattice $(S, 0, \vee, \prec)$ determines a strong continuous finitary cover $(S, \blacktriangleleft_{\vee}, \prec)$ where $\blacktriangleleft_{\vee}$ is given by (4.3). Moreover, we have $\ll_{\blacktriangleleft_{\vee}} = \ll_{\vee}$ where \ll_{\vee} is given by (4.5). Thus, we have the following.

Proposition 4.17. *The equivalence of $\mathbf{ContFCov}$ and \mathbf{PxJLat} restricts to their respective full subcategories of strong continuous finitary covers and strong proximity \vee -semilattices.*

Next, we characterise approximable maps between strong continuous finitary covers which correspond to join-approximable relations.

Definition 4.18. Let $(S, \blacktriangleleft, \sqsubset)$ and $(S', \blacktriangleleft', \sqsubset')$ be strong continuous finitary covers. An approximable map $r: (S, \blacktriangleleft, \ll\blacktriangleleft) \rightarrow (S', \blacktriangleleft', \ll\blacktriangleleft')$ is *join-preserving* if

$$a \ r \ B \rightarrow \exists A \in \text{Fin}(S) (a \ \blacktriangleleft \ A \ \& \ \forall a' \in A \exists b \in B (a' \ r \ \{b\})). \quad (4.8)$$

We call join-preserving approximable maps simply as *join-approximable maps*.

Lemma 4.19. Let $(S, \blacktriangleleft, \sqsubset)$ and $(S', \blacktriangleleft', \sqsubset')$ be strong continuous finitary covers. An approximable map $r: (S, \blacktriangleleft, \ll\blacktriangleleft) \rightarrow (S', \blacktriangleleft', \ll\blacktriangleleft')$ is join-preserving if and only if the corresponding approximable relation $\tilde{r}: (\mathbb{L}(S, \blacktriangleleft), \widetilde{\ll\blacktriangleleft}) \rightarrow (\mathbb{L}(S', \blacktriangleleft'), \widetilde{\ll\blacktriangleleft}')$ is join-preserving.

Proof. Suppose that r is join-approximable. We must show (App0) and (App \vee). For (App0), suppose $A \tilde{r} \emptyset$. Then (4.8) implies $A \tilde{\blacktriangleleft} \emptyset$. For (App \vee), suppose $A \tilde{r} B \cup C$. By (4.8), for each $a \in A$, there exist $B_a, C_a \in \text{Fin}(S)$ such that $a \ \blacktriangleleft \ B_a \cup C_a$ and that $\forall b' \in B_a \exists b \in B (b' \ r \ \{b\})$ and $\forall c' \in C_a \exists c \in C (c' \ r \ \{c\})$. Put $B' = \bigcup_{a \in A} B_a$ and $C' = \bigcup_{a \in A} C_a$. Then, $A \ \tilde{\blacktriangleleft} \ B' \cup C'$, $B' \tilde{r} B$, and $C' \tilde{r} C$.

Conversely, suppose that \tilde{r} is join-approximable, and let $a \ r \ B$. Then, $\{a\} \tilde{r} \bigcup_{b \in B} \{b\}$. Since \tilde{r} is join-preserving, there exist $A_0, \dots, A_{n-1} \in \text{Fin}(S)$ such that $a \ \blacktriangleleft \ \bigcup_{i < n} A_i$ and for each $i < n$, there exists $b \in B$ such that $A_i \tilde{r} \{b\}$. \square

Let SContFCov be the category in which objects are strong continuous finitary covers and morphisms are join-approximable maps between underlying continuous finitary covers of strong continuous finitary covers. Then, by Proposition 4.17 and Lemma 4.19, the functor $\overline{F}: \text{ContFCov} \rightarrow \text{PxJLat}$ restricts to a full and faithful functor from SContFCov to SPxJLat . The following lemma implies that this restriction of \overline{F} is essentially surjective.

Lemma 4.20. Every isomorphism in PxJLat between strong proximity \vee -semilattices is join-preserving.

Proof. Let $(S, 0, \vee, \prec)$ and $(S', 0', \vee', \prec')$ be strong proximity \vee -semilattices, and $r: S \rightarrow S'$ and $s: S' \rightarrow S$ be approximable relations such that $s \circ r = \prec$ and $r \circ s = \prec'$. We show that r is join-preserving. For (App0), suppose $a \ r \ 0'$. Since $0' \ s \ 0$ by (AppI), we have $a \ (s \circ r) \ 0$. Thus, $a \ \prec \ 0$ and so $a = 0$ by the strength of S . For (App \vee), suppose $a \ r \ (b \ \vee' \ c)$. Since r is approximable, there exists $d \ \prec' (b \ \vee' \ c)$ such that $a \ r \ d$. Since S' is strong, there exist $b' \ \prec' b$ and $c' \ \prec' c$ such that $d \leq (b' \ \vee' \ c')$. Thus, there exist $a_b, a_c \in S$ such that $b' \ s \ a_b \ r \ b$ and $c' \ s \ a_c \ r \ c$. Then $a \ r \ (b' \ \vee' \ c') \ s \ (a_b \ \vee \ a_c)$, and so $a \ \prec \ a_b \ \vee \ a_c$. Since S is strong, there exist $a', a'' \in S$ such that $a \leq a' \ \vee \ a''$, $a' \ \prec \ a_b$, and $a'' \ \prec \ a_c$. Then $a' \ r \ b$ and $a'' \ r \ c$. \square

Theorem 4.21. SContFCov is equivalent to SPxJLat .

4.4. Localized strong continuous finitary covers. We characterise a subcategory of strong continuous finitary covers which corresponds to LSPxJLat , the category of localized strong proximity \vee -semilattices and proximity relations.

In what follows, notations $a \ \downarrow_{\sqsubset} A$ and $A \ \downarrow_{\sqsubset} B$ are defined similarly as in $a \ \downarrow_{\prec} A$ and $A \ \downarrow_{\prec} B$, respectively.

Definition 4.22. A strong continuous finitary cover $(S, \blacktriangleleft, \sqsubset)$ is *localized* if

$$b \ \sqsubset \ a \ \blacktriangleleft \ A \rightarrow \exists B \in \text{Fin}(a \ \downarrow_{\sqsubset} A) (b \ \blacktriangleleft \ B).$$

The following characterisation is more convenient.

Lemma 4.23. *A strong continuous finitary cover $(S, \triangleleft, \sqsubset)$ is localized if and only if*

$$b \sqsubset a \triangleleft A \ \& \ a \triangleleft B \ \rightarrow \ \exists C \in \text{Fin}(A \downarrow_{\sqsubset} B) \ (b \triangleleft C).$$

Proof. “If” part follows from $a \triangleleft \{a\}$. Conversely, suppose $b \sqsubset a \triangleleft A$ and $a \triangleleft B$. Choose $b' \in S$ such that $b \sqsubset b' \sqsubset a$. Since S is localized, there exists $C \in \text{Fin}(a \downarrow_{\sqsubset} A)$ such that $b' \triangleleft C$, and since S is strong, there exists $D \sqsubset_L C$ such that $b \triangleleft D$. Then, for each $d \in D$, there exists $c \in a \downarrow_{\sqsubset} A$ such that $d \prec c$, and since S is strong, we find $B' \sqsubset_L B$ such that $c \triangleleft B'$. Since S is localized, there exists $E_d \in \text{Fin}(c \downarrow_{\sqsubset} B') \subseteq \text{Fin}(A \downarrow_{\sqsubset} B)$ such that $d \triangleleft E_d$. Then, by putting $E = \bigcup_{d \in D} E_d$, we have $b \triangleleft E \in \text{Fin}(A \downarrow_{\sqsubset} B)$. \square

Lemma 4.24. *For any localized strong continuous finitary cover $(S, \triangleleft, \sqsubset)$, the corresponding strong proximity \vee -semilattice $(L(S, \triangleleft), \widetilde{\llcorner}_{\triangleleft})$ is localized.*

Proof. Suppose that $(S, \triangleleft, \sqsubset)$ is localized, and let $A \widetilde{\llcorner}_{\triangleleft} B \triangleleft C \cup D$. There exists B' such that $A \sqsubset_L B' \triangleleft B$ by (4.7), and so $B' \triangleleft C \cup D$. Write $A = \{a_0, \dots, a_{n-1}\}$. For each $i < n$, there exist $e_i \in S$ and $b_i \in B'$ such that $a_i \sqsubset e_i \sqsubset b_i$. Since S is localized, there exists $A_i \in \text{Fin}(C \cup D \downarrow_{\sqsubset} b_i)$ such that $e_i \triangleleft A_i$. Split A_i into C_i and D_i so that $A_i = C_i \cup D_i$, $C_i \subseteq C \downarrow_{\sqsubset} b_i$, and $D_i \subseteq D \downarrow_{\sqsubset} b_i$. Put $E = \{e_i \mid i < n\}$, $C' = \bigcup_{i < n} C_i$, and $D' = \bigcup_{i < n} D_i$. Then, $A \sqsubset_L E \triangleleft C' \cup D'$, $C' \subseteq C \downarrow_{\sqsubset} B'$, and $D' \subseteq D \downarrow_{\sqsubset} B'$. Thus $A \widetilde{\llcorner}_{\triangleleft} C' \cup D'$, $C' \in C \downarrow_{\widetilde{\llcorner}_{\triangleleft}} B$, and $D' \in D \downarrow_{\widetilde{\llcorner}_{\triangleleft}} B$. Hence $(L(S, \triangleleft), \widetilde{\llcorner}_{\triangleleft})$ is localized. \square

Definition 4.25. An approximable map $r: (S, \triangleleft, \sqsubset) \rightarrow (S', \triangleleft', \sqsubset')$ between strong continuous finitary covers is *Lawson approximable* if

- (1) $a \sqsubset a' \rightarrow \exists B \in \text{Fin}(S') \ (a \ r \ B)$,
- (2) $a \sqsubset a' \ r \ B \ \& \ a' \ r \ C \rightarrow \exists D \in \text{Fin}(S') \ (D \widetilde{\llcorner}_{\triangleleft'} B \ \& \ D \widetilde{\llcorner}_{\triangleleft'} C \ \& \ a \ r \ D)$.

Remark 4.26. If S' is localized, then the condition 2 can be strengthened to

$$a \sqsubset a' \ r \ B \ \& \ a' \ r \ C \rightarrow \exists D \in \text{Fin}(B \downarrow_{\sqsubset'} C) \ (a \ r \ D).$$

If S and S' are both localized and r is join-preserving, then the condition 2 can be simplified further to

$$a \sqsubset a' \ r \ \{b\} \ \& \ a' \ r \ \{c\} \rightarrow \exists D \in \text{Fin}(b \downarrow_{\sqsubset'} c) \ (a \ r \ D). \quad (4.9)$$

Lemma 4.27. *An approximable map $r: (S, \triangleleft, \sqsubset) \rightarrow (S', \triangleleft', \sqsubset')$ is Lawson approximable if and only if the corresponding approximable relation $\tilde{r}: (L(S, \triangleleft), \widetilde{\llcorner}_{\triangleleft}) \rightarrow (L(S', \triangleleft'), \widetilde{\llcorner}_{\triangleleft'})$ is Lawson approximable.*

Proof. Suppose that r is Lawson approximable, and let $A \widetilde{\llcorner}_{\triangleleft} A'$. Then, there exists $A'' \in \text{Fin}(S)$ such that $A \triangleleft A'' \sqsubset_L A'$. Since r is Lawson approximable, there exists $B \in \text{Fin}(S')$ such that $A'' \tilde{r} B$. Then $A \tilde{r} B$. Next, suppose $A \widetilde{\llcorner}_{\triangleleft} A' \tilde{r} B$ and $A' \tilde{r} C$. Then, there exists $A'' \in \text{Fin}(S)$ such that $A \triangleleft A'' \sqsubset_L A'$. Since r is Lawson approximable, there exists $D \in \text{Fin}(S')$ such that $A'' \tilde{r} D$, $D \widetilde{\llcorner}_{\triangleleft'} B$, and $D \widetilde{\llcorner}_{\triangleleft'} C$. Then, $A \tilde{r} D$.

Conversely, suppose that \tilde{r} is Lawson approximable. If $a \sqsubset a'$, then $\{a\} \widetilde{\llcorner}_{\triangleleft} \{a'\}$. Since \tilde{r} is Lawson approximable, there exists $B \in \text{Fin}(S')$ such that $\{a\} \tilde{r} B$, and so $a \ r \ B$. Next, suppose $a \sqsubset a' \ r \ B$ and $a' \ r \ C$. Then, $\{a\} \widetilde{\llcorner}_{\triangleleft} \{a'\} \tilde{r} B$, and $\{a'\} \tilde{r} C$. Since \tilde{r} is Lawson approximable, there exists $D \in \text{Fin}(S')$ such that $D \widetilde{\llcorner}_{\triangleleft'} B$, $D \widetilde{\llcorner}_{\triangleleft'} C$, and $\{a\} \tilde{r} D$. Then $a \ r \ D$. \square

Definition 4.28. A join-preserving Lawson approximable map between localized strong continuous finitary covers is called a *proximity map*.

Let LSContFCov be the category of localized strong continuous finitary covers and proximity maps.

Proposition 4.29. *The equivalence between SContFCov and SPxJLat restricts to an equivalence between LSContFCov and LSPxJLat .*

Proof. By Lemma 4.24 and Lemma 4.27, the embedding of SContFCov into SPxJLat restricts to an embedding of LSContFCov into LSPxJLat . Moreover, for any localized strong proximity \vee -semilattice $(S, 0, \vee, \prec)$, the strong continuous finitary cover $(S, \blacktriangleleft_{\vee}, \prec)$ defined in (4.3) is localized by Lemma 3.33 and Lemma 4.23. As in Lemma 4.20, one can also show that every isomorphism in PxPos is Lawson approximable. Thus, the embedding of LSContFCov into LSPxJLat is essentially surjective. \square

4.5. Formal topologies. We relate the theory of strong continuous finitary covers to that of continuous lattices and locally compact locales in formal topology [Neg98].

Definition 4.30. A *basic cover* is a pair (S, \triangleleft) where \triangleleft is a relation between S and $\text{Pow}(S)$ satisfying

$$\frac{a \in U}{a \triangleleft U} \text{ (reflexivity)} \qquad \frac{a \triangleleft U \quad U \triangleleft V}{a \triangleleft V} \text{ (transitivity)}$$

where $U \triangleleft V \stackrel{\text{def}}{\iff} \forall a \in U (a \triangleleft V)$. A basic cover (S, \triangleleft) is *continuous* if it is equipped with a relation $\text{wb} \subseteq S \times S$ satisfying

$$a \triangleleft \text{wb}^{-1} a \qquad \frac{b \text{wb} a \quad a \triangleleft U}{\exists A \in \text{Fin}(U) b \triangleleft A}. \quad (4.10)$$

Among the various definitions of formal topology described in [CMS13], we prefer to work with the one with a preorder.

Definition 4.31. A *formal topology* is a triple (S, \triangleleft, \leq) where (S, \triangleleft) is a basic cover and (S, \leq) is a preorder satisfying

$$\frac{a \leq b}{a \triangleleft \{b\}} \text{ } (\leq\text{-left}) \qquad \frac{a \triangleleft U \quad a \triangleleft V}{a \triangleleft U \downarrow_{\leq} V} \text{ } (\downarrow\text{-right})$$

where $U \downarrow_{\leq} V \stackrel{\text{def}}{=} \downarrow_{\leq} U \cap \downarrow_{\leq} V$. A formal topology (S, \triangleleft, \leq) is *locally compact* if (S, \triangleleft) is a continuous basic cover.

A subset $U \subseteq S$ of a basic cover (S, \triangleleft) is *saturated* if

$$a \triangleleft U \rightarrow a \in U.$$

Negri [Neg98] showed that the collection of saturated subsets of a continuous basic cover and that of a locally compact formal topology are continuous lattice and locally compact locale, respectively. She also showed that every continuous lattice and locally compact locale can be represented in this way.⁸

Proposition 4.32. *Let $(S, \blacktriangleleft, \sqsupseteq)$ be a strong continuous finitary cover. Define a relation $\blacktriangleleft_{\sqsupseteq} \subseteq S \times \text{Pow}(S)$ by*

$$a \blacktriangleleft_{\sqsupseteq} U \stackrel{\text{def}}{\iff} \forall b \sqsupseteq a \exists B \in \text{Fin}(U) (b \ll_{\blacktriangleleft} B). \quad (4.11)$$

⁸In Negri [Neg98], continuous basic covers and locally compact formal topologies are called *locally Stone infinitary preorders* and *locally Stone formal topologies*, respectively.

- (1) The structure $(S, \blacktriangleleft_{\sqsubseteq})$, is a continuous basic cover with respect to \sqsubseteq .
(2) Let \sqsubseteq be the reflexive closure of \sqsubseteq . Then, $(S, \blacktriangleleft, \sqsubseteq)$ is localized if and only if the triple $(S, \blacktriangleleft_{\sqsubseteq}, \sqsubseteq)$ is a locally compact formal topology.

Proof. 1. This is straightforward to check.

2. Suppose that $(S, \blacktriangleleft, \sqsubseteq)$ is localized. Since $\sqsubseteq \circ \sqsubseteq \subseteq \sqsubseteq$, $(S, \blacktriangleleft_{\sqsubseteq}, \sqsubseteq)$ satisfies (\leq -left). As for (\downarrow -right), suppose $a \blacktriangleleft_{\sqsubseteq} U$ and $a \blacktriangleleft_{\sqsubseteq} V$. Let $b \sqsubseteq a$, and choose $c, d \in S$ such that $b \sqsubseteq c \sqsubseteq d \sqsubseteq a$. Then, there exist $A \in \text{Fin}(U)$ and $B \in \text{Fin}(V)$ such that $d \ll_{\blacktriangleleft} A$ and $d \ll_{\blacktriangleleft} B$. Thus, there exist $A' \sqsubseteq_L A$ and $B' \sqsubseteq_L B$ such that $d \blacktriangleleft A'$ and $d \blacktriangleleft B'$. By Lemma 4.23, there exists $C \in \text{Fin}(A' \downarrow_{\sqsubseteq} B') \subseteq \text{Fin}(A \downarrow_{\sqsubseteq} B)$ such that $c \blacktriangleleft C$. Then $b \ll_{\blacktriangleleft} C$, and so $a \blacktriangleleft_{\sqsubseteq} U \downarrow_{\sqsubseteq} V$. Conversely, suppose that $\blacktriangleleft_{\sqsubseteq}$ satisfies (\leq -left) and (\downarrow -right) with respect to \sqsubseteq . Let $b \sqsubseteq a \blacktriangleleft A$ and $a \blacktriangleleft B$. Since $\blacktriangleleft \subseteq \blacktriangleleft_{\sqsubseteq}$, we have $a \blacktriangleleft_{\sqsubseteq} A \downarrow_{\sqsubseteq} B$ by (\downarrow -right), so there exists $C \in \text{Fin}(A \downarrow_{\sqsubseteq} B)$ such that $b \ll_{\blacktriangleleft} C$. Then, there exists $C' \in \text{Fin}(A \downarrow_{\sqsubseteq} B)$ such that $b \blacktriangleleft C'$. \square

Next, we recall the notions of morphism between basic covers and formal topologies.

Definition 4.33. Let (S, \triangleleft) and (S', \triangleleft') be basic covers. A relation $r \subseteq S \times S'$ is a *basic cover map* from (S, \triangleleft) to (S', \triangleleft') if

- (BCM1) $a \triangleleft r^{-}b \rightarrow a r b$,
(BCM2) $b \triangleleft' V \rightarrow r^{-}b \triangleleft r^{-}V$.

Basic covers and basic cover maps form a category BCov . The identity $\text{id}_{(S, \triangleleft)}$ on a basic cover (S, \triangleleft) is defined by

$$a \text{id}_{(S, \triangleleft)} b \stackrel{\text{def}}{\iff} a \triangleleft \{b\},$$

and the composition $s * r$ of basic cover maps $r: S \rightarrow S'$ and $s: S' \rightarrow S''$ is defined by

$$a (s * r) c \stackrel{\text{def}}{\iff} a \triangleleft r^{-}s^{-}c.$$

A *formal topology map* between formal topologies (S, \triangleleft, \leq) and $(S', \triangleleft', \leq')$ is a basic cover map $r: (S, \triangleleft) \rightarrow (S', \triangleleft')$ such that

- (FTM1) $S \triangleleft r^{-}S'$,
(FTM2) $r^{-}a \downarrow_{\leq} r^{-}b \triangleleft r^{-}(a \downarrow_{\leq'} b)$.

Formal topologies and formal topology maps form a subcategory FTop of BCov .

In what follows, we extend Proposition 4.32 to morphisms, i.e., we show that there is a bijective correspondence between join-approximable maps and basic cover maps, or in the case of localized strong continuous finitary covers, between proximity maps and formal topology maps (cf. Proposition 4.35). The correspondence is mediated by an auxiliary notion given in the next lemma.

Lemma 4.34. Let $(S, \blacktriangleleft, \sqsubseteq)$ and $(S', \blacktriangleleft', \sqsubseteq')$ be strong continuous finitary covers.

- (1) There exists a bijective correspondence between join-approximable maps from S to S' and relations $r \subseteq S \times S'$ satisfying
- $a r b \leftrightarrow \exists a' \in S (a \sqsubseteq a' r b)$,
 - $a \blacktriangleleft A \subseteq r^{-}b \rightarrow a r b$,
 - $a r b \blacktriangleleft' B \rightarrow \exists A \in \text{Fin}(S) (a \blacktriangleleft A r_L B)$,
 - $a r b \sqsubseteq' b' \rightarrow a r b'$,
 - $a r b \rightarrow \exists A \in \text{Fin}(S) \exists B \in \text{Fin}(S') (a \blacktriangleleft A r_L B \sqsubseteq'_L \{b\})$.

(2) If $(S, \blacktriangleleft, \sqsubset)$ and $(S', \blacktriangleleft', \sqsubset')$ are localized, the above bijection restricts to a bijection between proximity maps and relations between S and S' satisfying (a)–(e) above and the following properties:

- (f) $a \sqsubset a' \rightarrow \exists A \in \text{Fin}(S) \exists B \in \text{Fin}(S')(a \blacktriangleleft A r_L B)$,
- (g) $a \sqsubset a' r b \ \& \ a' r c \rightarrow \exists A \in \text{Fin}(S) \exists D \in \text{Fin}(b \downarrow_{\sqsubset'} c) (a \blacktriangleleft A r_L D)$.

Proof. 1. If $r: (S, \blacktriangleleft, \sqsubset) \rightarrow (S', \blacktriangleleft', \sqsubset')$ is a join-approximable map, then the relation $r^\dagger \subseteq S \times S'$ defined by

$$a r^\dagger b \stackrel{\text{def}}{\iff} a r \{b\} \quad (4.12)$$

satisfies (a)–(e). Conversely, given a relation $r \subseteq S \times S'$ satisfying (a)–(e), define a relation $r^* \subseteq S \times \text{Fin}(S')$ by

$$a r^* B \stackrel{\text{def}}{\iff} \exists A \in \text{Fin}(S) (a \blacktriangleleft A r_L B). \quad (4.13)$$

Clearly, r^* satisfies (4.8). Moreover, one can easily show $r^* \cdot \ll_{\blacktriangleleft} = r^* = \ll_{\blacktriangleleft'} \cdot r^*$ using (a), (c), (d), and (e). Thus r^* is a join-approximable map from $(S, \blacktriangleleft, \sqsubset)$ to $(S', \blacktriangleleft', \sqsubset')$. Then, it is straightforward to check that above correspondence is bijective (note that the fact $(r^*)^\dagger = r$ requires (b)).

2. Suppose that $(S, \blacktriangleleft, \sqsubset)$ and $(S', \blacktriangleleft', \sqsubset')$ are localized. First, if $r: (S, \blacktriangleleft, \sqsubset) \rightarrow (S', \blacktriangleleft', \sqsubset')$ is a proximity map, then the relation r^\dagger given by (4.12) satisfies (f) and (g) by the fact that r is Lawson and join-approximable (cf. Remark 4.26). Conversely, let $r \subseteq S \times S'$ be a relation satisfying (a)–(g). Then, the relation r^* given by (4.13) satisfies the first property of Lawson approximable map by (f). For the second property, it suffices to show (4.9). Suppose $a \sqsubset a' r^* \{b\}$ and $a' r^* \{c\}$. Then, $a' r b$ and $a' r c$ by (b). Thus, there exists $D \in \text{Fin}(b \downarrow_{\sqsubset'} c)$ such that $a r^* D$ by (g). \square

Proposition 4.35. *Let $(S, \blacktriangleleft, \sqsubset)$ and $(S', \blacktriangleleft', \sqsubset')$ be strong continuous finitary covers.*

- (1) *There exists a bijective correspondence between join-approximable maps from $(S, \blacktriangleleft, \sqsubset)$ and $(S', \blacktriangleleft', \sqsubset')$ and basic cover maps from $(S, \blacktriangleleft_{\sqsubset})$ to $(S', \blacktriangleleft'_{\sqsubset'})$.*
- (2) *If $(S, \blacktriangleleft, \sqsubset)$ and $(S', \blacktriangleleft', \sqsubset')$ are localized, then the above bijection restricts to a bijection between proximity maps and formal topology maps.*

Proof. 1. We establish a bijection between basic cover maps from $(S, \blacktriangleleft_{\sqsubset})$ to $(S', \blacktriangleleft'_{\sqsubset'})$ and relations between S and S' satisfying (a)–(e) in Lemma 4.34. In what follows, we take Proposition 4.32 as given.

First, for a relation $r \subseteq S \times S'$ satisfying (a)–(e), define $\check{r} \subseteq S \times S'$ by

$$a \check{r} b \stackrel{\text{def}}{\iff} a \blacktriangleleft_{\sqsubset} r^- b. \quad (4.14)$$

We show that \check{r} satisfies (BCM1) and (BCM2).

(BCM1) This follows from (transitivity) of $\blacktriangleleft_{\sqsubset}$.

(BCM2) Suppose $b \blacktriangleleft'_{\sqsubset'} V$ and $a \check{r} b$. Since $a \blacktriangleleft_{\sqsubset} r^- b$, we may assume $a r b$ by (transitivity). By (e), there exists $A \in \text{Fin}(S)$ and $B \in \text{Fin}(S')$ such that $a \blacktriangleleft A r_L B \sqsubset'_L \{b\}$. Since $b \blacktriangleleft'_{\sqsubset'} V$, there exist $B', B'' \in \text{Fin}(S')$ such that $B \widetilde{\blacktriangleleft}' B' \sqsubset'_L B'' \subseteq V$. By (c), there exists $A' \in \text{Fin}(S)$ such that $A \widetilde{\blacktriangleleft} A' r_L B'$, and so $A' r_L B''$ by (d). Then, $a \blacktriangleleft A' \in \text{Fin}(\check{r}^- V)$ so that $a \blacktriangleleft_{\sqsubset} \check{r}^- V$. Hence, $\check{r}^- b \blacktriangleleft_{\sqsubset} \check{r}^- V$.

Conversely, for a basic cover map $r: (S, \blacktriangleleft_{\sqsubset}) \rightarrow (S', \blacktriangleleft'_{\sqsubset'})$, define a relation $r^\ddagger \subseteq S \times S'$ by

$$a r^\ddagger b \stackrel{\text{def}}{\iff} a \in \downarrow_{\sqsubset} r^- b. \quad (4.15)$$

We show that r^\ddagger satisfies (a)–(e) in Lemma 4.34.

(a) This follows from the idempotency of \sqsubset .

(b) Let $a \blacktriangleleft A \subseteq (r^\ddagger)^-b$. Then, there exists $A' \in \text{Fin}(r^-b)$ such that $A \sqsubset_L A'$. Since S is strong, there exists a' such that $a \sqsubset a' \blacktriangleleft A'$. Then, $a' \blacktriangleleft_{\sqsubset} A'$ and so $a' r b$ by (BCM1). Hence $a r^\ddagger b$.

(c) Let $a r^\ddagger b \blacktriangleleft' B$. Then, there exists a' such that $a \sqsubset a' r b$. Since $b \blacktriangleleft'_{\sqsubset} B$, we have $r^-b \blacktriangleleft_{\sqsubset} r^-B$ by (BCM2), so there exists $A \in \text{Fin}(r^-B)$ such that $a \ll_{\blacktriangleleft} A$. Thus, there exists $A' \sqsubset_L A$ such that $a \blacktriangleleft A'$ and $A' (r^\ddagger)_L B$.

(d) Let $a r^\ddagger b \sqsubset' b'$. Then, there exists a' such that $a \sqsubset a' r b$. Since $b \sqsubset' b'$ implies $b \blacktriangleleft'_{\sqsubset} \{b'\}$, we have $r^-b \blacktriangleleft_{\sqsubset} r^-b'$ by (BCM2). Then $a' r b'$ by (BCM1) so that $a r^\ddagger b'$.

(e) Let $a r^\ddagger b$. Then, there exists a' such that $a \sqsubset a' r b$. Since $b \blacktriangleleft'_{\sqsubset} \downarrow_{\sqsubset} b$, we have $a' \blacktriangleleft_{\sqsubset} r^- \downarrow_{\sqsubset} b$ by (BCM2). Thus, there exists $A \in \text{Fin}(r^- \downarrow_{\sqsubset} b)$ such that $a \ll_{\blacktriangleleft} A$, and so there exist $A' \in \text{Fin}(S)$ and $B \sqsubset'_L \{b\}$ such that $a \blacktriangleleft A' \sqsubset_L A$ and $A r_L B$. Then, $A' (r^\ddagger)_L B$.

Next, we show that the above correspondence is bijective. First, note that

$$a \blacktriangleleft_{\sqsubset} U \iff a \blacktriangleleft_{\sqsubset} \downarrow_{\sqsubset} U \quad (4.16)$$

for each $a \in S$ and $U \subseteq S$. Thus, for any basic cover map $r: (S, \blacktriangleleft_{\sqsubset}) \rightarrow (S', \blacktriangleleft'_{\sqsubset})$, we have

$$\begin{aligned} a \check{r}^\ddagger b &\iff a \blacktriangleleft_{\sqsubset} (r^\ddagger)^-b \\ &\iff a \blacktriangleleft_{\sqsubset} r^-b && \text{(by (4.16))} \\ &\iff a r b && \text{(by (BCM1)).} \end{aligned}$$

Conversely, for any relation $r \subseteq S \times S'$ satisfying (a)–(e), we have

$$\begin{aligned} a (\check{r}^\ddagger)^\ddagger b &\iff \exists a' \in S (a \sqsubset a' \blacktriangleleft_{\sqsubset} r^-b) \\ &\iff \exists A \in \text{Fin}(r^-b) a \ll_{\blacktriangleleft} A && \text{(by } \blacktriangleleft \subseteq \blacktriangleleft_{\sqsubset} \text{)} \\ &\iff \exists A \in \text{Fin}(r^-b) a \blacktriangleleft A && \text{(by (a))} \\ &\iff a r b && \text{(by (b)).} \end{aligned}$$

2. Suppose that $(S, \blacktriangleleft, \sqsubset)$ and $(S', \blacktriangleleft', \sqsubset')$ are localized. First, let $r \subseteq S \times S'$ be a relation satisfying (a)–(g) in Lemma 4.34. We must show that \check{r} given by (4.14) satisfies (FTM1) and (FTM2).

(FTM1) Let $a \in S$ and $a' \sqsubset a$. Choose $a'' \in S$ such that $a' \sqsubset a'' \sqsubset a$. By (f), there exist $A \in \text{Fin}(S)$ and $B \in \text{Fin}(S')$ such that $a'' \blacktriangleleft A r_L B$. Then, $a' \ll_{\blacktriangleleft} A$ and $A \subseteq (\check{r})^-B$. Thus $a \blacktriangleleft_{\sqsubset} (\check{r})^-B \subseteq (\check{r})^-S'$, and hence $S \blacktriangleleft_{\sqsubset} (\check{r})^-S'$.

(FTM2) Let $a \in \check{r}^-b \downarrow_{\sqsubset} \check{r}^-c$. By (\leq -left), we have $a \blacktriangleleft_{\sqsubset} r^-b$ and $a \blacktriangleleft_{\sqsubset} r^-c$, and so $a \blacktriangleleft_{\sqsubset} r^-b \downarrow_{\sqsubset} r^-c$ by (\downarrow -right). By (transitivity), it suffices to show $r^-b \downarrow_{\sqsubset} r^-c \blacktriangleleft_{\sqsubset} (\check{r})^-(b \downarrow_{\sqsubset} c)$. Let $a' \in r^-b \downarrow_{\sqsubset} r^-c$, and $a'' \sqsubset a'$. Choose $a''' \in S$ such that $a'' \sqsubset a''' \sqsubset a'$. We have $r^-b \downarrow_{\sqsubset} r^-c = r^-b \cap r^-c$ by (a), so by applying (g), we find $A \in \text{Fin}(S)$ and $D \in \text{Fin}(b \downarrow_{\sqsubset} c)$ such that $a''' \blacktriangleleft A r_L D$. Then, $a'' \ll_{\blacktriangleleft} A$ and $A \subseteq (\check{r})^-D \subseteq (\check{r})^-(b \downarrow_{\sqsubset} c)$, and so $a' \blacktriangleleft_{\sqsubset} (\check{r})^-(b \downarrow_{\sqsubset} c)$. Hence, $r^-b \downarrow_{\sqsubset} r^-c \blacktriangleleft_{\sqsubset} (\check{r})^-(b \downarrow_{\sqsubset} c)$.

Conversely, let r be a formal topology map from $(S, \blacktriangleleft_{\sqsubset}, \sqsubset)$ to $(S', \blacktriangleleft'_{\sqsubset}, \sqsubset')$. We must show that r^\ddagger given by (4.15) satisfies (f) and (g).

- (f) Let $a \sqsubset a'$. Since $S \blacktriangleleft_{\sqsubset} r^{-} S'$ by (FTM1), there exists $A \in \text{Fin}(r^{-} S')$ such that $a \ll_{\blacktriangleleft} A$, so there exist $A' \sqsubset_L A$ and $B \in \text{Fin}(S')$ such that $a \blacktriangleleft A'$ and $A r_L B$. Then $A' (r^{\dagger})_L B$.
- (g) Suppose $a \sqsubset a' r^{\dagger} b$ and $a' r^{\dagger} c$. Then, $a' \blacktriangleleft_{\sqsubset} r^{-} b$ and $a' \blacktriangleleft_{\sqsubset} r^{-} c$. Since $b \blacktriangleleft'_{\sqsubset'} \downarrow_{\sqsubset'} b$ and $c \blacktriangleleft'_{\sqsubset'} \downarrow_{\sqsubset'} c$, we have $a' \blacktriangleleft_{\sqsubset} r^{-} \downarrow_{\sqsubset'} b$ and $a' \blacktriangleleft_{\sqsubset} r^{-} \downarrow_{\sqsubset'} c$ by (BCM2). Then, by (\downarrow -right) and (FTM2), we have $a' \blacktriangleleft_{\sqsubset} r^{-} ((\downarrow_{\sqsubset'} b) \downarrow_{\sqsubset'} (\downarrow_{\sqsubset'} c)) \subseteq r^{-} (b \downarrow_{\sqsubset'} c)$. Thus, there exists $A \in \text{Fin}(r^{-} (b \downarrow_{\sqsubset'} c))$ such that $a \ll_{\blacktriangleleft} A$. Then, as in the proof of (f) above, we find $D \in \text{Fin}(b \downarrow_{\sqsubset'} c)$ and $A' \in \text{Fin}(S)$ such that $a \blacktriangleleft A'$ and $A' (r^{\dagger})_L D$. \square

Let ContBCov be the full subcategory of BCov consisting of continuous basic covers, and let LKFTop be the full subcategory of FTop consisting of locally compact formal topologies. Then, it is straightforward to show that the composition of the assignments $r \mapsto r^{\dagger}$ and $r \mapsto \check{r}$ given by (4.12) and (4.14), respectively, determines a functor $F: \text{SContFCov} \rightarrow \text{ContBCov}$. By Proposition 4.35, F is full and faithful, and it restricts to a full and faithful functor from LSContFCov to LKFTop .

Theorem 4.36. *SContFCov is equivalent to ContBCov. The equivalence restricts to an equivalence between LSContFCov and LKFTop.*

Proof. First, we show that the functor $F: \text{SContFCov} \rightarrow \text{ContBCov}$ described above is essentially surjective. Let (S, \blacktriangleleft) be a continuous basic cover with a relation $\text{wb} \subseteq S \times S$ satisfying (4.10). Define a finitary cover \blacktriangleleft on $\text{Fin}(S)$ by

$$A \blacktriangleleft \mathcal{U} \stackrel{\text{def}}{\iff} A \blacktriangleleft \bigcup \mathcal{U}. \quad (4.17)$$

Then, define a relation \sqsubset on $\text{Fin}(S)$ by

$$A \sqsubset B \stackrel{\text{def}}{\iff} \exists C \in \text{Fin}(S) (A \blacktriangleleft C \text{wb}_L B). \quad (4.18)$$

One can easily show that \sqsubset is idempotent by noting that $a \blacktriangleleft \text{wb}^{-} a \blacktriangleleft \{a\}$. We show that $(\text{Fin}(S), \blacktriangleleft, \sqsubset)$ satisfies (4.6). First, suppose $A \sqsubset B \blacktriangleleft \mathcal{U}$. Then, there exists $C \in \text{Fin}(S)$ such that $A \blacktriangleleft C \text{wb}_L B$. Since $B \blacktriangleleft \text{wb}^{-} \bigcup \mathcal{U}$, there exists $D \in \text{Fin}(\text{wb}^{-} \bigcup \mathcal{U})$ such that $C \blacktriangleleft D$. Then, there exists $\mathcal{V} \in \text{Fin}(\text{Fin}(S))$ such that $D = \bigcup \mathcal{V}$ and $\mathcal{V} (\text{wb}_L)_L \mathcal{U}$. Thus $A \blacktriangleleft \mathcal{V} \sqsubset_L \mathcal{U}$. Conversely, if $A \blacktriangleleft \mathcal{V} \sqsubset_L \mathcal{U}$, then $A \sqsubset \bigcup \mathcal{U} \blacktriangleleft \mathcal{U}$. Therefore, $(\text{Fin}(S), \blacktriangleleft, \sqsubset)$ is a strong continuous finitary cover.

It is straightforward to show that (S, \blacktriangleleft) is isomorphic to the basic cover $(\text{Fin}(S), \blacktriangleleft_{\sqsubset})$ determined by $(\text{Fin}(S), \blacktriangleleft, \sqsubset)$ as in (4.11). Specifically, we have an isomorphism $r: (S, \blacktriangleleft) \rightarrow (\text{Fin}(S), \blacktriangleleft_{\sqsubset})$ defined by

$$a r A \stackrel{\text{def}}{\iff} a \blacktriangleleft A$$

with an inverse $s: (\text{Fin}(S), \blacktriangleleft_{\sqsubset}) \rightarrow (S, \blacktriangleleft)$ defined by

$$A s a \stackrel{\text{def}}{\iff} A \blacktriangleleft \{a\}.$$

Hence, the functor $F: \text{SContFCov} \rightarrow \text{ContBCov}$ is essentially surjective.

Next, we show that for any locally compact formal topology $(S, \blacktriangleleft, \leq)$ equipped with a relation wb satisfying (4.10), the strong continuous finitary cover $(\text{Fin}(S), \blacktriangleleft, \sqsubset)$ defined by (4.17) and (4.18) is localized. Suppose $A \sqsubset B \blacktriangleleft \mathcal{U}$. Then $B \blacktriangleleft \text{wb}^{-} \bigcup \mathcal{U} \downarrow_{\leq} \text{wb}^{-} B$, so there exists $C \in \text{Fin}(\text{wb}^{-} \bigcup \mathcal{U} \downarrow_{\leq} \text{wb}^{-} B)$ such that $A \blacktriangleleft C$. Thus, there exists $\mathcal{V} \in \text{Fin}(\text{Fin}(S))$ such that $C = \bigcup \mathcal{V}$, $\mathcal{V} \sqsubset_L \mathcal{U}$, and $\mathcal{V} \sqsubset_L \{B\}$. Hence $A \blacktriangleleft \mathcal{V} \in \text{Fin}(B \downarrow_{\sqsubset} \mathcal{U})$, and so $(\text{Fin}(S), \blacktriangleleft, \sqsubset)$ is localized.

As we have shown in the first part, as a basic cover, (S, \triangleleft, \leq) is isomorphic to the locally compact formal topology $(\text{Fin}(S), \triangleleft_{\square}, \sqsubseteq)$ determined by $(\text{Fin}(S), \triangleleft, \square)$. Since any isomorphism between the underlying basic covers is a formal topology map [CMS13, Proposition 5.6], the restriction of $F: \text{SContFCov} \rightarrow \text{ContBCov}$ to LSContFCov and LKFTop is essentially surjective. \square

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