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LOGICAL REASONING FOR HIGHER-ORDER FUNCTIONS WITH LOCAL STATE

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ABSTRACT. We introduce an extension of Hoare logic for call-by-value higher-order functions with ML-like local reference generation. Local references may be generated dynamically and exported outside their scope, may store higher-order functions and may be used to construct complex mutable data structures. This primitive is captured logically using a predicate asserting reachability of a reference name from a possibly higher-order datum and quantifiers over hidden references. We explore the logic's descriptive and reasoning power with non-trivial programming examples combining higher-order procedures and dynamically generated local state. Axioms for reachability and local invariant play a central role for reasoning about the examples.

CONTENTS

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1. Introduction

Reference Generation in Higher-Order Programming. This paper proposes an extension of Hoare Logic [17] for call-by-value higher-order functions with ML-like new reference generation [1, 2], and demonstrates its use through non-trivial reasoning examples. New reference generation, embodied for example in ML's `ref`-construct, is a highly expressive programming primitive. The first key functionality of this construct is to introduce local state into the dynamics of programs by generating a fresh reference inaccessible from the outside. Consider the following program:

$$\mathtt{Inc} \stackrel{\text{def}}{=} \mathtt{let}\ x = \mathtt{ref}(0)\ \mathtt{in}\ \lambda().(x := !x+1;\ !x) \tag{1.1}$$

where "$\mathtt{ref}(M)$" returns a fresh reference whose content is the value which M evaluates to; "$!x$" denotes dereferencing the imperative variable x; and ";" is sequential composition. In (1.1), a reference with content 0 is newly created, but never exported to the outside. When the anonymous function in `Inc` is invoked, it increments the content of the local variable x, and returns the new content. The procedure returns a different result at each call, whose source is hidden from external observers. This is different from $\lambda().(x := !x+1;\ !x)$ where x is globally accessible.

Secondly, local references may be exported outside of their original scope and be shared, contributing to the expressivity of significant imperative idioms. Let us show how stored procedures interact with new reference generation and sharing of references. We consider the following program from [49, § 6]:

$$\mathtt{incShared} \stackrel{\text{def}}{=} a := \mathtt{Inc}; b := !a; z_1 := (!a)(); z_2 := (!b)(); (!z_1 + !z_2) \tag{1.2}$$

The initial content of the hidden x is 0. Following the standard semantics of ML [38], the assignment $b := !a$ copies the code (or a pointer to the code) from a to b while sharing the store x. Hence the content of x is incremented every time the functions stored in a and b, sharing the same store x, are called, returning 3 at the end of the program `incShared`. To understand the behaviour of `incShared` precisely and give it an appropriate specification, we must capture the sharing of x between the procedures assigned to a and b. From the viewpoint of visibility, the scope of x is originally restricted to the function stored in a but gets extruded to and shared by the one stored in b. If we replace $b := !a$ by $b := \mathtt{Inc}$ as follows, two separate instances of `Inc` (hence with separate hidden stores) are assigned to a and b, and the final result is not 3 but 2.

$$\mathtt{incUnShared} \stackrel{\text{def}}{=} a := \mathtt{Inc}; b := \mathtt{Inc}; z_1 := (!a)(); z_2 := (!b)(); (!z_1 + !z_2) \tag{1.3}$$

Controlling the sharing of local references is essential for writing concise algorithms that manipulate functions with shared store, or mutable data structures such as trees and graphs, but complicates formal reasoning, even for relatively small programs [18, 34, 36].

Thirdly, through information hiding, local references can be used for efficient implementations of highly regular observable behaviour, for example, purely functional behaviour. The following program, taken from [49, § 1], called `memFact`, is a simple memoised factorial.

$$\begin{aligned} \mathtt{memFact} \ \stackrel{\text{def}}{=}\ & \mathtt{let}\ a = \mathtt{ref}(0),\ \ b = \mathtt{ref}(1)\ \mathtt{in} \\ & \lambda x.\mathtt{if}\ x = !a\ \mathtt{then}\ !b\ \mathtt{else}\ (a := x;\ b := \mathtt{fact}(x);\ !b) \end{aligned} \tag{1.4}$$

Here `fact` is the standard factorial function. To external observers, `memFact` behaves purely functionally. The program implements a simple case of memoisation: when `memFact` is called with a stored argument in a, it immediately returns the stored value $!b$ without calculation. If x differs from a's content, the factorial fx is calculated and the new pair is stored. For complex functions, memoisation can lead to substantial speedups, but for this to be meaningful we need a memoised function to behave indistinguishably from the original function except for efficiency. So we ask: why can we say `memFact` is indistinguishable from the pure factorial function? The answer to this question can be articulated clearly through a *local invariant property* [49] which can be stated informally as follows:

> *Throughout all possible invocations of* `memFact`, *the content of b is the factorial of the content of a.*

Such local invariants capture one of the basic patterns in programming with local state, and play a key role in preceding studies of operational reasoning about program equivalence in the presence of local state [27, 48, 49, 59]. Can we distill this principle axiomatically and use it to validate efficiently properties of higher-order programs with local state such as `memFact`?

As a further example of local invariants, this time involving mutually recursive stored functions, consider the following program:

$$\begin{array}{rl}\mathtt{mutualParity} \stackrel{\text{def}}{=} & x := \lambda n.\mathtt{if}\ n{=}0\ \mathtt{then}\ \mathtt{f}\ \mathtt{else}\ \mathtt{not}((!y)(n{-}1)); \\ & y := \lambda n.\mathtt{if}\ n{=}0\ \mathtt{then}\ \mathtt{t}\ \mathtt{else}\ \mathtt{not}((!x)(n{-}1))\end{array} \tag{1.5}$$

After running `mutualParity`, the application $(!x)n$ returns `t` if n is odd and otherwise `f`; $(!y)n$ acts dually. But since x and y are free, a program may disturb `mutualParity`'s functioning by inappropriate assignment: if a program reads from x and stores it in another variable, say z, assigns a diverging function to x, and feeds the content of z with 7, then the program diverges rather than returning `t`.

With local state, we can avoid unexpected interference at x and y.

$$\mathtt{safeOdd} \stackrel{\text{def}}{=} \mathtt{let}\ x = \mathtt{ref}(\lambda n.\mathtt{t}),\ y = \mathtt{ref}(\lambda n.\mathtt{t})\ \mathtt{in}\ (\mathtt{mutualParity}; !x) \tag{1.6}$$

$$\mathtt{safeEven} \stackrel{\text{def}}{=} \mathtt{let}\ x = \mathtt{ref}(\lambda n.\mathtt{t}),\ y = \mathtt{ref}(\lambda n.\mathtt{t})\ \mathtt{in}\ (\mathtt{mutualParity}; !y) \tag{1.7}$$

(Here $\lambda n.\mathtt{t}$ can be any initialising value.) Now that x, y are inaccessible, the programs behave like pure functions, e.g. $\mathtt{safeOdd}(3)$ always returns `true` without any side effects. Similarly $\mathtt{safeOdd}(16)$ always returns `f`. In this case, the invariant says:

> *Throughout all possible invocations,* `safeOdd` *is a procedure which checks if its argument is odd, provided y stores a procedure which does the dual, whereas* `safeEven` *is a procedure which checks if its argument is even, whenever x stores a dual procedure*.

Later we present general reasoning principles for local invariants which can verify properties of these two and many other non-trivial examples [27, 31, 32, 34, 48, 49].

Contribution. This paper studies a Hoare logic for imperative higher-order functions with dynamic reference generation, a core part of ML-like languages. Starting from their origins in the λ-calculus, the syntactic and semantic properties of typed higher-order functional programming languages such as Haskell and ML have been studied extensively, making them an ideal target for the formal validation of properties of programs on a rigorous semantic basis. Further, given the expressive power of imperative higher-order functions (attested to by the encodability of objects [10, 46, 47] and of low-level idioms [58]), a study of logics for these languages may have wide repercussions on logics of programming languages in general.

Such languages [1, 2] combine higher-order functions and imperative features including new reference generation. Extending Hoare logic to these languages leads to technical difficulties due to three fundamental features:

- Higher-order functions, including stored ones.

- General forms of aliasing induced by nested reference types.
- Dynamically generated local references and scope extrusion.

The first is the central feature of these languages; the second arises by allowing reference types to occur in other types; the third feature has been discussed above. In preceding studies, we built Hoare logics for core parts of ML which cover the first two features [6, 22, 24, 25]. On the basis of these works, the present work introduces an extension of Hoare logic for ML-like local reference generation. As noted above, this construct enriches programs' behaviour radically, and has so far defied clean logical and axiomatic treatment. A central challenge is to identify simple but expressive logical primitives, proof rules (for Hoare triples) and axioms (for assertions), enabling tractable assertions and verification.

The program logic proposed in the present paper introduces a predicate representing reachability of a reference from an arbitrary datum in order to represent new reference generation. Since we are working with higher-order programs, a datum and a reference may as well be, or store, a higher-order function. We shall show that this predicate is fully axiomatisable using (in)equality when it only involves first-order data types (the result is closely related with known axiomatisations of reachability [45]). However we shall also show that the predicate becomes undecidable when higher-order types are involved, indicating an inherent intractability.

A good news is, however, that this predicate enables us, when combined with a pair of mutually dual hiding quantifiers (i.e. quantifiers ranging over variables denoting hidden references), to obtain a simple compositional proof rule for new reference generation, preserving all the compositional proof rules for the remaining constructs from our foregoing program logics.

At the level of assertions, we can find a set of useful axioms for (un)reachability and the hiding quantifiers, which are effectively combined with logical primitives and associated axioms for higher-order functions and aliasing studied in our preceding works [6, 25]. These axioms for reachability and hiding quantifiers are closely related with reasoning principles studied in existing semantic studies on local state, such as the principle of local invariants [49] . The local invariant axioms capture common patterns in reasoning about local state, and enable us to verify the examples in [27, 31, 32, 34, 48, 49] axiomatically, including programs discussed above. The program logic also satisfies strong completeness properties including the standard relative completeness as discussed later. As a whole, our program logic offers an expressive reasoning framework where (relatively) simple programs such as pure functions can be reasoned about using simpler primitives while programs with more complex behaviour such as those with non-trivial use of local state are reasoned about using incrementally more involved logical constructs and axioms.

Outline. This paper is a full version of [63], with complete definitions and detailed explanations and proofs. The present version not only gives more detailed analysis for the properties of the models, axioms and proof rules, but also more examples with full derivations and comprehensive comparisons with related work.

Section 2 presents the programming language and the assertion language. Section 3 gives the semantics of the logic. Section 4 proposes the proof rules and proves soundness. Section 5 explores axioms of the assertion language. Sections 6 discusses the use of the logic through non-trivial reasoning examples centring on local invariants. Section 7 summarises extensions including the three completeness results of the logic, gives the comparisons with related works, and concludes with further topics. Appendix lists auxiliary definitions and detailed proofs. Larger examples of reasoning about mutable data structures can be found in [62].

2. ASSERTIONS FOR LOCAL STATE

2.1. **A Programming Language.** Our target programming language is call-by-value PCF with unit, sums, products and recursive types, augmented with imperative constructs. Let $a,b,...,x,y,\ldots$ range over an infinite set of variables, and $\mathrm{X},\mathrm{Y},\ldots$ over an infinite set of type variables.[1] Then types, values and programs are given by:

$$\begin{array}{rcl}
\alpha,\beta & ::= & \mathsf{Unit} \mid \mathsf{Bool} \mid \mathsf{Nat} \mid \alpha\Rightarrow\beta \mid \alpha\times\beta \mid \alpha+\beta \mid \mathsf{Ref}(\alpha) \mid \mathrm{X} \mid \mu\mathrm{X}.\alpha \\
V,W & ::= & \mathtt{c} \mid x^{\alpha} \mid \lambda x^{\alpha}.M \mid \mu f^{\alpha\Rightarrow\beta}.\lambda y^{\alpha}.M \mid \langle V,W\rangle \mid \mathtt{inj}_i^{\alpha+\beta}(V) \\
M,N & ::= & V \mid MN \mid M:=N \mid \mathtt{ref}(M) \mid !M \mid \mathtt{op}(\tilde{M}) \mid \pi_i(M) \mid \langle M,N\rangle \mid \mathtt{inj}_i^{\alpha+\beta}(M) \\
 & \mid & \mathtt{if}\ M\ \mathtt{then}\ M_1\ \mathtt{else}\ M_2 \mid \mathtt{case}\ M\ \mathtt{of}\ \{\mathtt{in}_i(x_i^{\alpha_i}).M_i\}_{i\in\{1,2\}}
\end{array}$$

We use standard notation [14, 46] like constants $\mathtt{c}$ (unit $()$; booleans $\mathtt{t}$, $\mathtt{f}$; numbers $\mathtt{n}$; and location labels also called simply *locations* $l,l',...$) and first-order operations $\mathtt{op}$ $(+,-,\times,=,\neg,\wedge,\ldots)$. Locations only appear at runtime when references are generated. $\tilde{M}$ etc. denotes a vector and ε the empty vector. A program is *closed* if it has no free variables. Note that a closed program might contain free locations. We use abbreviations such as:

$$\begin{array}{rcll}
\lambda().M & \stackrel{\text{def}}{=} & \lambda x^{\mathsf{Unit}}.M & (x\notin \mathsf{fv}(M)) \\
M;N & \stackrel{\text{def}}{=} & (\lambda().N)M & \\
\mathtt{let}\ x=M\ \mathtt{in}\ N & \stackrel{\text{def}}{=} & (\lambda x.N)M & (x\notin \mathsf{fv}(M))
\end{array}$$

We use the standard notion of types for imperative λ-calculi [14, 46] and use the equi-isomorphic approach [46] for recursive types. Nat, Bool and Unit are called *base types*. We leave the illustration of each language construct to standard textbooks [46], except for reference generation $\mathtt{ref}(M)$, the focus of the present study. $\mathtt{ref}(M)$ behaves as follows: first M of type α is evaluated and becomes a value V; then a *fresh* reference of type $\mathsf{Ref}(\alpha)$ with initial content V is generated.

The behaviour of the programs is formalised by the reduction rules. Let σ denote a *store*, a finite map from locations to closed values. We use $\sigma\uplus[l\mapsto V]$ to denote the result of disjointly adding a pair (l,V) to σ. A *configuration* is of the form $(\nu\tilde{l})(M,\sigma)$ where M is a program, σ a store, and $\tilde{l}$ a vector of distinct locations (the order is irrelevant) occurring in σ, and hidden by ν. The need of ν-biniding is discussed in § 2.3 and Remark 3.4.

A *reduction relation*, or often *reduction* for short, is a binary relation between configurations, written

$$(\nu\tilde{l})(M,\sigma_1)\ \longrightarrow\ (\nu\tilde{l}')(N,\sigma_2)$$

The relation is generated by the following rules. First we have the standard rules for call-by-value PCF:

$$\begin{array}{rcl}
(\lambda x.M)V & \to & M[V/x] \\
\pi_1(\langle V_1,V_2\rangle) & \to & V_1 \\
\mathtt{if}\ \mathtt{t}\ \mathtt{then}\ M_1\ \mathtt{else}\ M_2 & \to & M_1 \\
(\mu f.\lambda g.N)W & \to & N[W/g][\mu f.\lambda g.N/f] \\
\mathtt{case}\ \mathtt{in}_1(W)\ \mathtt{of}\ \{\mathtt{in}_i(x_i).M_i\}_{i\in\{1,2\}} & \to & M_1[W/x_1]
\end{array}$$

[1] For simplicity, we omit the polymorphism from the language, see [24].

Then we have the reduction rules for imperative constructs, i.e. assignment, dereference and new-name generation.

$$\begin{array}{rcl} (!l,\ \sigma) & \rightarrow & (\sigma(l),\ \sigma) \\ (l := V,\ \sigma) & \rightarrow & ((),\ \sigma[l \mapsto V]) \\ (\mathtt{ref}(V),\ \sigma) & \rightarrow & (\nu\, l)(l,\ \sigma \uplus [l \mapsto V]) \end{array}$$

In the reduction rule for references, the resulting configuration uses a ν-*binder*, which lets us directly capture the observational meaning of programs. Finally we close $\longrightarrow$ under evaluation contexts and ν-binders.

$$\frac{(\nu\tilde{l}_1)(M,\sigma) \rightarrow (\nu\tilde{l}_2)(M',\sigma')}{(\nu\tilde{l}\tilde{l}_1)(\mathcal{E}[M],\sigma) \rightarrow (\nu\tilde{l}\tilde{l}_2)(\mathcal{E}[M'],\sigma')}$$

where $\tilde{l}$ are disjoint from both $\tilde{l}_1$ and $\tilde{l}_2$, $\mathcal{E}[\cdot]$ is the left-to-right call-by-value evaluation context (with eager evaluation), inductively given by:

$$\begin{array}{rcl} \mathcal{E}[\cdot] & ::= & (\mathcal{E}[\cdot]M) \ \mid\ (V\mathcal{E}[\cdot]) \ \mid\ \langle V,\mathcal{E}[\cdot]\rangle \ \mid\ \langle \mathcal{E}[\cdot],M\rangle \ \mid\ \pi_i(\mathcal{E}[\cdot]) \ \mid\ \mathtt{in}_i(\mathcal{E}[\cdot]) \\ & \mid & \mathtt{op}(\tilde{V},\mathcal{E}[\cdot],\tilde{M}) \ \mid\ \mathtt{if}\ \mathcal{E}[\cdot]\ \mathtt{then}\ M\ \mathtt{else}\ N \ \mid\ \mathtt{case}\ \mathcal{E}[\cdot]\ \mathtt{of}\ \{\mathtt{in}_i(x_i).M_i\}_{i\in\{1,2\}} \\ & \mid & !\mathcal{E}[\cdot] \ \mid\ \mathcal{E}[\cdot] := M \ \mid\ V := \mathcal{E}[\cdot] \ \mid\ \mathtt{ref}(\mathcal{E}[\cdot]) \end{array}$$

We write (M,σ) for $(\nu\,\varepsilon)(M,\sigma)$ with ε denoting the empty vector. We define:

- $(\nu\tilde{l})(M,\sigma) \Downarrow (\nu\tilde{l}')(V,\sigma')$ means $(\nu\tilde{l})(M,\sigma) \rightarrow^* (\nu\tilde{l}')(V,\sigma')$
- $(\nu\tilde{l})(M,\sigma) \Downarrow$ means $(\nu\tilde{l})(M,\sigma) \Downarrow (\nu\tilde{l}')(V,\sigma')$ for some $(\nu\tilde{l}')(V,\sigma')$

An *environment* $\Gamma,\Delta,\ldots$ is a finite map from variables to types and from locations to reference types. The typing rules are standard [46] and are left to Appendix A. Sequents have the form $\Gamma \vdash M : \alpha$, to be read: M has type α under Γ. A store σ is typed under Δ, written $\Delta \vdash \sigma$, when, for each l in its domain, $\sigma(l)$ is a closed value which is typed α under Δ, where we assume $\Delta(l) = \mathsf{Ref}(\alpha)$. A configuration (M,σ) is *well-typed* if for some Γ and α we have $\Gamma \vdash M : \alpha$ and $\Gamma \vdash \sigma$. Standard type safety holds for well-typed configurations. *Henceforth we only consider well-typed programs and configurations.*

We define the observational congruence between configurations. Assume $\Gamma, \tilde{l}_{1,2} : \tilde{\alpha}_{1,2} \vdash M_{1,2} : \alpha$ and $\Gamma, \tilde{l}_{1,2} : \tilde{\alpha}_{1,2} \vdash \sigma_{1,2}$. Write

$$\Gamma \vdash (\nu\tilde{l}_1)(M_1,\sigma_1) \cong (\nu\tilde{l}_2)(M_2,\sigma_2)$$

if, for each typed context $C[\,\cdot\,]$ which produces a closed program which is typed as Unit under Δ and in which no labels from $\tilde{l}_{1,2}$ occur, the following holds:

$$(\nu\tilde{l}_1)(C[M_1],\ \sigma_1) \Downarrow \quad \text{iff} \quad (\nu\tilde{l}_2)(C[M_2],\ \sigma_2) \Downarrow$$

which we often write $(\nu\tilde{l}_1)(M_1,\sigma_1) \cong (\nu\tilde{l}_2)(M_2,\sigma_2)$ leaving type information implicit. We also write $\Gamma \vdash M_1 \cong M_2$, or simply $M_1 \cong M_2$ leaving type information implicit, if, $\tilde{l}_i = \sigma_i = \emptyset$ $(i = 1,2)$.

2.2. **A Logical Language.** The logical language we shall use is that of standard first-order logic with equality [33, § 2.8], extended with the constructs for (1) higher-order application [24, 25] (for imperative higher-order functions); (2) quantification over store content [6] (for aliasing); (3) reachability and quantifications over hidden names (for local state). For (1) we decompose the original construct [24, 25] into more elementary constructs, which becomes important for precisely capturing the semantics of higher-order programs with local state and for obtaining strong completeness properties of the logic, as we shall discuss in later sections.

The grammar follows, letting $\star \in \{\wedge, \vee, \supset\}$, $\mathcal{Q} \in \{\exists, \forall, \nu, \overline{\nu}\}$ and $\mathcal{Q}' \in \{\exists, \forall\}$.

$$
\begin{array}{rcl}
e & ::= & x \mid \mathsf{c} \mid \mathsf{op}(\tilde{e}) \mid \langle e, e' \rangle \mid \mathsf{inj}_i^{\alpha_1+\alpha_2}(e) \mid\ !e \\
C & ::= & e = e' \mid \neg C \mid C \star C' \mid \mathcal{Q} x^{\alpha}.C \mid \mathcal{Q}' \mathrm{X}.C \mid [!e]C \mid \langle !e \rangle C \\
& \mid & e \bullet e' = x\{C\} \mid \Box C \mid \Diamond C \mid e \hookrightarrow e' \mid e \# e'
\end{array}
$$

The first grammar $(e, e', \ldots)$ defines *terms*; the second *formulae* $(A, B, C, C', E, \ldots)$. Terms include variables, constants c (unit $()$, numbers n, booleans t, f and locations $l, l', \ldots$), pairing, injection and standard first-order operations. $!e$ denotes the dereference of a reference e. Formulae include standard logical connectives and first-order quantifiers [33].

The remaining constructs in the logical language are for capturing the behaviour of imperative higher-order functions with local state. First, the universal and existential quantifiers, $\forall x.C$ and $\exists x.C$, are standard. We include, following [6, 24], quantification over type variables $(\mathrm{X}, \mathrm{Y}, \ldots)$. We also use the two quantifiers for aliasing introduced in [6]. $[!x]C$ is *universal content quantification of x in C*, while $\langle !x \rangle C$ is *existential content quantification of x in C*. In both, x should have a reference type. $[!x]C$ says C holds regardless of the value stored in a memory cell named x; and $\langle !x \rangle C$ says C holds for some value that may be stored in the memory cell named x. In both, what is being quantified is the content of a store, *not* the name of that store. In $[!x]C$ and $\langle !x \rangle C$, C is the *scope* of the quantification. The free variable x is not a binder: we have $\mathsf{fv}(\langle !x \rangle C) = \mathsf{fv}([!x]C) = \{x\} \cup \mathsf{fv}(C)$ where $\mathsf{fv}(C)$ denotes the set of free variables in C. We define $\langle !e \rangle C$ as a shorthand for $\exists x.(x = e \wedge \langle !x \rangle C)$, assuming $x \notin \mathsf{fv}(C)$. Likewise, $[!e]C$ is short for $\forall x.(x = e \supset [!x]C)$ with x being fresh. The scope of a content quantifier is as small as possible, e.g. $[!x]C \supset C'$ stands for $([!x]C) \supset C'$.

Decomposing the original evaluation formulae [24, 25] into $e \bullet e' = x\{C\}$ and $\Box C$, is used for describing the behaviour of functions.[2] $e \bullet e' = x\{C\}$, which we call (one-sided) *evaluation formula*, intuitively says:

> *The application of a function e to an argument e′ starting from the* present *state will terminate with a resulting value (name it x) and a final state, together satisfying C.*

whereas $\Box C$, which we read *always C*, intuitively means:

> *C holds in any possible state reachable from the current one.*

Its dual is written $\Diamond C$ (defined as $\neg \Box \neg C$), which we read *someday C*. We call $\Box$ (resp. $\Diamond$) *necessity* (resp. *possibility*) operators. As a typical usage of these primitives, consider:

$$\Box (C \supset f \bullet x = y\{C'\}) \tag{2.1}$$

This can be read: "for now or any future state, once C holds, then the application of f to x terminates, with both a return value y and a final state satisfying C'". Note that (2.1) corresponds to the original evaluation formula in [24, 25]. Further, in the presence of local state, (2.1) can describe situations which cannot be represented using the original evaluation formula (see § 2.3 for examples). The decomposition (2.1) can also generalise the local invariant axiom in Proposition 5.15 from [63]. Thus this decomposed form is strictly more expressive. It also allows a more streamlined theory.

There are two new logical primitives for representing local state — in other words, for describing the effects of generating and using a fresh reference. First, the *hiding-quantifiers*, $\nu x.C$ (*for some hidden reference x, C holds*) and $\overline{\nu} x.C$ (*for each hidden reference x, C holds*), quantify over reference variables, i.e. the type of x above should be of the form $\mathsf{Ref}(\beta)$. These quantifiers range over hidden references, such as x generated by `Inc` in (1.1) in § 1. The need for having these quantifiers

[2] We later show $\Box C$ is expressible by $e \bullet e' = x\{C\}$: nevertheless treating $\Box C$ independently is convenient for our technical development.

in addition to the standard ones is illustrated in § 2.3 and Remark 3.4. The formal difference of ν as a quantifier from $\exists$ will be clarified in § 5.3, Proposition 5.8.

The second new primitive for local state is $e_1 \hookrightarrow e_2$ (with e_2 of a reference type), which we call *reachability predicate*. This predicate says:

We can reach the reference denoted by e_2 from a datum denoted by e_1.

As an example, if x denotes a starting point of a linked list, $x \hookrightarrow y$ says a reference y occurs in one of the cells reachable from x. We set its dual [12, 55], written $e\#e'$, to mean $\neg e' \hookrightarrow e$. This negative form says:

One can never reach a reference e starting from a datum denoted by e'.

is frequently used for representing freshness of new references.

Note that expressions of our logical language do not include arbitrary programs. If we enlarge terms in the present logical language to encompass arbitrary programs, then terms in the logic will have effects when being evaluated (such as $\lambda y.x := 3$). In addition, the axiomatisation of equality would feature involved axioms like $() = (x := 3)$. Note also that the inclusion of application leads to expressions whose evaluation may be non-terminating. Excluding such arbitrary terms means that we can use standard first-order logic with equality and its usual axiomatisation as its basis, avoiding non-termination and side-effects when calculating assertions.

Terms are typed inductively starting from types for variables and constants and signatures for operators. The typing rules for terms follow the standard ones for programs [46] and are given in Figure 3 in Appendix A. We write $\Gamma \vdash e : \alpha$ when e has type α such that free variables in e have types following Γ; and $\Gamma \vdash C$ when all terms in C are well-typed under Γ.

Equations between terms of different types will always evaluate to F.[3] The falsity F is definable as $1 \neq 1$, and its dual $\mathsf{T} \stackrel{\text{def}}{=} \neg\mathsf{F}$. The *syntactic substitution* $C[e/!x]$ is also used frequently: the definition is standard, save for some subtlety regarding substitution into the post-condition of evaluation formulae, details can be found in Appendix B in [6]. *Henceforth we only treat well-typed terms and formulae.*

Further notational conventions follow.

Notation 2.1 (Assertions)**.**

(1) In the subsequent technical development, logical connectives are used with their standard precedence/association, with content quantification given the same precedence as standard quantification (i.e. they associate stronger than binary connectives). For example,

$$\neg A \wedge B \supset \forall x.C \vee \langle !e\rangle D \supset E$$

is a shorthand for $((\neg A) \wedge B) \supset (((\forall x.C) \vee (\langle !e\rangle D)) \supset E)$. The standard binding convention is always assumed.

(2) $C_1 \equiv C_2$ stands for $(C_1 \supset C_2) \wedge (C_2 \supset C_1)$, stating the logical equivalence of C_1 and C_2.

(3) $e \neq e'$ stands for $\neg e = e'$.

(4) Logical connectives are used not only syntactically but also semantically, i.e. when discussing meta-logical and other notions of validity.

(5) We write $\{C\}\ e_1 \bullet e_2 = z\ \{C'\}$ for $C \supset e_1 \bullet e_2 = z\{C'\}$.

(6) $e_1 \bullet e_2 = e'\{C\}$ stands for $e_1 \bullet e_2 = x\{x = e' \wedge C\}$ where x is fresh and e' is not a variable; $e_1 \bullet e_2\{C\}$ stands for $e_1 \bullet e_2 = ()\{C\}$; and $e_1 \bullet e_2 \Downarrow$ stands for the convergence $e_1 \bullet e_2 = x\{\mathsf{T}\}$. We apply the same abbreviations to $\{C\}\ e_1 \bullet e_2 = z\ \{C'\}$.

[3] To be precise, "terms of unmatchable types": this is because of the presence of type variables. For example, the equation "$x^X = 1^{\mathsf{Nat}}$" can hold depending on models but "$x^{\mathsf{Ref}(X)} = 1^{\mathsf{Nat}}$" never holds.

(7) For convenience of rule presentation we will use projections $\pi_i(e)$ as a derived term. They are redundant in that any formula containing projections can be translated into one without: for example $\pi_1(e) = e'$ can be expressed as $\exists y.e = \langle e', y\rangle$.
(8) We denote $\mathsf{fv}(C)$ (resp. $\mathsf{fl}(C)$) for the set of the free variables (resp. free locations) in C.
(9) $[!x_1..x_n]C$ for $[!x_1]..[!x_n]C$. Similarly for $\langle !x_1..x_n\rangle C$.
(10) We write $\tilde{e}\#e$ for $\wedge_i e_i\#e$; $e\#\tilde{e}$ for $\wedge_i e\#e_i$; and $\tilde{e}\#\tilde{e}'$ for $\wedge_{ij} e_i\#e'_j$.

2.3. **Assertions for Local State.** We explain assertions with examples.

(1) The assertion $x = 6$ says that x of type Nat is equal to 6.
(2) Assuming x has type $\mathsf{Ref}(\mathsf{Nat})$, $!x = 2$ means x stores 2. Next assume that e_1 and e_2 have a reference type carrying a functional type, say $\mathsf{Ref}(\mathsf{Nat} \to \mathsf{Nat})$. Then we can specify equality of the contents of the reference as: $!e_1 = !e_2$. Note that neither e_1 nor e_2 contains λ-expressions. Section 5.1 shall show that the standard axioms for the equality hold in our logic.
(3) Consider a simple command $x := y; y := z; w := 1$. After its run, we can reach reference name z by dereferencing y, and y by dereferencing x. Hence z is reachable from y, y from x, hence z from x. So the final state satisfies $x \hookrightarrow y \wedge y \hookrightarrow z \wedge x \hookrightarrow z$ which implies by transitivity.
(4) Next, assuming w is newly generated, we may wish to say w is *unreachable* from x, to ensure freshness of w. For this we assert $w\#x$, which, as noted, stands for $\neg(x \hookrightarrow w)$. $x\#y$ always implies $x \neq y$. Note that $x \hookrightarrow x \equiv x \hookrightarrow !x \equiv \mathsf{T}$ and $x\#x \equiv \mathsf{F}$. But $!x \hookrightarrow x$ may or may not hold (since there may be a cycle between x's content and x in the presence of recursive types).
(5) We consider reachability in procedures. Assume $\lambda().(x := 1)$ is named as f_w, similarly $\lambda().!x$ as f_r. Since f_w can write to x, we have $f_w \hookrightarrow x$. Similarly $f_r \hookrightarrow x$. Next suppose $\mathtt{let}\ x = \mathtt{ref}(z)\ \mathtt{in}\ \lambda().x$ has name f_c and z's type is $\mathsf{Ref}(\mathsf{Nat})$. Then $f_c \hookrightarrow z$ (e.g. consider $!(f_c()) := 1$). However x is *not* reachable from $\lambda().((\lambda y.())(\lambda().x))$ since semantically, this function never touches x.
(6) $\Box !x = 1$ says that x's content is unchanged from 1 forever, which is logically equivalent to F (since x might be updated in the future). Instead $\Diamond !x = 1 \equiv \mathsf{T}$. On the other hand, $\Box x = 1 \equiv \Diamond x = 1 \equiv x = 1$ (since a value of a functional variable is not affected by the state).
(7) The following program:
$$f \stackrel{\text{def}}{=} \lambda().(x := !x + 1; !x) \tag{2.2}$$
satisfies the following assertion, when named u:
$$\Box \forall i^{\mathsf{Nat}}.\{!x = i\} u \bullet () = z \{!x = z \wedge !x = i + 1\}$$
saying:

now or for any future state, invoking the function named u increments the content of x and returns that content.

Stating it for a future state is important since a closure is potentially invoked many times in different states.
(8) We often wish to say that the write effects of an application are restricted to specific locations. The following *located assertion* [6] is used for this purpose: $e \bullet e' = x\{C\}@\tilde{e}$ where each e_i is of reference type and does not contain a dereference. $\tilde{e}$ is called *effect set*, which might be modified by the evaluation. As an example:
$$\mathsf{inc}(u,x) \stackrel{\text{def}}{=} \Box \forall i.\{!x = i\} u \bullet () = z\{z = !x \wedge !x = i + 1\}@x \tag{2.3}$$
is satisfied by f in (2.2), saying that a function named u, when invoked, will: (1) increment the content of x and (2) return the original content of x, without modifying (in an observational

fashion) any state except x. As in [6], located assertions can be translated into non-located evaluation formulae together with content quantification in § 2.2, see Proposition 5.5.

(9) Assuming f denotes the result of evaluating `Inc` in the introduction, we can assert, using the existential hiding quantifier and naming by u:

$$\nu x.(!x = 0 \wedge \mathsf{inc}(u,x)) \tag{2.4}$$

which says: there is a hidden reference x storing 0 such that, whenever u is invoked, it writes at x and returns the increment of the value stored in x at the time of invocation.

(10) We illustrate that combining hiding quantifiers and the non-reachability predicate is necessary for describing the effects and use of newly generated references. Consider:

$$\mathtt{let}\ x = \mathtt{ref}(2)\ \mathtt{in}\ y := x \tag{2.5}$$

The location denoted by the bound variable x is, at the time when the new reference is generated, hidden and disjoint from any existing datum. The location represented by x is still hidden but it has now become accessible from a variable y, and this location is still unreachable from other references. Thus hiding and disjointness are separate concerns, and, assuming z to be a reference disjoint from y, the post-state of (2.5) can be described as:

$$\nu x.(!y = x \wedge !x = 2 \wedge z\#x) \tag{2.6}$$

(11) The function $f_1 \stackrel{\text{def}}{=} \lambda n^{\mathsf{Nat}}.\mathtt{ref}(n)$, named u, meets the following specification. Let i and X be fresh.

$$\mathsf{fresh} \quad \stackrel{\text{def}}{=} \quad \Box \forall n^{\mathsf{Nat}}.\forall \mathrm{X}.\forall i^{\mathrm{X}}.u \bullet n = z\{\nu x.(!z = n \wedge z\#i \wedge z = x)\}@\emptyset. \tag{2.7}$$

The above assertion says that u, when applied to n, will always return a hidden fresh reference z whose content is n and which is unreachable from any datum existing at the time of the invocation; and in the execution it will leave no writing effects to the existing state. Since i ranges over arbitrary data, unreachability of x from each such i in the post-condition indicates that x is freshly generated and is not stored in any existing reference.

(12) Now let us consider the following three formulae:

$$\mathsf{fresh}_1 \quad \stackrel{\text{def}}{=} \quad \forall n^{\mathsf{Nat}}.\forall \mathrm{X}.\forall i^{\mathrm{X}}.u \bullet n = z\{\nu x.(!z = n \wedge z\#i \wedge z = x)\}@\emptyset \tag{2.8}$$

$$\mathsf{fresh}_2 \quad \stackrel{\text{def}}{=} \quad \forall n^{\mathsf{Nat}}.\forall \mathrm{X}.\forall i^{\mathrm{X}}.\Box u \bullet n = z\{\nu x.(!z = n \wedge z\#i \wedge z = x)\}@\emptyset \tag{2.9}$$

$$\mathsf{fresh}_3 \quad \stackrel{\text{def}}{=} \quad \Box \forall n^{\mathsf{Nat}}.\forall \mathrm{X}.\forall i^{\mathrm{X}}.\Box u \bullet n = z\{\nu x.(!z = n \wedge z\#i \wedge z = x)\}@\emptyset \tag{2.10}$$

Each formula is read as follows:

- fresh_1 means that the procedure named by u, when invoked in the present state with number n, will create a cell with that content which is fresh *in the current state*.
- fresh_2 means that the procedure u, when invoked with number n in the present or any future state, will create a cell with content n which is fresh *in the current state*. For example the following program satisfies this assertion (naming it as u):

$$f_2 \stackrel{\text{def}}{=} \mathtt{let}\ x = \mathtt{ref}(0)\ \mathtt{in}\ \lambda y^{\mathsf{Nat}}.(x := y; x) \tag{2.11}$$

  The function returned by (2.11) does return a fresh reference upon initial invocation: but from the next time this function returns the same reference cell albeit with the new value specified. So it will be fresh with respect to the current state (for which we are asserting this formula) but *not necessarily* with respect to each initial state of invocation.
- fresh_3 means that if we invoke the procedure u in the current state or in any further future state, it will create a cell which is fresh in that state.

Then we have:

$$\mathsf{fresh} \equiv \mathsf{fresh}_3 \supset \mathsf{fresh}_2 \supset \mathsf{fresh}_1 \tag{2.12}$$

which we shall prove by the axioms for $\Box$ later. The program (2.11) satisfies fresh_1 and fresh_2, but does *not* satisfy fresh (nor fresh_3) since f_2 returns the same location. On the other hand, f_1 satisfies all of fresh, fresh_1, fresh_2 and fresh_3. This example demonstrates that a combination of $\Box$ and a decomposed evaluation formula gives precise specifications in the presence of the local state.[4]

3. Models and Semantics

3.1. **Models.** We introduce the semantics of the logic based on the operational semantics of programs, using partially hidden stores. Our purpose is to have a *precise* and *clear* correspondence between programs' operational behaviour (and the induced observational semantics) and the semantics of assertions. This is the reason for defining our models operationally. This approach offers a simple framework to reason about the semantic effects of hidden (and/or newly generated) stores on higher-order imperative programs (for further discussions, see Remark 3.3 later). For capturing local state, our models incorporate hidden locations using ν-binders, suggested by the π-calculus [37]. For example, consider the program `Inc` from the introduction.

$$\mathtt{Inc} \stackrel{\text{def}}{=} \mathtt{let}\, x = \mathtt{ref}(0)\ \mathtt{in}\ \lambda().(x := !x+1;\, !x) \tag{3.1}$$

Recall that after running `Inc`, we reach a state where a hidden name stores 0, to be used by the resulting procedure when invoked. Hence, `Inc` named u, is modelled as:

$$(\nu l)(\{u : \lambda().(l := !l+1;\, !l)\},\ \{l \mapsto 0\}) \tag{3.2}$$

which says that the appropriate behaviour is at u, in addition to a hidden reference l storing 0.

Definition 3.1. (models) An *open model of type* Γ is a tuple (ξ, σ) where:

- ξ, called *environment*, is a finite map from variables in $\mathsf{dom}(\Gamma)$ to closed values such that, for each $x \in \mathsf{dom}(\Gamma)$, $\xi(x)$ is typed as $\Gamma(x)$ under Γ, i.e. $\Gamma \vdash \xi(x) : \Gamma(x)$.
- σ, called *store*, is a finite map from labels in $\{l \mid l \in \mathsf{dom}(\Gamma)\}$ to closed values such that for each $l \in \mathsf{dom}(\sigma)$, $\Gamma(l)$ has type $\mathsf{Ref}(\alpha)$, then $\sigma(l)$ has type α under Γ, i.e. $\Gamma \vdash \sigma(l) : \alpha$.

When Γ includes free type variables, ξ maps them to closed types, with the obvious corresponding typing constraints. A *model* of type Γ is a structure $(\nu\tilde{l})(\xi, \sigma)$ with (ξ, σ) being an open model of type Γ, Δ with $\{\tilde{l}\} = \mathsf{dom}(\Delta)$. $(\nu\tilde{l})$ acts as binders. $\mathcal{M}, \mathcal{M}', \ldots$ range over models.

An open model maps variables and locations to closed values: a model then specifies part of the locations as "hidden". For example, $(\nu l)(x : l \cdot y : l', [l \mapsto 3] \cdot [l' \mapsto 3])$ is a model with a typing environment: $\Gamma = \{x : \mathsf{Ref}(\mathsf{Nat}), y : \mathsf{Ref}(\mathsf{Nat}), l' : \mathsf{Ref}(\mathsf{Nat})\}$. We often omit Γ and a mapping from type variables to closed types from $\mathcal{M}$.

Since assertions in the present logic are intended to capture observable program behaviour, the semantics of the logic uses models quotiented by an observationally sound equivalence, which we choose to be the standard contextual congruence itself.

[4] Note that in fresh and fresh_3, it is essential that we put universal quantifications $\forall \mathrm{X}$ and $\forall i^{\mathrm{X}}$ *after* $\Box$. This has not been possible in the two-sided evaluation formulae used in the logics for pure and imperative higher-order functions without local state in [6, 22, 24, 25]. See (2.1).

Definition 3.2. Assume $\mathcal{M}_i \stackrel{\text{def}}{=} (\nu\tilde{l}_i)(\tilde{x}:\tilde{V}_i,\sigma_i)$ typable under Γ. Then we write $\mathcal{M}_1 \approx \mathcal{M}_2$ if the following clause holds for each typed context $C[\,\cdot\,]$ which is typable under Γ and in which no labels from $\tilde{l}_{1,2}$ occur:

$$(\nu\tilde{l}_1)(C[\langle\tilde{V}_1\rangle],\sigma_1)\Downarrow \quad \text{iff} \quad (\nu\tilde{l}_2)(C[\langle\tilde{V}_2\rangle],\sigma_2)\Downarrow \tag{3.3}$$

where $\langle\tilde{V}\rangle$ is the n-fold pairings of a vector of values.

Definition 3.2 in effect takes models up to the standard contextual congruence. We could have used a different program equivalence (for example call-by-value $\beta\eta$ convertibility), as far as it is observationally sound. Note that we have

$$(\nu\tilde{l})(\xi\cdot x{:}V_1,\sigma\cdot l\mapsto W_1)\approx(\nu\tilde{l})(\xi\cdot x{:}V_2,\sigma\cdot l\mapsto W_2) \tag{3.4}$$

whenever $V_1\cong V_2$ and $W_1\cong W_2$, where $\cong$ is the contextual congruence on programs defined in § 2.1.

To see the reason why we take the models up to observational congruence, let us consider the following program:

$$\texttt{Inc2} \stackrel{\text{def}}{=} \texttt{let}\; x=\texttt{ref}(0),\; y=\texttt{ref}(0)\;\texttt{in}\;\lambda().(x:=!x+1;y:=!y+1;\;(!x+!y)/2) \tag{3.5}$$

which is contextually equivalent to `Inc`. Then we have the following model for `Inc2`.

$$(\nu ll')(\{u:\lambda().(x:=!x+1;y:=!y+1;\;(!x+!y)/2),\; x:l,\;\; y:l'\},\;\{l\mapsto 0,\; l'\mapsto 0\}) \tag{3.6}$$

Since the two programs originate in the same abstract behaviour, we wish to identify the model in (3.2) and the above model, taking them up to the equivalence.

Remark 3.3. (presentation of models) The model as given above can be presented algebraically using the language of categories [59]. One method, which can treat hiding as above categorically, uses a class of toposes which treat renaming through symmetries [20]. We can also use the "swapping"-based treatment of binding based on [13]. Note however that the use of such different presentations (with respective merits) does *not* alter the equational and other properties of models and the satisfaction relation, as far as we wish to use the standard observational semantics (Morris-like contextual congruence) or the equivalent models (so-called fully abstract models) as a basis of our logic. Another significant point is that the game-based model in [4] is the only known model satisfying this (full abstraction) criteria, whose morphisms are isomorphic to a class of typed π-calculus processes [21]. The presented "operational" model is hinted at by, and is close to, the π-calculus presentation of semantics of the target language. The present approach allows us to have models which are automatically faithful to the standard observational semantics of the language, directly capturing the effects of hidden stores by semantics of the logic. Other models may as well be used for exploring various aspects of the presented logic.

Remark 3.4. (hidden locations) Following standard textbooks [14, 46], we treat locations as values (which is natural from the viewpoint of reduction). A significant point is that distinctions among these values (locations) matter even if they are hidden. For example if we have:

$$M \stackrel{\text{def}}{=} \langle\texttt{ref}(2),\texttt{ref}(2)\rangle \tag{3.7}$$

and evaluate M, we get a pair of two fresh locations both storing 2. For the denotation of this resulting value, it is essential that these two references are distinct. For example the program:

$$N \stackrel{\text{def}}{=} \texttt{let}\; x=\texttt{ref}(2)\;\texttt{in}\;\langle x,x\rangle \tag{3.8}$$

has a different observable behaviour, as justified by a context $C[\,] \stackrel{\text{def}}{=} \texttt{if}\;\pi_1[\,]=\pi_2[\,]\;\texttt{then}\;1\;\text{else}\;2$. Thus distinctions matter, even if locations are hidden.

3.2. **Semantics of Equality.** For the rest of this section, we give semantics to assertions, mainly focussing on key features concerning local state and which therefore differ from the previous logics [6]. We start with the semantics of equality.

A key example are the programs `incShared` in (1.2) and `incUnShared` in (1.3) from the introduction. After the second assignment of (1.2) and (1.3), we consider whether we can assert "$!a = !b$" (i.e. the content of a and b are equal). For this inquiry, let us first recall the following defining clause for the satisfaction of equality of two logical terms from [6] which follows the standard definition of logical equality. First we set, with $\Gamma \vdash e : \alpha$, $\Gamma \vdash \mathcal{M}$ and an open model $\mathcal{M} = (\xi, \sigma)$, an interpretation of e under $\mathcal{M}$ as follows.[5]

$$[\![x]\!]_{\xi,\sigma} = \xi(x) \quad [\![!e]\!]_{\xi,\sigma} = \sigma([\![e]\!]_{\xi,\sigma}) \quad [\![\mathsf{c}]\!]_{\xi,\sigma} = \mathsf{c} \quad [\![\mathsf{op}(\tilde{e})]\!]_{\xi,\sigma} = \mathsf{op}([\![\tilde{e}]\!]_{\xi,\sigma})$$

$$[\![\langle e, e' \rangle]\!]_{\xi,\sigma} = \langle [\![e]\!]_{\xi,\sigma}, [\![e']\!]_{\xi,\sigma} \rangle \quad [\![\mathtt{inj}_i(e)]\!]_{\xi,\sigma} = \mathsf{inj}_i([\![e]\!]_{\xi,\sigma})$$

which are all standard. Then we define:

$$\text{(the definition from [6])} \qquad \mathcal{M} \models e_1 = e_2 \quad \stackrel{\text{def}}{\equiv} \quad [\![e_1]\!]_{\mathcal{M}} \approx [\![e_2]\!]_{\mathcal{M}} \tag{3.9}$$

Note that (3.9) says that $e_1 = e_2$ is true under an open model $\mathcal{M}$ iff their interpretations in $\mathcal{M}$ are congruent. Now suppose we apply (3.9) to the question of $!a = !b$ in `incUnShared`. Since the two instances of `Inc` stored in a and b have the identical denotation (or identical behaviour: because they are exactly the same programs), the equality $!a = !b$ holds for `incUnShared` if we use (3.9). *However this interpretation is wrong:* we observe that, in `incUnShared`, running $!a$ twice and running $!a$ and $!b$ consecutively lead to different observable behaviours, due to their distinct local states (which can be easily represented using evaluation formulae). Hence we must have $!a \neq !b$, which says the standard definition (3.9) is not applicable in the presence of the local state. On the other hand, running $!a$ and running $!b$ have always identical observable effects: that is we can always replace the content of a with the content of b in `incShared`, hence the equality $!a = !b$ should hold for `incShared`.

The reason that the standard equality does not hold is because two currently identical stateful procedures will in future demonstrate distinct behaviour. On the other hand, two identical functions which share the same local state always show the same behaviour hence in `incShared` we obtain equality.

This analysis indicates that we need to consider programs placed in contexts to compare them precisely, leading to the following extension for the semantics for the equality, assuming $\mathcal{M} \stackrel{\text{def}}{=} (\nu \tilde{l})(\xi, \sigma)$:

$$\mathcal{M} \models e_1 = e_2 \quad \stackrel{\text{def}}{\equiv} \quad \mathcal{M}[u : e_1] \approx \mathcal{M}[u : e_2] \tag{3.10}$$

where $\mathcal{M}[u : e]$ denotes $(\nu \tilde{l})(\xi \cdot u : [\![e]\!]_{\xi,\sigma}, \sigma)$ with u fresh and the variables and labels in e should be free in $\mathcal{M}$. Note that $\mathcal{M}[u : e]$ offers the notion of a "program-in-context" when e denotes a program. For example let us consider a model for the state immediately after the assignment $b :=!a$ in `incShared`. Then the model may be written as (taking a and b to be locations):

$$\mathcal{M}_{\mathtt{incShared}} = (\nu l) \left(\emptyset, \quad \begin{array}{l} a \mapsto \lambda().(l :=!l + 1; !l), \\ b \mapsto \lambda().(l :=!l + 1; !l), \\ l \mapsto n \end{array} \right) \tag{3.11}$$

We obtain (writing the map for a, b, l above as σ for brevity):

$$\mathcal{M}_{\mathtt{incShared}}[u : !a] = (\nu l)\,(u : \lambda().(l :=!l + 1; !l), \quad \sigma) \tag{3.12}$$

[5]Since a model in [6] does not have local state, it suffices to consider open models.

Notice that the function assigned to u shares l in the environment: we are interpreting the dereference $!a$ "in context". Similarly we obtain:

$$\mathcal{M}_{\texttt{incShared}}[u:!b] \;=\; (\nu l)\,(u:\lambda().(l:=!l+1;!l), \quad \sigma) \tag{3.13}$$

By which we conclude $\mathcal{M}_{\texttt{incShared}} \models !a = !b$: if the results of interpreting two terms in context are equal then we know their effects to the model are equal. We leave it to the reader to check the inequality between $!a$ and $!b$ for the corresponding model representing `incUnShared`.

The definition of equality above satisfies the standard axioms of equality as we shall see in § 5. It is also accompanied by a notion of *symmetry* which can be used for checking (in)equality, introduced below.

Definition 3.5 (permutation)**.** Let $\mathcal{M} \stackrel{\text{def}}{=} (\nu\tilde{l})(\xi\cdot v{:}V\cdot w{:}W,\ \sigma)$ where $\mathcal{M}$ is typed under Γ and v,w have the same type under Γ. Then, we set:

$$\mathcal{M}\binom{vw}{wv} \stackrel{\text{def}}{=} (\nu\tilde{l})(\xi\cdot v{:}W\cdot w{:}V,\ \sigma) \tag{3.14}$$

called a *permutation of* $\mathcal{M}$ *at* v *and* w . We extend the notion to an arbitrary bijection ρ on $\mathsf{dom}(\Gamma)$, writing $\mathcal{M}[\rho]$. A permutation ρ on $\mathcal{M}$ is a *symmetry on* $\mathcal{M}$ when $\mathcal{M}[\rho] \approx \mathcal{M}$.

Proposition 3.6 (symmetries)**.**

(1) *Given* $\mathcal{M}_{1,2}$ *and a bijection* ρ *on free variables in the domain of* $\mathcal{M}_{1,2}$*, we have* $\mathcal{M}_1 \approx \mathcal{M}_2$ *iff* $\mathcal{M}_1[\rho] \approx \mathcal{M}_2[\rho]$.

(2) *If* $\mathcal{M}_1 \approx \mathcal{M}_2$ *and* ρ *is symmetry of* $\mathcal{M}_1$*, then* ρ *is symmetry of* $\mathcal{M}_2$.

Proof. Obvious by definition. □

We illustrate how we can use the result above to model the subtlety of equality of behaviours with shared local state. Let us consider the following models $\mathcal{M}_1$ and $\mathcal{M}_2$, which represent the situations analogous to `incShared` and `incUnShared` (again after running the second assignment). The defining clause for equality gives , using $\mathcal{M}_1[u{:}v] \approx \mathcal{M}_1[u{:}w]$:

$$\mathcal{M}_1 \;=\; (\nu l)\begin{pmatrix} v:\lambda().(l:=!l+1;!l), & \\ w:\lambda().(l:=!l+1;!l), & l\mapsto 0 \end{pmatrix} \models v = w \tag{3.15}$$

On the other hand, we have:

$$\mathcal{M}_2 \;=\; (\nu l l')\begin{pmatrix} v:\lambda().(l:=!l+1;!l), & l\mapsto 0, \\ w:\lambda().(l':=!l'+1;!l'), & l'\mapsto 0 \end{pmatrix} \models v \neq w \tag{3.16}$$

This is because $\binom{uv}{vu}$ is a symmetry of $\mathcal{M}_2[u{:}v]$, but *not* of $\mathcal{M}_2[u{:}w]$. The latter can be examined by comparing the following two models (writing "$u,w:V$" to denote "$u:V,w:V$"):

$$\mathcal{M}_2[u{:}w] \quad = \quad (\nu l l')\begin{pmatrix} v:\lambda().(l:=!l+1;!l), & l\mapsto 0, \\ u,w:\lambda().(l':=!l'+1;!l'), & l'\mapsto 0 \end{pmatrix} \tag{3.17}$$

$$(\mathcal{M}_2[u{:}w])\binom{uv}{vu} \quad = \quad (\nu l l')\begin{pmatrix} u:\lambda().(l:=!l+1;!l), & l\mapsto 0, \\ v,w:\lambda().(l':=!l'+1;!l'), & l'\mapsto 0 \end{pmatrix} \tag{3.18}$$

which differ semantically when e.g. v and w are invoked consecutively. Hence by Proposition 3.6 (2), $\mathcal{M}_2[u{:}v] \not\approx \mathcal{M}_2[u{:}w]$, justifying the above inequality $v \neq w$. The permutations also help to prove the axioms of equality in § 5.

3.3. **Semantics of Necessity and Possibility Operators.** We define, with u fresh,

$$\mathcal{M}[u\!:\!N]\Downarrow\mathcal{M}' \quad \text{when } (N\xi,\sigma)\Downarrow(\nu\tilde{l}')(V,\sigma') \text{ with } \mathcal{M}=(\nu\tilde{l})(\xi,\ \sigma) \text{ and } \mathcal{M}'=(\nu\tilde{l}\tilde{l}')(\xi\cdot u\!:\!V,\ \sigma')$$

where we always assume u is fresh and the variables and labels in N are free in $\mathcal{M}$. The above definition intuitively means that $\mathcal{M}$ *can* reduce to $\mathcal{M}'$ through arbitrary effects on $\mathcal{M}$ by an external program: in other words, $\mathcal{M}'$ is a hypothetical future state (or "possible world") of $\mathcal{M}$. Then we generate $\mathcal{M}\rightsquigarrow\mathcal{M}'$ by

(1) $\mathcal{M}\rightsquigarrow\mathcal{M}$

(2) if $\mathcal{M}\rightsquigarrow\mathcal{M}_0$ and $\mathcal{M}_0[u:N]\Downarrow\mathcal{M}'$, then $\mathcal{M}\rightsquigarrow\mathcal{M}'$

Thus $\mathcal{M}\rightsquigarrow\mathcal{M}'$ reads:

> $\mathcal{M}$ *may evolve to* $\mathcal{M}'$ *by interaction with zero or more typable programs.*

Note that $\rightsquigarrow$ is reflexive and transitive. If $\mathcal{M}\rightsquigarrow\mathcal{M}'$ and $\mathcal{M}'$ adds the new domain $\{x_1..x_n\}$, then $x_1..x_n$ is its *increment* and we often explicitly write $\mathcal{M}\overset{x_1..x_n}{\rightsquigarrow}\mathcal{M}'$.

The semantics of $\Box C$ says that for any target of evolution, C should hold:

$$\mathcal{M}\models\Box C \quad \stackrel{\text{def}}{\equiv} \quad \forall\mathcal{M}'.(\mathcal{M}\rightsquigarrow\mathcal{M}'\supset\mathcal{M}'\models C). \tag{3.19}$$

Dually we set:

$$\mathcal{M}\models\Diamond C \quad \stackrel{\text{def}}{\equiv} \quad \exists\mathcal{M}'.(\mathcal{M}\rightsquigarrow\mathcal{M}'\ \wedge\ \mathcal{M}'\models C). \tag{3.20}$$

3.4. **Semantics of Evaluation Formulae.** The semantics of the evaluation formula is given below:

$$\mathcal{M}\models e\bullet e'=x\{C\} \quad \stackrel{\text{def}}{\equiv} \quad \exists\mathcal{M}'.(\mathcal{M}[x:ee']\Downarrow\mathcal{M}'\ \wedge\ \mathcal{M}'\models C)$$

which says that in the *current* state, if we apply e to e', then the return value (named x) and the resulting state together satisfy C.

We already motivated the decomposition of the original evaluation formulae [6] into the simplified evaluation formulae and the necessity operator from § 2.3. Let us write the original evaluation formulae in [6, 25] as $\{C\}e\bullet e'=x\{C'\}^\star$. Then we can translate this in the present language as:

$$\{C\}e\bullet e'=x\{C'\}^\star \stackrel{\text{def}}{\equiv} \exists f,g.(f=e\wedge g=e'\wedge\Box\{C\}f\bullet g=x\{C'\})$$

that is, we interpret e and e' in the present state and name them f and g, and assert that, now or in any future state in which C is satisfied, if we apply f to g, then it returns x which, together with the resulting state, satisfies C'. The original clause says:

> *In any initial hypothetical state which is reachable from the present state and which satisfies C, the application of e to e′ terminates and both the result x and the final state satisfy C′.*

To see the reason why we require $\Box$ in the specification of functions , we set:

$$\mathcal{M} \quad \stackrel{\text{def}}{=} \quad (\nu l)(u:\lambda().!l,\ w:\lambda().l:=!l+1,\ \ l\mapsto 5) \tag{3.21}$$

We can check that the set of all legitimate hypothetical states from this state (i.e. $\mathcal{M}'$ such that $\mathcal{M}[z:N]\Downarrow\mathcal{M}'$) can be enumerated by:

$$\mathcal{M}'/z \quad \stackrel{\text{def}}{=} \quad (\nu l)(u:=\lambda().!l,\ w:\lambda().l:=!l+1,\ \ l\mapsto m) \tag{3.22}$$

for each $m\geq 5$ (since these are essentially all the models reachable from $\mathcal{M}$, as outside programs can create new references).

Thus we have, for $\mathcal{M}$ in (3.21):

$$\mathcal{M} \models \Box w \bullet () = x\{x \geq 5\} \tag{3.23}$$

which says in any *future* state where w is invoked, it always returns something no less than 5, which is operationally reasonable.

We can use this formula for specifying the following program:

$$\begin{array}{lll} L & \stackrel{\text{def}}{=} & \texttt{let}\ x = \texttt{ref}(5)\ \texttt{in} \\ & & \quad \texttt{let}\ u = \lambda().!x\ \texttt{in} \\ & & \qquad \texttt{let}\ w = \lambda().x :=!x+1\ \texttt{in} \\ & & \qquad\quad (fw)\ ;\ \texttt{if}\ x \geq 5\ \texttt{then}\ \texttt{t}\ \texttt{else}\ \texttt{f} \end{array} \tag{3.24}$$

When the application fw takes place, some unknown computation occurs which may change the value of x: but as far as fw terminates, it always returns $\texttt{t}$. To reach (3.23), we need to consider *all possible* $\mathcal{M}'$ with the effect from the outside. Since such $\mathcal{M}'$ satisfies (3.22), we can conclude the program L always returns $\texttt{t}$ (if fw terminates).

3.5. **Semantics of Universal and Existential Quantification.** The universal and existential quantifiers also need to incorporate local state. We need one definition to identify a set of terms which do not change the state of any models. Below $\mathcal{M}^\Gamma$ indicates that $\mathcal{M}$ is typable under Γ.

Definition 3.7 (Functional Terms)**.** We define the set of *functional terms* of type Γ, denoted $\mathcal{F}^\Gamma$, or often simply $\mathcal{F}$ leaving its typing implicit, as:

$$\mathcal{F} \stackrel{\text{def}}{=} \{N \mid \forall \mathcal{M}^\Gamma.(\mathcal{M}[u:N] \Downarrow \mathcal{M}' \supset \mathcal{M} \cong \mathcal{M}'/u)\}$$

where $\mathcal{M}/u \stackrel{\text{def}}{=} (\nu\tilde{l})(\xi,\sigma)$ if $\mathcal{M} = (\nu\tilde{l})(\xi \cdot u{:}V,\sigma)$; and $\mathcal{M}/u \stackrel{\text{def}}{=} \mathcal{M}$ when $u \notin \mathsf{fv}(\mathcal{M})$. We write $L, L', \ldots$ for functional terms, often leaving their types implicit.

Above $\mathcal{M} \cong \mathcal{M}'/u$ ensures that L does not affect $\mathcal{M}$ during evaluation of L in $\mathcal{M}$. Note that values are always functional terms. In a context of reasoning for object-oriented languages, a similar formulation (called strong purity) is used in [44] for justifying the semantics of method invocations whose evaluation has no effect on the state of existing objects.

Now we define:

$$\mathcal{M} \models \forall x.C \quad \stackrel{\text{def}}{\equiv} \quad \forall L \in \mathcal{F}.(\mathcal{M}[x{:}L] \Downarrow \mathcal{M}' \supset \mathcal{M}' \models C) \tag{3.25}$$

Dually, we have:

$$\mathcal{M} \models \exists x.C \stackrel{\text{def}}{\equiv} \exists L \in \mathcal{F}.(\mathcal{M}[x{:}L] \Downarrow \mathcal{M}' \wedge \mathcal{M}' \models C) \tag{3.26}$$

If we restrict L above to a value, then the definition coincides with the original one in [6]. We need to extend values to functional terms so that a term can read information from hidden locations (cf. the semantics of equality $e_1 = e_2$). As a simple example, consider:

$$\mathcal{M} \stackrel{\text{def}}{=} (\nu l_1, l_2)(y : l_1,\ l_1 \mapsto l_2, l_2 \mapsto 2)$$

Under this model, we wish to say $\mathcal{M} \models \exists x.x = !y$. But if we only allow x to range over values, this standard tautology does not hold for $\mathcal{M}$. Using the functional term $!y \in \mathcal{F}$, we can expand the entry x with $!y$, and we have:

$$\mathcal{M}[x:!y] \Downarrow (\nu l_1 l_2)(x : l_1 \cdot y : l_1,\ l_1 \mapsto l_2, l_2 \mapsto 2) \stackrel{\text{def}}{=} \mathcal{M}' \wedge \mathcal{M}' \models x = y$$

Thus using a functional term L instead of a value V for a quantified variable is necessary for reasons similar to those that required modifying the semantics of equality. Universal and existential quantifiers satisfy the standard axioms familiar from first-order logic, some of which are studied later.

3.6. **Semantics of Hiding.** The universal hiding-quantifier has the following semantics.

$$\mathcal{M} \models \overline{\nu}x.C \quad \stackrel{\text{def}}{\equiv} \quad \forall \mathcal{M}'.((\nu l)\mathcal{M}' \approx \mathcal{M} \supset \mathcal{M}'[x:l] \models C) \tag{3.27}$$

where l is fresh, i.e. $l \notin \mathsf{fl}(\mathcal{M})$ where $\mathsf{fl}(\mathcal{M})$ denotes free labels in $\mathcal{M}$. The notation $(\nu l)\mathcal{M}'$ denotes addition of the hiding of l to $\mathcal{M}'$, as well as indicating that l occurs free in $\mathcal{M}'$. $\mathcal{M}[x:l]$ adds $x:l$ to the environment part of $\mathcal{M}$.

Dually, with l fresh again:

$$\mathcal{M} \models \nu x.C \quad \stackrel{\text{def}}{\equiv} \quad \exists \mathcal{M}'.((\nu l)\mathcal{M}' \approx \mathcal{M} \wedge \mathcal{M}'[x:l] \models C) \tag{3.28}$$

which says that x denotes a hidden reference, say l, and the result of taking it off from $\mathcal{M}$ satisfies C.

As an example of satisfaction, let:

$$\mathcal{M} \stackrel{\text{def}}{=} (\nu l)(\{u : \lambda().(l := !l+1; !l)\},\ \{l \mapsto 0\}) \tag{3.29}$$

then we have:

$$\mathcal{M} \models \nu x.C \tag{3.30}$$

with

$$C \stackrel{\text{def}}{=} !x = 0 \ \wedge\ \Box \forall i.\{!x = i\} u \bullet () = z\{z = !x \wedge\ !x = i+1\} \tag{3.31}$$

using the definition in (3.28) above. To see this holds, let

$$\mathcal{M}' \stackrel{\text{def}}{=} (\{u : \lambda().(l := !l+1; !l)\},\ \{l \mapsto 0\}) \tag{3.32}$$

We have $(\nu l)\mathcal{M}' \stackrel{\text{def}}{\equiv} \mathcal{M}$ and $\mathcal{M}'[x:l] \models C$. Here $\mathcal{M}$ represents a situation where l is hidden and u denotes a function which increments and returns the content of l; whereas $\mathcal{M}'$ is the result of taking off this hiding, exposing the originally local state, cf. [11].

Despite x's type being a reference, $\forall x.C$ differs substantially from $\overline{\nu}x.C$. The former says that for any reference x, which can be either (1) an existing free reference; (2) an existing hidden reference reachable through dereferences; or (3) a fresh reference with arbitrary content, the model satisfies C. On the other hand, the latter means that for any reference x which is hidden in the present model, C should hold: in this case x cannot be a free reference name hence (1) is not included. Similarly for their dual existential versions.

3.7. **Semantics of Content Quantification.** Next we define the semantics of the content quantification. Let us write $\mathcal{M}[x \mapsto V]$ for $(\nu \tilde{l})(\xi, \sigma[l \mapsto V])$ with $\mathcal{M} = (\nu \tilde{l})(\xi, \sigma)$ and $\xi(x) = l$. In [6] (without local state), $\mathcal{M} \models [!x]C$ is defined as $\forall V.\mathcal{M}[x \mapsto V] \models C$ which means that for all content of x, C holds. In the presence of the local state, we simply extend the use of values to the use of functional terms in the sense of Definition 3.7 as follows:

$$\mathcal{M} \models [!e]C \quad \stackrel{\text{def}}{\equiv} \quad \forall L \in \mathcal{F}.\mathcal{M}[e \mapsto L] \models C \tag{3.33}$$

where we write $\mathcal{M}[e \mapsto L]$ for $(\nu \tilde{l})(\xi, \sigma[l' \mapsto V])$, assuming $\mathcal{M} = (\nu \tilde{l})(\xi, \sigma)$, $[\![e]\!]_{\xi,\sigma} = l'$, $(\nu \tilde{l})(L\xi, \sigma) \Downarrow \mathcal{M}'$ and $\mathcal{M}' \approx (\nu \tilde{l})(V, \sigma)$. Thus we consider an update through the assignment of an external functional term L to a location in $\mathcal{M}$ under local names. With this definition, all the axioms and invariant rules in [6] stay unchanged.

3.8. **Semantics of Reachability.** We now define the semantics of reachability. Let σ be a store and $S \subset \mathsf{dom}(\sigma)$. Then the *label closure of S in σ*, written $\mathsf{lc}(S, \sigma)$, is the minimum set S' of locations such that: (1) $S \subset S'$ and (2) If $l \in S'$ then $\mathsf{fl}(\sigma(l)) \subset S'$. The label closure satisfies the following natural properties.

Lemma 3.8. *For all σ, we have:*

(1) $S \subset \mathsf{lc}(S, \sigma)$*;* $S_1 \subset S_2$ *implies* $\mathsf{lc}(S_1, \sigma) \subset \mathsf{lc}(S_2, \sigma)$*; and* $\mathsf{lc}(S, \sigma) = \mathsf{lc}(\mathsf{lc}(S, \sigma), \sigma)$
(2) $\mathsf{lc}(S_1, \sigma) \cup \mathsf{lc}(S_2, \sigma) = \mathsf{lc}(S_1 \cup S_2, \sigma)$
(3) $S_1 \subset \mathsf{lc}(S_2, \sigma)$ *and* $S_2 \subset \mathsf{lc}(S_3, \sigma)$*, then* $S_1 \subset \mathsf{lc}(S_3, \sigma)$
(4) *there exists* $\sigma' \subset \sigma$ *such that* $\mathsf{lc}(S, \sigma) = \mathsf{fl}(\sigma') = \mathsf{dom}(\sigma')$.

Proof. (1,2) are direct from the definition. (3) follows immediately from (1,2). For (4), take $\sigma' = \cup_{l \in \mathsf{lc}(S,\sigma)}[l \mapsto \sigma(l)]$. Then obviously $\sigma' \subset \sigma$ and $\mathsf{lc}(S, \sigma) = \mathsf{fl}(\sigma') = \mathsf{dom}(\sigma')$. ☐

For reachability, we define:

$$\mathcal{M} \models e_1 \hookrightarrow e_2 \quad \text{if} \quad [\![e_2]\!]_{\xi,\sigma} \in \mathsf{lc}(\mathsf{fl}([\![e_1]\!]_{\xi,\sigma}), \sigma) \;\; \text{for each } (\nu \tilde{l})(\xi, \sigma) \approx \mathcal{M}$$

The clause says the set of all reachable locations from e_1 includes e_2 modulo $\approx$.

For the programs in § 2.3 (5), we can check $f_w \hookrightarrow x$, $f_r \hookrightarrow x$ and $f_c \hookrightarrow z$ hold under $f_w : \lambda().(x := 1)$, $f_r : \lambda().!x$, $f_c : \mathtt{let}\ x = \mathtt{ref}(z)\ \mathtt{in}\ \lambda().x$ (regardless of the store part).

The following characterisation of # is often useful for justifying fresh name axioms. Below $\sigma = \sigma_1 \uplus \sigma_2$ indicates that σ is the union of σ_1 and σ_2, assuming $\mathsf{dom}(\sigma_1) \cap \mathsf{dom}(\sigma_2) = \emptyset$.

Proposition 3.9 (partition). *$\mathcal{M} \models x \# u$ if and only if for some $\tilde{l}, V, l$ and $\sigma_{1,2}$, we have $\mathcal{M} \approx (\nu \tilde{l})(\xi \cdot u : V \cdot x : l,\ \sigma_1 \uplus \sigma_2)$ such that $\mathsf{lc}(\mathsf{fl}(V), \sigma_1 \uplus \sigma_2) = \mathsf{fl}(\sigma_1) = \mathsf{dom}(\sigma_1)$ and $l \in \mathsf{dom}(\sigma_2)$.*

Proof. For the only-if direction, assume $\mathcal{M} \models x \# u$. By the definition of (un)reachability, we can set (up to $\approx$) $\mathcal{M} \stackrel{\text{def}}{=} (\nu \tilde{l}')(\xi \cdot u : V \cdot x : l,\ \sigma)$ such that $l \notin \mathsf{lc}(\mathsf{fl}(V), \sigma)$. Now take σ_1 such that $\mathsf{lc}(\mathsf{fl}(V), \sigma) = \mathsf{lc}(\mathsf{fl}(V), \sigma_1) = \mathsf{fl}(\sigma_1) = \mathsf{dom}(\sigma_1)$ by Lemma 3.8. Note by definition $l \notin \mathsf{dom}(\sigma_1)$. Now let $\sigma = \sigma_1 \uplus \sigma_2$. Since $l \in \mathsf{dom}(\sigma)$, we know $l \in \mathsf{dom}(\sigma_2)$, hence done. The if-direction is obvious by definition of reachability. ☐

The characterisation says that if x is unreachable from u then, up to $\approx$, the store can be partitioned into one covering all reachable names from u and another containing x.

Now we give the full definition of the satisfaction relation. For readability, we first list the auxiliary definitions many of which have already been stated before.

Notation 3.10.

(a) $\mathcal{M}[u : e]$ denotes $(\nu \tilde{l})(\xi \cdot u : [\![e]\!]_{\xi,\sigma}, \sigma)$ where we always assume u is fresh and the variables and labels in e are free in $\mathcal{M}$.
(b) $\mathcal{M}/u$ denotes $(\nu \tilde{l})(\xi, \sigma)$ if $\mathcal{M} = (\nu \tilde{l})(\xi \cdot u : V, \sigma)$; and if $u \notin \mathsf{fv}(\mathcal{M})$ we set $\mathcal{M}/u = \mathcal{M}$.
(c) $\mathcal{M}[u : N] \Downarrow \mathcal{M}'$ when $(N\xi, \sigma) \Downarrow (\nu \tilde{l}')(V, \sigma')$ and $\mathcal{M}' = (\nu \tilde{l}\tilde{l}')(\xi \cdot u : V,\ \sigma')$ with $\mathcal{M} = (\nu \tilde{l})(\xi,\ \sigma)$ where we always assume u is fresh and the variables and labels in N are free in $\mathcal{M}$.
(d) $\mathcal{M} \rightsquigarrow \mathcal{M}'$ is generated by: (1) $\mathcal{M} \rightsquigarrow \mathcal{M}$; and (2) if $\mathcal{M} \rightsquigarrow \mathcal{M}_0$ and $\mathcal{M}_0[u : N] \Downarrow \mathcal{M}'$, then $\mathcal{M} \rightsquigarrow \mathcal{M}'$.

(e) We write $\mathcal{M}[e \mapsto V]$ for $(\nu\tilde{l})(\xi, \sigma[l \mapsto V])$ with $\mathcal{M} = (\nu\tilde{l})(\xi,\sigma)$ and $[\![e]\!]_{\xi,\sigma} = l$.

(f) We write $\mathcal{M}\cdot\mathrm{X}:\alpha$ for $\mathcal{M} = (\nu\tilde{l})(\xi\cdot\mathrm{X}:\alpha,\sigma)$ with $\mathcal{M} = (\nu\tilde{l})(\xi,\sigma)$ where X is not in $\mathcal{M}$ and α is closed.

Definition 3.11 (Satisfaction)**.** The semantics of the assertions follows. All omitted cases are by de Morgan duality.

(1) $\mathcal{M} \models e_1 = e_2$ if $\mathcal{M}[u : e_1] \approx \mathcal{M}[u : e_2]$.
(2) $\mathcal{M} \models C_1 \wedge C_2$ if $\mathcal{M} \models C_1$ and $\mathcal{M} \models C_2$.
(3) $\mathcal{M} \models \neg C$ if not $\mathcal{M} \models C$.
(4) $\mathcal{M} \models \Box C$ if $\forall \mathcal{M}'.(\mathcal{M} \rightsquigarrow \mathcal{M}' \supset \mathcal{M}' \models C)$.
(5) $\mathcal{M} \models \forall x.C$ if $\forall L \in \mathcal{F}.(\mathcal{M}[x{:}L] \Downarrow \mathcal{M}' \supset \mathcal{M}' \models C)$
(6) $\mathcal{M} \models \overline{\nu}x.C$ if $\forall \mathcal{M}'.((\nu l)\mathcal{M}' \approx \mathcal{M} \supset\ \mathcal{M}'[x : l] \models C)$
(7) $\mathcal{M} \models \forall \mathrm{X}.C$ if for all closed types α, $\mathcal{M}\cdot\mathrm{X}:\alpha \models C$.
(8) $\mathcal{M} \models [!e]C$ if for each $\forall L \in \mathcal{F}.\mathcal{M}[e \mapsto L] \models C$.
(9) $\mathcal{M} \models e_1 \hookrightarrow e_2$ if for each $(\nu\tilde{l})(\xi,\sigma) \approx \mathcal{M}$, $[\![e_2]\!]_{\xi,\sigma} \in \mathsf{lc}(\mathsf{fl}([\![e_1]\!]_{\xi,\sigma}),\sigma)$.
(10) $\mathcal{M} \models e \bullet e' = z\{C\}$ if $\exists \mathcal{M}'.(\mathcal{M}[x : ee'] \Downarrow \mathcal{M}' \ \wedge\ \mathcal{M}' \models C)$.
(11) $\mathcal{M} \models e \bullet e' = z\{C\}@\tilde{w}$ if

$$\begin{array}{ll}\exists \mathcal{M}'.(& \mathcal{M}[z : ee'] \Downarrow \mathcal{M}' \ \wedge\ \mathcal{M}' \models C' \ \wedge \\ & \forall \mathcal{M}''.(\mathcal{M}[z : \mathtt{let}\ \tilde{x} = !\tilde{w}\ \mathtt{in}\ \mathtt{let}\ y = ee'\ \mathtt{in}\ \tilde{w} := \tilde{x}] \Downarrow \mathcal{M}'' \ \supset\ \mathcal{M}'' \approx \mathcal{M}[z : ()]))\end{array}$$

In the defining clauses above, we assume $\mathsf{fv}(e, e_{1,2}, e') \subset \mathsf{fv}(\mathcal{M})$, $\mathsf{fl}(e, e_{1,2}, e') \subset \mathsf{fl}(\mathcal{M})$, $\mathsf{fv}(L) \subset \mathsf{fv}(\mathcal{M})$ and $\mathsf{fl}(L) \subset \mathsf{fl}(\mathcal{M})$, as well as well-typedness of models and formulae.

In Definition 3.11, (2) and (3) are standard. (7) is from [24]. Others have already been explained. In (11), the program $\mathtt{let}\ \tilde{x} = !\tilde{w}\ \mathtt{in}\ \mathtt{let}\ y = ee'\ \mathtt{in}\ \tilde{w} := \tilde{x}$ first keeps the content of $\tilde{w}$ in $\tilde{x}$ and executes the application ee'; then finally restores the original content in $\tilde{w}$. By $\mathcal{M}'' \approx \mathcal{M}[z : ()]$ the resulting model $\mathcal{M}''$ has no state change w.r.t. the original model $\mathcal{M}$, this means ee' only updates at $\tilde{w}$ up to $\approx$.

This concludes the introduction of the satisfaction relation for the present logic. The properties of models are explored further in the rest of this section and in § 5.

3.9. **Thin and Stateless Formulae.** In this subsection, we introduce two kinds of formulae which play a key role in the reasoning principles of the present logic, in particular the proof rules discussed in the next section.

The first definition introduces formulae in which the thinning of unused variables from models can be done as in first-order logic.

Definition 3.12 (Thin Formula)**.** Let $\Gamma \vdash C$ and $y \in \mathsf{dom}(\Gamma)$ such that $y \notin \mathsf{fv}(C)$. Then we say that C is *thin with respect to* y if for each $\mathcal{M}$ typable under Γ, $\mathcal{M} \models C$ implies $\mathcal{M}/y \models C$. We say C is *thin* if under each typing and for each $y \notin \mathsf{fv}(C)$, C is thin w.r.t. y.

In a thin formula C, reference names which do not appear in C do not affect the meaning of C. There are formulae which are not thin (we see some examples below) but they are of a very special kind. In our experience they never appear in practical reasoning including our reasoning examples in § 6.

As examples of formulae which are not thin, when an evaluation formula occurs negatively, formulae may cease to be thin. Consider the following satisfaction:

$$(\nu l l')(u : \lambda().!l',\ x : l, l \mapsto l', l' \mapsto 1)\ \models\ \Diamond\, u \bullet () = z\{z = 2\}$$

which means that u is a function which might return 2 *someday* since a value stored in l' can be changed via x (for example, by the command $!x := 2$). When we delete x from the above model, the behaviour of u will change as follows.

$$(\nu l')(u : \lambda().!l', l' : 1) \models \Box u \bullet () = z\{z = 1\}$$

since now u *always* returns 1 when it is invoked. The above judgement entails:

$$(\nu l')(u : \lambda().!l', l' : 1) \not\models \Diamond u \bullet () = z\{z = 2\}$$

Hence $\Diamond u \bullet () = z\{z = 2\}$ is not thin. Similarly $\Diamond\Box u \bullet () = z\{z = 0\}$ is not a thin formula.

As noted, formulae which are not thin hardly appear in reasoning; all formulae appearing in § 6 are thin; the proof rules always generate thin formulae from thin formulae. We shall however work with general formulae since many results hold for none-thin formulae too.

The following syntactic characterisation of thin formulae is useful.

Proposition 3.13 (Syntactically Thin Formula)**.** (1) *If* $\Gamma \vdash C$, $\Gamma \vdash y : \alpha$ *and* $\alpha \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$, *then* C *is thin with respect to* y.
(2) $e = e', e \neq e', e \hookrightarrow e'$ *and* $e \# e'$ *are thin.*
(3) *If* C, C' *are thin w.r.t.* y, *then* $C \wedge C'$, $C \vee C'$, $\forall x^{\alpha}.C$ *for all* α, $\exists x^{\alpha}.C$ *with* $\alpha \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$, $\exists \mathrm{X}.C$, $\forall \mathrm{X}.C$, $\nu x.C$, $\overline{\nu} x.C$, $\Box C$, $[!x]C$ *and* $e \bullet e' = x\{C'\}$ *are thin w.r.t.* y.

Proof. (1,2) are immediate. For (3), suppose C and C' are thin w.r.t. y, $y \notin \mathsf{fv}(C, C')$ and $\mathcal{M} \models C \wedge C'$. Then $\mathcal{M} \models C$ hence $\mathcal{M}/y \models C$, similarly for C', hence $\mathcal{M}/y \models C \wedge C'$. Similarly for other cases. Next let C be thin w.r.t. y and $\mathcal{M} \models \nu x.C$. Then there exists $\mathcal{M}'$ such that $(\nu l)\mathcal{M}' \approx \mathcal{M}$ and $\mathcal{M}'[x : l] \models C$. Then $(\nu l)\mathcal{M}'/y \approx \mathcal{M}/y$. By assumption, $\mathcal{M}'[x : l]/y \models C$, and hence $\mathcal{M}/y \models \nu x.C$, as desired. Next let C be thin w.r.t. y. Suppose $\mathcal{M} \models e \bullet e' = z\{C\}$, i.e. $\mathcal{M}[z : ee'] \Downarrow \mathcal{M}'$ and $\mathcal{M}' \models C$. Then we have $\mathcal{M}/y[z : ee'] \Downarrow \mathcal{M}'/y$. Since C is thin w.r.t. y, we have $\mathcal{M}'/y \models C$, as required. $\Box$

The next set of formulae are *stateless formulae* whose validity does not depend on the state part of the model, cf. stateless formulae in [6, 25].

Definition 3.14 (Stateless Formula)**.** C is *stateless* iff $C \supset \Box C$ is valid. We let $A, B, A', B', \ldots$ range over stateless formulae.

Proposition 3.15 (Stateless Formulae)**.** (1) *For all* C, $\Box C$ *is stateless.*
(2) *If* C *is stateless then* $C \equiv \Box C \equiv \Box\Box C$.

Proof. Both are immediate from the definition, see also § 5.2 for further related results. $\Box$

The above proposition says that if C is stateless then C holds in any future state starting from the present state. The following generalisation of this notion says that the validity of a formula does not depend on the stateful part of models *except at specific locations*. This notion is used by the axioms for local invariants later.

Definition 3.16 (Stateless Formula Except $\tilde{x}$)**.** We say that C is *stateless except* $\tilde{x}$ if, whenever $\mathcal{M} \models C$ and $\mathcal{M} \rightsquigarrow \mathcal{M}'$ such that $\mathcal{M}$ and $\mathcal{M}'$ coincide in their content at $\tilde{x}$ of reference types, i.e.
(1) $\mathcal{M} \approx (\nu \tilde{l}_0)(\xi, \sigma)$;
(2) $\mathcal{M}' \approx (\nu \tilde{l}_0 \tilde{l}_1)(\xi \cdot \xi', \sigma')$; and
(3) $\sigma(\xi(x_i)) = \sigma'(\xi(x_i))$ for each $x_i \in \{\tilde{x}\}$,
then $\mathcal{M}' \models C$.

Definition 3.16 uses the internal representation of models. Alternatively we may define an $\tilde{x}$-*preserving term* which has the shape:

$$\mathtt{let}\ y_1 = !x_1\ \mathtt{in}\ ...\mathtt{let}\ y_n = !x_n\ \mathtt{in}\ \mathtt{let}\ z = N'\ \mathtt{in}\ (x_1 := y_1; ...; x_n := y_n; z) \tag{3.34}$$

then say C is stateless except $\tilde{x}$ if whenever $\mathcal{M} \models C$ and $\mathcal{M}[u : N] \Downarrow \mathcal{M}'$ where N is a $\tilde{x}$-preserving term we have $\mathcal{M}' \models C$.

Note if $\tilde{x}$ is empty in Definition 3.16 then the third clause is vacuous: hence in this case the definition means that for each $\mathcal{M}$ such that $\mathcal{M} \models C$ we have $\mathcal{M} \rightsquigarrow \mathcal{M}'$ implies $\mathcal{M}' \models C$, that is C is stateless.

It is convenient to be able to check the statelessness of formulae (relative to references) syntactically. For an inductive characterisation, we introduce the following notion. As always we assume the standard bound name convention.

Definition 3.17 (Tame Formulae)**.** The set of *tame formulae* is generated by the following rules:

- $e_1 = e_2$ and $e_1 \neq e_2$ are tame.
- $e_1 \hookrightarrow e_2$ and $e_1 \# e_2$ are tame.
- For any C, $\Box C$ is tame.
- if C is tame then $\forall y^{\alpha}.C$, $\exists y^{\alpha}.C$, $\exists \mathrm{X}.C$, $\forall \mathrm{X}.C$, $[!y]C$ and $\langle !y \rangle C$ are all tame.
- if C, C' are tame then $C \wedge C'$ and $C \vee C'$ are tame.

We say that $!x$ is an *active dereference* in C if C is tame and $!x$ (with x being free or bound) occurs neither in the scope of $\Box$, $[!x]$ nor $\langle !x \rangle$.

The following result (though not used in the present work) is notable for carrying over reasoning techniques from the logic for aliasing [6].

Proposition 3.18 (Decomposition)**.** *Suppose C is tame. Then there is tame C' such that $C \equiv C'$ and C' does not contain content quantifications except under the scope of $\Box$.*

Proof. The proof follows precisely that of [6, §6.1, Theorem 1]. □

We can now introduce syntactic stateless formulae.

Definition 3.19 (Syntactic Stateless Formulae)**.** We say *C is syntactically stateless except $\tilde{x}$* if C is tame and only names from $\tilde{x}$ are among the active dereferences in C.

Proposition 3.20.

(1) *If C is syntactically stateless except $\tilde{x}$ then C is stateless except $\tilde{x}$.*
(2) *If $[!\tilde{x}]C$ is syntactically stateless then C is stateless except $\tilde{x}$.*

Proof. (1) is by induction of the generation of tame formulae. Base cases and $\Box C$ are immediate. Among the inductive cases the only non-trivial case is quantifications of references. Suppose C is tame and contains active dereferences at $\tilde{x}y$.

- If the validity of C relies on y (i.e. for some $\mathcal{M}_{1,2}$ which differ only at y we have $\mathcal{M}_1 \models C$ and $\mathcal{M}_2 \not\models C$) then $\forall y^{\alpha}.C$ is falsity: if not $\forall y^{\alpha}.C$ and C are equivalent. In either case we know C is stateless except $\tilde{x}$.
- If validity of C relies on y then $\exists y^{\alpha}.C$ is truth: if not $\exists y^{\alpha}.C$ and C are equivalent. The rest is the same.
- If validity of $[!y]C$ relies on the content of y then $[!y]C$ is falsity: the rest is the same. Similarly for $\langle !y \rangle C$.

The cases of $C \wedge C'$ and $C \vee C'$ are immediate by induction. (2) is an immediate corollary of (1). □

4. Proof Rules and Soundness

4.1. **Hoare Triples.** This subsection summarises judgements and proof rules for local state. The main judgement consists of a program and a pair of formulae following Hoare [17], augmented with a fresh name called *anchor* [22, 24, 25],

$$\{C\}\ M :_u \{C'\}$$

which says:

> *If we evaluate M in the initial state satisfying C, then it terminates with a value, name it u, and a final state, which together satisfy C'.*

Note that our judgements are about total correctness. Sequents have identical shape as those in [6, 25]: the computational situations described is however quite different, in that both C and C' may now describe behaviour and data structures with local state.

The same sequent is used for both validity and provability. If we wish to be specific, we prefix it with either $\vdash$ (for provability) or $\models$ (for validity). We assume that judgements are well-typed in the sense that, in $\{C\}\ M :_u \{C'\}$ with $\Gamma \vdash M : \alpha$, $\Gamma, \Delta \vdash C$ and $u : \alpha, \Gamma, \Delta \vdash C'$ for some Δ such that $\mathsf{dom}(\Delta) \cap (\mathsf{dom}(\Gamma) \cup \{u\}) = \emptyset$.

In $\{C\}\ M :_u \{C'\}$, the name u is the *anchor* of the judgement, which should *not* be in $\mathsf{dom}(\Gamma) \cup \mathsf{fv}(C)$; and C is the *pre-condition* and C' is the *post-condition*. The *primary names* are $\mathsf{dom}(\Gamma) \cup \{u\}$, while the *auxiliary names* (ranged over by $i, j, k, ...$) are those free names in C and C' which are not primary. An anchor is used for naming the value from M and for specifying its behaviour. We use the abbreviation $\{C\} M \{C'\}$ to denote $\{C\} M :_u \{u = () \wedge C'\}$.

4.2. **Proof Rules.** The full compositional proof rules and new structure rules are given in Figure 1. In each proof rule, we assume all occurring judgements to be well-typed and no primary names in the premise(s) to occur as auxiliary names in the conclusion. We write $C^{\text{-}\tilde{x}}$ to indicate $\mathsf{fv}(C) \cap \{\tilde{x}\} = \emptyset$. Despite our semantic enrichment, all compositional proof rules in the base logic [6] (and [*Rec-Ren*] from [23]) syntactically stay as they are, except for:

- adding a rule for the reference generation,
- revising [*Abs*] and [*App*] so they use one-sided evaluation formulae ,
- adding the thinness condition in the post-condition of the conclusion in [*Case*], [*App*], [*Assign*] and [*Deref*]

The thinness condition is required when the anchor names used in the premise contribute to C' in the conclusion. The reason for this becomes clearer when we prove soundness . This condition does not jeopardise the completeness of our logic. All reasoning examples we have explored meet this condition including those in § 6.

Note that in [*Add*], since C' is always thin with respect to m_i by Proposition 3.13 (1), we do not have to state this condition explicitly. Similarly for [*If*] since C' is always thin with respect to b.

[*Assign*] uses *logical substitution* which is built with content quantification to represent substitution of content of a possibly aliased reference [6].

$$C\{\!|e_2/!e_1|\!\} \stackrel{\text{def}}{=} \forall m.(m = e_2 \supset [!e_1](!e_1 = m \supset C)).$$

with m fresh (we have a dual characterisation by $\langle !e_1 \rangle$). Intuitively $C\{\!|e_2/!e_1|\!\}$ describes the situation where a model satisfying C is updated at a memory cell referred to by e_1 (of a reference type) with a value e_2 (of its content type), with $e_{1,2}$ interpreted in the current model.

$$[\mathit{Var}]\ \frac{-}{\{C[x/u]\}\ x :_u \{C\}} \qquad [\mathit{Const}]\ \frac{-}{\{C[\mathsf{c}/u]\}\ \mathsf{c} :_u \{C\}} \qquad [\mathit{Add}]\ \frac{\{C\}\, M_1 :_{m_1} \{C_0\}\ \{C_0\}\, M_2 :_{m_2} \{C'[m_1+m_2/u]\}}{\{C\}\, M_1+M_2 :_u \{C'\}}$$

$$[\mathit{In}_1]\ \frac{\{C\}\ M :_v \{C'[\mathsf{inj}_1(v)/u]\}}{\{C\}\ \mathtt{inj}_1(M) :_u \{C'\}} \qquad [\mathit{Case}]\ \frac{\{C^{\text{-}\tilde{x}}\}\ M :_m \{C_0^{\text{-}\tilde{x}}\} \quad \{C_0[\mathsf{inj}_i(x_i)/m]\}\ M_i :_u \{C'^{\,\text{-}\tilde{x}}\}}{\{C\}\ \mathtt{case}\ M\ \mathtt{of}\ \{\mathtt{in}_i(x_i).M_i\}_{i\in\{1,2\}} :_u \{C'\}}$$

$$[\mathit{Proj}_1]\ \frac{\{C\}\ M :_m \{C'[\pi_1(m)/u]\}}{\{C\}\ \pi_1(M) :_u \{C'\}} \qquad [\mathit{Pair}]\ \frac{\{C\}\ M_1 :_m \{C_0\} \quad \{C_0\}\ M_2 :_n \{C'[\langle m,n\rangle/u]\}}{\{C\}\ \langle M_1,M_2\rangle :_u \{C'\}}$$

$$[\mathit{Abs}]\ \frac{\{A^{\text{-}x\tilde{i}} \wedge C\}\ M :_m \{C'\}}{\{A\}\ \lambda x.M :_u \{\Box\forall x\tilde{i}.(\{C\}u\bullet x = m\{C'\})\}} \qquad [\mathit{App}]\ \frac{\{C\}\ M :_m \{C_0\}\ \{C_0\}\ N :_n \{m\bullet n = u\{C'\}\}}{\{C\}\ MN :_u \{C'\}}$$

$$[\mathit{If}]\ \frac{\{C\}\ M :_b \{C_0\} \quad \{C_0[\mathsf{t}/b]\}\ M_1 :_u \{C'\} \quad \{C_0[\mathsf{f}/b]\}\ M_2 :_u \{C'\}}{\{C\}\ \mathtt{if}\ M\ \mathtt{then}\ M_1\ \mathtt{else}\ M_2 :_u \{C'\}}$$

$$[\mathit{Deref}]\ \frac{\{C\}\ M :_m \{C'[!m/u]\}}{\{C\}\ !M :_u \{C'\}} \qquad [\mathit{Assign}]\ \frac{\{C\}\ M :_m \{C_0\} \quad \{C_0\}\ N :_n \{C'\{\!|n/\,!m|\!\}\}}{\{C\}\ M := N\ \{C'\}}$$

$$[\mathit{Rec\text{-}Ren}]\ \frac{\{A^{\text{-}f}\}\ \lambda x.M :_u \{B\}}{\{A\}\ \mu f.\lambda x.M :_u \{B[u/f]\}} \qquad [\mathit{Ref}]\ \frac{\{C\}\ M :_m \{C'\}}{\{C\}\ \mathtt{ref}(M) :_u \{\nu x.(C'[!u/m] \wedge u\#i^{\mathrm{X}} \wedge u = x)\}}$$

$$[\mathit{Aux}_\forall V]\ \frac{\{C^{\text{-}i}\}\ V :_u \{C'\}}{\{C\}\ V :_u \{\forall i.C'\}} \qquad [\mathit{Aux}_\forall]\ \frac{\{C^{\text{-}i}\}\ M :_u \{C'\} \quad \alpha \text{ is base type}}{\{C\}\ M :_u \{\forall i^{\alpha}.C'\}}$$

$$[\mathit{Conseq}]\ \frac{C \supset C_0\ \{C_0\}\, M :_u \{C_0'\}\ C_0' \supset C'}{\{C\}\ M :_u \{C'\}} \qquad [\mathit{Subs}]\ \frac{\{C\}\ M :_u \{C'\} \quad u \notin \mathsf{fpn}(e)}{\{C[e/i]\}\ M :_u \{C'[e/i]\}}$$

$$[\mathit{Cons\text{-}Eval}]\ \frac{\begin{array}{c}\{C_0\}\ M :_m \{C_0'\} \quad x \text{ fresh;}\ \ \tilde{i} \text{ auxiliary} \\ \Box\forall\tilde{\mathrm{X}}.\forall\tilde{i}.\{C_0\}x\bullet() = m\{C_0'\}\ \supset\ \Box\forall\tilde{\mathrm{X}}.\forall\tilde{i}.\{C\}x\bullet() = m\{C'\}\end{array}}{\{C\}\ M :_m \{C'\}}$$

We require C' is thin w.r.t. m in [*Case*] and [*Deref*], and C' is thin w.r.t. m,n in [*App*, *Assign*].

Figure 1: Proof Rules

In rule [*Ref*], $u\#i$ indicates that the newly generated cell u is unreachable from any i of arbitrary type X in the initial state: then the result of evaluating M is stored in that cell.[6] Here i is a(ny) fresh variable denoting an arbitrary datum which already exited in the pre-state. Just as the standard auxiliary variable in Hoare-like logics, this i is semantically bound at the sequent level. In a large proof, we may want each instance of [*Ref*] to use a fresh and distinct variable, even though in practice we usually apply the substitution rule discussed below to instantiate this "bound" variable into an appropriate expression so name clash may not occur.[7]

For the structural rules (i.e. those which only manipulate assertions), those given in [6, §7.3] for the base logic stay valid except that the universal abstraction rule [$\mathit{Aux}_\forall$] in [6, §7.3] needs to be weakened as [$\mathit{Aux}_\forall$] and [$\mathit{Aux}_\forall V$] in Figure 1. Note that the original structural rule [$\mathit{Aux}_\forall$], which does not have this condition, is not valid in the presence of new reference generation. For example we can take:

$$\{\mathsf{T}\}\ \mathtt{ref}(3) :_u \{u\#i \wedge !u = 3\} \tag{4.1}$$

[6]One may write the conclusion of this rule as $\{C\}\ \mathtt{ref}(M) :_u \{(C'[!u/m] \wedge u\#i^{\mathrm{X}})\}$ which may be useful for readability. In this paper however we intentionally do not introduce this or other abbreviations for the sake of clarity.

[7]The treatment of a fresh variable as an input binder in [*Ref*] is useful for mechanisation of reasoning, just like auxiliary variables in Hoare triples.

which is surely valid. But without the side condition, we can infer the following from (4.1).

$$\{\top\}\ \mathtt{ref}(3) :_u \{\forall i.(u\#i \wedge !u = 3)\}$$

which does not make sense (just substitute u for i). This is because i cannot range over newly generated names: such an interplay with new name generation is not possible if the target program is a value, or if i is of base type.

We also have two useful structural rules added in the present logic. The first rule is [*Subs*] in Figure 1, which can be used to instantiate the fresh variable i in [*Ref*] with an arbitrary datum. The rule uses the following set of reference names.

Definition 4.1 (Plain Name)**.** We write $\mathsf{fpn}(e)$ for the set of *free plain names* of e, defined as: $\mathsf{fpn}(x) = \{x\}$, $\mathsf{fpn}(\mathsf{c}) = \mathsf{fpn}(!e) = \emptyset$, $\mathsf{fpn}(\langle e, e'\rangle) = \mathsf{fpn}(e) \cup \mathsf{fpn}(e')$, and $\mathsf{fpn}(\mathsf{inj}_i(e)) = \mathsf{fpn}(e)$.

In brief, the set of free plain names of e contains reference names in e that do not occur dereferenced, as first described in Definition 4.1. As we shall see later, the side condition for [*Subs*] using $\mathsf{fpn}(e)$ is necessary for soundness.

As an example usage of [*Subs*], consider:

$$\{!z = 2\}\mathtt{ref}(2) :_m \{!m = 2 \wedge i\#m\} \tag{4.2}$$

where we take off ν by an axiom later. We can then use [*Subs*] to show:

$$\{!z = 2\}\mathtt{ref}(2) :_m \{!m = 2 \wedge z\#m\} \tag{4.3}$$

Note $m \in \mathsf{fpn}(m)$: hence we *cannot* use m instead of z in (4.3), which is obviously unsound. As another use of [*Subs*], consider a judgement:

$$\{\top\}\langle\mathtt{ref}(2), \mathtt{ref}(2)\rangle :_m \{!\pi_1(m) = 2 \wedge !\pi_2(m) = 2 \wedge \pi_1(m) \neq \pi_2(m)\} \tag{4.4}$$

In order to derive (4.4), we simply combine (4.2) with the following judgement:

$$\{!m = 2 \wedge i\#m\}\mathtt{ref}(2) :_n \{!m = 2 \wedge !n = 2 \wedge j\#n\} \tag{4.5}$$

where we use a different fresh variable j. We can now replace j with m using [*Subs*], and via [*Cons*] we obtain:

$$\{!m = 2 \wedge i\#m\}\mathtt{ref}(2) :_n \{!m = 2 \wedge !n = 2 \wedge m \neq n\} \tag{4.6}$$

from which we can infer (4.4) by pairing, combined with (4.2).

Another significant additional rule is [*Cons-Eval*], also given in Figure 1. This is a strengthened version of the standard consequence rule, and is used when incorporating the local invariant axiom of the evaluation formula for derivations of the examples in § 6. Technically, this is a consequence of (a) having a proof system by which we can compositionally build proofs; and (b) representing fresh generation of references by disjointness from fresh variables. We shall see in examples that it is useful in reasoning.

The full list of structural rules can be found in Appendix B.

4.3. **Located Judgements.** Proof rules which contain an explicit effect set (similar to located evaluation formulae) were introduced in [6] and are of substantial help in reasoning about programs. Located Hoare triples take the form:

$$\{C\}M :_u \{C'\}@\tilde{e}$$

where each e_i is of a reference type and does not contain (sub)expressions of the form $!g$.[8] $\tilde{e}$ is called *effect set*. We prefix it with either $\vdash$ (for provability) or $\models$ (for validity) if we wish to be specific.

The full rules are listed in Figure 4 (proof rules) and Figure 5 (structure rules) in Appendix B. All rules come from [6] except for the new name generation rule and the universal quantification rule, both corresponding to the new rules in the basic proof system. The structures rules are also revised along the lines of Figure 1.

4.4. **Invariance Rules for Reachability.** Invariance rules are useful for modular reasoning. A simple form is when there is no state change:

$$[\mathit{Inv\text{-}Val}]\ \frac{\{C\}\ V :_m \{C'\}}{\{C \wedge C_0\}\ V :_m \{C' \wedge C_0\}}$$

Alternatively if a formula is stateless it continues to hold irrespective of state change.

$$[\mathit{Inv\text{-}Stateless}]\ \frac{\{C\}\ M :_m \{C'\}}{\{C \wedge \Box C_0\}\ M :_m \{C' \wedge \Box C_0\}}$$

When it is formulated with (un)reachability predicates, however, one needs some care. Since reachability is a stateful property, it is generally *not* invariant under state change. For example, suppose x is unreachable from y; after running $y := x$, x becomes reachable from y. Hence the following rule is unsound.

$$[\mathit{Unsound\text{-}Inv}]\ \frac{\{C\}\ M :_m \{C'\}}{\{C \wedge e\#e'\}\ M :_m \{C' \wedge e\#e'\}} \qquad \text{(unsound)}$$

From the following general invariance rule [*Inv*], we can derive an invariance rule for $\#$.

$$[\mathit{Inv}]\ \frac{\{C\}\ M :_m \{C'\}@\tilde{w} \qquad C_0 \text{ is tame}}{\{C \wedge [!\tilde{w}]C_0\}\ M :_m \{C' \wedge [!\tilde{w}]C_0\}@\tilde{w}}$$

In [*Inv*], the effect set $\tilde{w}$ gives the minimum information by which the assertion we wish to add, C_0, can be stated as an invariant since $[!\tilde{w}]C_0$ says that C_0 holds regardless of the content of $\tilde{w}$. Thus C_0 can stay invariant after execution of M. Unlike the existing invariance rules as found in standard Hoare logic or in Separation Logic [56], we need no side condition "M does not modify stores mentioned in C_0": C and C_0 may even overlap in their mentioned references, and C does not have to mention all references M may read or write.

The following instance of [*Inv*] is useful.

$$[\mathit{Inv\text{-}\#}]\ \frac{\{C\}\ M :_m \{C'\}@x \quad \text{no dereference occurs in } \tilde{e}}{\{C \wedge x\#\tilde{e}\}\ M :_m \{C' \wedge x\#\tilde{e}\}@x}$$

In $[\mathit{Inv\text{-}\#}]$, we note $[!x]x\#\tilde{e} \equiv x\#\tilde{e}$ is always valid if $\tilde{e}$ contains no dereference $!e$, cf. Proposition 5.9 3-(5) later. Hence $x\#\tilde{e}$ is stateless except at x. The side condition is indispensable: consider $\{\mathsf{T}\}x := x\{\mathsf{T}\}@x$ (which is typable with recursive types), which does not imply $\{x\#!x\}x := x\{x\#!x\}@x$.

One of the important aspects of these invariance rules is that the effect set of a located judgement or assertion can contain a hidden name – a name which has been created and which is (partially) accessible. For example, we can infer (using $[\mathit{LetRef}]$ in § 6.1):[9]

$$\{!y = h\}\ \mathtt{let}\ x = \mathtt{ref}(2)\ \mathtt{in}\ !y = x :_u \{\nu x.(!h = x \wedge !x = 2 \wedge !y = h \wedge x\#i)\}@h$$

[8] This restriction is for a simplification of the interpretation, and can be taken off if $\tilde{e}$ is interpreted in the pre-condition.

[9] Since $!y$ is stated in the pre-condition, we can also write $\{\mathsf{T}\}\ \mathtt{let}\ x = \mathtt{ref}(2)\ \mathtt{in}\ !y = x :_u \{\nu x.(!x = 2 \wedge !!y = x \wedge x\#i)\}@!y$, cf. footnote 8.

4.5. **Soundness.** Let $\mathcal{M}$ be a model $(\nu\tilde{l})(\xi,\sigma)$ of type Γ, and $\Gamma \vdash M : \alpha$ with u fresh. Then *validity* $\models \{C\}M :_u \{C'\}$ is given by (with $\mathcal{M}$ including all variables in M, C and C' except u):

$$\models \{C\}M :_u \{C'\} \stackrel{\text{def}}{\equiv} \forall \mathcal{M}.(\mathcal{M} \models C \supset (\mathcal{M}[u\!:\!M] \Downarrow \mathcal{M}' \wedge \mathcal{M}' \models C'))$$

where the notation $\mathcal{M}[u\!:\!N] \Downarrow \mathcal{M}'$ appeared in Definition 3.11(c). This is equivalent to, with $V \stackrel{\text{def}}{=} \lambda().M$:

$$\forall \mathcal{M}.(\mathcal{M}[m:V] \models \Box\{C\}m \bullet () = u\{C'\}) \tag{4.7}$$

Similarly the semantics of the located judgement:

$$\models \{C\}\ M :_u\ \{C'\}@\tilde{x} \tag{4.8}$$

is given through the corresponding located assertion, using the following term (let z be fresh):

$$V \stackrel{\text{def}}{=} \texttt{let}\ z = \texttt{ref}(0)\ \texttt{in}\ \lambda().\texttt{if}\ !z = 0\ \texttt{then}\ \texttt{let}\ m = M\ \texttt{in}\ (z := !z+1; m)\ \texttt{else}\ \Omega \tag{4.9}$$

where Ω is a diverging closed term (in fact any closed program works). The use of z is to prevent leakage of information from m after the evaluation: after evaluation m can never reveal any information thus it is the same thing as evaluating M once.

With this V we set the definition of (4.8) as follows:

$$\forall \mathcal{M}.(\mathcal{M}[m:V] \models \Box\{C\}m \bullet () = u\{C'\}@\tilde{x}) \tag{4.10}$$

Among the proof rules the only non-trivial addition from the preceding systems (in fact the only difference) is the rule for reference generation. For its soundness we use the free plain names as defined in Definition 4.1 (recall $\mathsf{fpn}(e)$ is the set of reference names in e that do not occur dereferenced). For free plain names we note:

Lemma 4.2. *Let $u \notin \mathsf{fpn}(e)$. Then for all M, with u fresh, we have: $\mathcal{M}[u\!:\!\texttt{ref}(M)] \Downarrow \mathcal{M}'$ implies $\mathcal{M}' \models u\#e$.*

Proof. Suppose $\mathcal{M} = (\nu\tilde{l})(\xi,\sigma)$ and $\mathcal{M}[u\!:\!\texttt{ref}(M)] \Downarrow \mathcal{M}'$. Then $\mathcal{M}' = (\nu\tilde{l}l)(\xi \cdot u : l, \sigma \cdot [l \mapsto V])$ with $u \notin \mathsf{fv}(\xi)$, $l \notin \mathsf{fl}(\sigma,\xi)$ and $(\nu\tilde{l_0})(M\xi,\sigma_0) \Downarrow (\nu\tilde{l_0})(V,\sigma)$. Then one can check $[\![i]\!]_{\xi\cdot u:l,\sigma\cdot[l\mapsto V]} = [\![i]\!]_{\xi,\sigma}$ and $[\![i]\!]_{\xi,\sigma} \notin \mathsf{lc}(l,\sigma\cdot[l \mapsto V]) = \mathsf{lc}(l,[l \mapsto V])$. ☐

We can now establish:

Theorem 4.3 (Soundness). $\vdash \{C\}M :_u \{C'\}$ *implies* $\models \{C\}M :_u \{C'\}$.

Proof. Except [*Ref*], all rules precisely follow [6, §8.2] (except for the use of thinness which allows the same reasoning as in [6, §8.2] to go through). For [*Ref*], we have, with l fresh:

$$\begin{array}{lll}
\mathcal{M} \models C & \Rightarrow \ \mathcal{M}[m\!:\!M] \Downarrow \mathcal{M}' \wedge \mathcal{M}' \models C' & \text{Hypothesis} \\
& \Rightarrow \ \mathcal{M}[m\!:\!M][u\!:\!\texttt{ref}(m)] \Downarrow (\nu l)\mathcal{M}'' \wedge \mathcal{M}'' \models C' \wedge !u = m & \\
& \quad \text{with } \mathcal{M}'' \stackrel{\text{def}}{=} \mathcal{M}'[u:l][l \mapsto V] & \\
& \Rightarrow \ \mathcal{M}[u\!:\!\texttt{ref}(M)] \Downarrow (\nu l)\mathcal{M}''/m \wedge \mathcal{M}''/m \models C'[!m/u] \wedge u\#i & \text{Lemma 4.2} \\
& \Rightarrow \ \mathcal{M}''/m[x:l] \models C'[!m/u] \wedge u\#i \wedge x = u & \\
& \Rightarrow \ (\nu l)\mathcal{M}''/m \models \nu x.(C'[!m/u] \wedge u\#i \wedge x = u) &
\end{array}$$

See Appendix B.1 for the full proofs. ☐

Theorem 4.4 (Soundness). $\vdash \{C\}M :_u \{C'\}@\tilde{e}$ *implies* $\models \{C\}M :_u \{C'\}@\tilde{e}$.

Proof. As above (and for remaining rules as in [6, §8.2]). See Appendix B.1 for [*Ref*] and the invariant rules. ☐

5. Axioms and Local Invariants

This section studies the basic axioms for the logical constructs, including those for local state.

5.1. **Axioms for Equality.**

Equality, logical connectives and quantifiers satisfy the standard axioms (quantifications need a modest use of thinness, see Proposition 5.8 later). For logical connectives, this is direct from the definition. For equality and quantification, however, this is not immediate, due to the non-standard definition of their semantics.

First we check the equality indeed satisfies the standard axioms for equality. We start from the following lemmas. $C[u/v;v/u]$ denotes a simultaneous substitution.

Lemma 5.1. *Let $\mathcal{M}$ have type Γ.*

(1) *(injective renaming) Let $u,v \in \mathsf{dom}(\Gamma)$. Then $\mathcal{M} \models C$ iff $\mathcal{M}[u/v;v/u] \models C[u/v;v/u]$.*
(2) *(permutation) Let $u,v \in \mathsf{dom}(\Gamma)$. Then we have $\mathcal{M} \models C$ iff $\binom{uv}{vu}\mathcal{M} \models C[u/v;v/u]$.*
(3) *(exchange) Let $u,v \notin \mathsf{fv}(e,e')$. Then we have $\mathcal{M}[u\!:\!e][v\!:\!e'] \models C$ iff $\mathcal{M}[v\!:\!e'][u\!:\!e] \models C$.*
(4) *(partition and monotonicity) Let $\mathcal{M} = (\nu\tilde{l})(\xi,\sigma)$ be of type Γ and $\mathcal{M}' = (\nu\tilde{l}\tilde{l}')(\xi\cdot\xi',\sigma\cdot\sigma')$ be such that $(\mathsf{fl}(\sigma')\cup\mathsf{fl}(\xi'))\cap\{\tilde{l}\} = \emptyset$. Further let $\Gamma \vdash C$. Then $\mathcal{M} \models C$ iff $\mathcal{M}' \models C$. In particular with $u \notin \mathsf{fv}(C)$ we have $\mathcal{M} \models C$ iff $\mathcal{M}[u\!:\!V] \models C$.*
(5) *(symmetry) $\mathcal{M} \models e_1 = e_2$ iff for fresh and distinct u,v: $\mathcal{M}[u\!:\!e_1][v\!:\!e_2] \approx \mathcal{M}[u\!:\!e_2][v\!:\!e_1]$.*
(6) *(substitution) $\mathcal{M}[u\!:\!x][v\!:\!e] \approx \mathcal{M}[u\!:\!x][v\!:\!e[u/x]]$; and $\mathcal{M}[u\!:\!e][v\!:\!e'] \approx \mathcal{M}[u\!:\!e][v\!:\!e'[e/u]]$.*

Proof. All are elementary, mostly by (simultaneous) induction on C. □

In (4) above, note that the extended part in $\mathcal{M}'$ on the top of $\mathcal{M}$ may refer to free labels of $\mathcal{M}$ but (since $\mathcal{M}$ is a model) no labels in $\mathcal{M}$ can ever refer to (free or bound) labels in $\mathcal{M}'$.

We are now ready to establish the standard axioms for equality.

Lemma 5.2. (axioms for equality) *For any model $\mathcal{M}$ and x, y, z and C:*

(1) $\mathcal{M} \models x = x$, $\mathcal{M} \models x = y \supset y = x$ *and* $\mathcal{M} \models (x = y \wedge y = z) \supset x = z$.
(2) $\mathcal{M} \models (C(x,y) \wedge x = y) \supset C(x,x)$.

where $C(x,y)$ indicates C together with some of the occurrences of x and y, while $C(x,x)$ is the result of substituting x for y, i.e. $C(x,y)[x/y]$ see [33, §2.4].

Proof. For the first clause, reflexivity is because $\mathcal{M}[u\!:\!x] \approx \mathcal{M}[u\!:\!x]$, while symmetry and transitivity are from those of $\approx$. For the second clause, we proceed by induction on C. We show the case where C is $e_1 = e_2$. The case C is $e_1 \hookrightarrow e_2$ is straightforward by definition. Other claims are by induction on C.

It suffices to prove $\mathcal{M} \models x = y$ and $\mathcal{M} \models C$ imply $\mathcal{M} \models C[x/y]$.

$$\mathcal{M} \models x = y \Rightarrow \mathcal{M}[u\!:\!x][v\!:\!y] \approx \mathcal{M}[u\!:\!y][v\!:\!x] \tag{5.1}$$
$$\Rightarrow \mathcal{M}[u\!:\!x][v\!:\!y][w\!:\!e_i] \approx \mathcal{M}[u\!:\!y][v\!:\!x][w\!:\!e_i] \tag{5.2}$$

Here (5.1) is by Lemma 5.1.5 and (5.2) follows from the congruency of $\approx$.

$$\begin{aligned}
\mathcal{M}[u\!:\!x][v\!:\!y][w\!:\!e_i] &\approx \mathcal{M}[u\!:\!x][v\!:\!y][w\!:\!e_i[v/y]] && \text{(Lem. 5.1(6))}\\
&\approx \mathcal{M}[u\!:\!y][v\!:\!x][w\!:\!e_i[v/y]] && (5.1)\\
&\approx \mathcal{M}[u\!:\!y][v\!:\!x][w\!:\!e_i[v/x][v/y]] && \text{(Lem. 5.1(6))}\\
&\approx \mathcal{M}[u\!:\!y][v\!:\!x][w\!:\!e_i[x/x][x/y]] && \text{(Lem. 5.1(6))}\\
&\approx \mathcal{M}[w\!:\!e_i[x/x][x/y]][u\!:\!y][v\!:\!x] && \text{(Lem. 5.1(3))}
\end{aligned}$$

$$\mathcal{M} \models e_1 = e_2 \Rightarrow \mathcal{M}[u\!:\!x][v\!:\!y] \models e_1 = e_2 \qquad \text{(Lem. 5.1(4))}$$
$$\Rightarrow \mathcal{M}[u\!:\!x][v\!:\!y][w\!:\!e_1] \approx \mathcal{M}[u\!:\!x][v\!:\!y][w\!:\!e_2]$$

Thus we get

$$\begin{aligned}\mathcal{M}[w\!:\!e_1[x/x][x/y]][u\!:\!y][v\!:\!x] &\approx \mathcal{M}[u\!:\!x][v\!:\!y][w\!:\!e_1]\\ &\approx \mathcal{M}[u\!:\!x][v\!:\!y][w\!:\!e_2]\\ &\approx \mathcal{M}[w\!:\!e_2[x/x][x/y]][u\!:\!y][v\!:\!x]\end{aligned}$$

This allows to conclude to:

$$\mathcal{M}[w\!:\!e_1[x/x][x/y]] \approx \mathcal{M}[w\!:\!e_2[x/x][x/y]]$$

which is equivalent to $\mathcal{M} \models C(x,x)$, as required. □

5.2. **Axioms for Necessity Operators.** We list basic axioms for Necessity and Possibility Operators. Below recall that $\Diamond C \stackrel{\mathsf{def}}{\equiv} \neg(\Box\neg C)$.

Proposition 5.3 (Necessity Operator)**.**

(1) $\Box(C_1 \supset C_2) \supset \Box C_1 \supset \Box C_2$; $\Box C \supset C$; $\Box\Box C \equiv \Box C$; $C \supset \Diamond C$. *Hence* $\Box C \supset \Diamond C$.

(2) (permutation and decomposition)

- (a) $\Box e_1 = e_2 \equiv e_1 = e_2$ *and* $\Box e_1 \neq e_2 \equiv e_1 \neq e_2$ *if* e_i *does not contain dereference.*
- (b) $\Box(C_1 \wedge C_2) \equiv \Box C_1 \wedge \Box C_2$.
- (c) $\Box C_1 \vee \Box C_2 \supset \Box(C_1 \vee C_2)$.
- (d) $\Box\forall x.C \supset \forall x.\Box C$ *and* $\Box\forall x.\Box C \equiv \Box\forall x.C$.
- (e) $\exists x.\Box C \supset \Box\exists x.C$ *and* $\Box\exists x^{\alpha}.C \equiv \exists x^{\alpha}.\Box C$ *with* $\alpha \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$.
- (f) $\Box\overline{\nu}x.C \equiv \overline{\nu}x.\Box C$ *and* $\nu x.\Box C \supset \Box\nu x.C$.
- (g) $\Box\exists \mathrm{X}.C \equiv \exists \mathrm{X}.\Box C$; *and* $\Box\forall \mathrm{X}.C \equiv \forall \mathrm{X}.\Box C$.
- (h) $\Box[!x]C \equiv [!x]\Box C \equiv \Box C$ *and* $\langle !x\rangle\Box C \equiv \Box C \supset \Box\langle !x\rangle C$.

Proof. See Appendix C.2. □

By the second axiom in (d), we can derive $\mathsf{fresh} \equiv \mathsf{fresh}_3$ in the last example of § 2.3.

The following proposition clarifies the interplay between $\Box C$ and evaluation formulae, and is useful in many examples. Recall below that $e \bullet e' \Uparrow$ (defined in Notation 2.1) means the application leads to the divergence.

Proposition 5.4 (Perpetuity)**.** *With z fresh,* $\Box C \equiv \forall \mathrm{X},\mathrm{Y}.f^{\mathrm{X}\Rightarrow\mathrm{Y}}.x^{\mathrm{X}}.(f \bullet x \Downarrow\supset f \bullet x = z\{\Box C\})$. *Again with z fresh,* $\Box C \equiv \forall \mathrm{X},\mathrm{Y}.f^{\mathrm{X}\Rightarrow\mathrm{Y}}.x^{\mathrm{X}}.(f \bullet x \Downarrow\supset f \bullet x = z\{C\})$.

Proof. Throughout we use $\Box C \equiv \Box\Box C$. For the first equivalence suppose $\mathcal{M} \models \Box C$ and $\mathcal{M}[f : L][x : L'][z : fx] \Downarrow \mathcal{M}'$. Then step by step we reach $\mathcal{M}' \models \Box C$ by the definition of $\Box C$. For the other direction, suppose $\mathcal{M} \models \Box C$ and for all N, N', we have $\mathcal{M}[u : N][w : N'] \Downarrow \mathcal{M}'$. By assumption $\mathcal{M}[f : \lambda().N][z : f()][w : N'] \Downarrow \mathcal{M}'$ such that $\mathcal{M}' \models C$ with $\mathcal{M} \rightsquigarrow \mathcal{M}'$. Since $\mathcal{M}[u : N][w : N'] \models C$, we have $\mathcal{M}[u : N] \models \Box C$, as required. For the second equivalence, the "only-if" direction is immediate, while the "if" direction is proved as in the previous "if" direction, observing that we can combine an arbitrary number of applications into a single one. □

The first logical equivalence of Proposition 5.4 allows us to say that if $\Box C$ holds and if a procedure is executed and if the evaluation terminates then $\Box C$ (hence in particular C) holds again. In essence, this is why a specification using $\Box C$ (or the equivalent) is useful: it allows us specify a behaviour which holds regardless of execution of other procedures and resulting state change. The second logical equivalence shows that, in addition, we can in fact define $\Box C$ via evaluation formulae (which in fact directly corresponds to the semantics of $\Box C$ in § 3).

Next, the following proposition says that located assertions are derived constructs, definable by combining non-located assertions and content quantification.

Proposition 5.5 (Decomposition of Located Evaluation Formula)**.** $x \bullet y = z\{C\}@\tilde{w} \;\equiv\; \forall u^{\mathsf{Unit}\Rightarrow\mathsf{Unit}}.(u \bullet () \Downarrow \supset\; x \bullet y = z\{C \;\wedge\; \langle !\tilde{w}\rangle u \bullet () \Downarrow\})$

Proof. In the following discussions we consider $\tilde{w}$ to be a singleton w for simplicity. First assume the left-hand side holds for a model say $\mathcal{M}$. Then the application only changes the content of w, hence if $u \bullet () \Downarrow$ then by restoring the content of w we again have $u \bullet () \Downarrow$. Secondly assume the right-hand side holds but the left-hand side does not. Then there must be some u which uses this difference at w to change its diverging behaviour, hence a contradiction. $\Box$

This decomposition uses content quantification to define located evaluation formulae where the effect set is restricted to specified finite locations. We can generalise located assertions to those which can specify the range of effects by formulae, which is sometimes useful. Such formulae can also be decomposed in the same way using an extended form of content quantification.

5.3. **Axioms for Hiding.**

Next we list basic axioms for hiding quantifiers. The most convenient axiom is about the elimination of hiding quantifiers, introduced by reference generation. To formulate this, we need some preparation.

Definition 5.6 (Monotone/Anti-Monotone Formulae)**.** C is *monotone* if $\mathcal{M} \models C$ and $l \notin \mathsf{fl}(C)$ imply $(\nu l)\mathcal{M} \models C$. C is *anti-monotone* if $\neg C$ is monotone.

The proof of the following proposition is similar to Proposition 3.13.

Proposition 5.7 (Syntactic Monotone/Antimonotone Formulae)**.**

(1) *T, F, $e = e'$, $e \neq e'$, $e \hookrightarrow e'$ and $e \# e'$ are monotone.*
(2) *If C, C' are monotone, then $C \wedge C'$, $C \vee C'$, $\forall x^{\alpha}.C$ for all α, $\exists x^{\alpha}.C$ with $\alpha \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$, $\exists \mathrm{X}.C$, $\forall \mathrm{X}.C$, $\nu x.C$, $\overline{\nu} x.C$, $\Box C$, $[!x]C$, and $e \bullet e' = x\{C'\}$ are monotone.*
(3) *The conditions exactly dual to 1 and 2 give antimonotone formulae.*

Proposition 5.8 (Axioms for $\forall$, $\exists$ and ν)**.** *Below we assume there is no capture of variables in types and formulae.*

(1) (introduction) $C \supset \nu x.C$ *if* $x \notin \mathsf{fv}(C)$
(2) (elimination) $\nu x.C \equiv C$ *if* $x \notin \mathsf{fv}(C)$ *and* C *is monotone.*
(3) *For any C we have $C \supset \exists x.C$. Given C such that $x \notin \mathsf{fv}(C)$ and C is thin with respect to x, we have $\exists x.C \supset C$.*
(4) *For any C we have $\forall x.C \supset C$. For C such that $x \notin \mathsf{fv}(C)$ and C is thin with respect to x, we have $C \supset \forall x.C$.*
(5) $\nu x.(C_1 \wedge C_2) \supset \nu x.C_1 \wedge \nu x.C_2$.
(6) $\nu x.(C_1 \vee C_2) \equiv \nu x.C_1 \vee \nu x.C_2$.
(7) $\nu y.\forall x.C \supset \forall x.\nu y.C$
(8) $\exists x.\nu y.C \supset \nu y.\exists x.C$ *and* $\exists x^{\alpha}.\nu y.C \equiv \nu y.\exists x^{\alpha}.C$ *with* $\alpha \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$.

(9) $\nu y.\overline{\nu}x.C \supset \overline{\nu}x.\nu y.C$; *and* $\nu y.\nu x.C \equiv \nu x.\nu y.C$.
(10) $\nu y.\exists \mathrm{X}.C \equiv \exists \mathrm{X}.\nu y.C$; *and* $\nu y.\forall \mathrm{X}.C \supset \forall \mathrm{X}.\nu y.C$.
(11) $\nu y.[!x]C \supset [!x]\nu y.C$ *and* $\nu y.\langle !x\rangle C \supset \langle !x\rangle \nu y.C$

Proof. See Appendix C.4. $\square$

For (1) and (2), it is notable that we do *not* generally have $C \supset \nu x.C$ even if C is thin. Neither $\nu x.C \supset C$ with $x \notin \mathsf{fv}(C)$ holds generally.

For the counterexample of $C \supset \nu x.C$ without the side condition, let $\mathcal{M} \stackrel{\text{def}}{=} (\{x : l,\ x' : l\},\ \{l \mapsto 5\})$. Then $\mathcal{M} \models x = x'$ but we do *not* have $\mathcal{M} \models \nu y.y = x'$ since l is certainly not hidden (x is renamed to fresh y to avoid confusion).

For the counterexample of $\nu x.C \supset C$ with $x \notin \mathsf{fv}(C)$, let $\mathcal{M} \stackrel{\text{def}}{=} (\nu l)(\{u : \lambda().!l\},\ \{l \mapsto 5\})$. Then we have:

$$\mathcal{M} \models \Box u \bullet () = z\{z = 5\}$$

Also we have:

$$\mathcal{M} \models (\nu x)\Diamond u \bullet () = z\{z = 0\}$$

with $\mathcal{M}[x : l] \models \Diamond u \bullet () = z\{z = 0\}$. If we apply $\nu x.C \equiv C$ to the above formula, we have $\mathcal{M} \models \Diamond u \bullet () = z\{z = 0\}$, which contradicts $\mathcal{M} \models \Box\{\mathsf{T}\}u \bullet () = z\{z = 5\}$.

Note this shows that integrating these quantifiers with the standard universal and existential quantifiers lets the latter loose their standard axioms, motivating the introduction of the ν-operator : from Proposition 5.8 (1,2,3), either $\exists x.C \supset \nu x.C$ or $\nu x.C \supset \exists x.C$ (with x typed by a reference type) does not hold in general (if $x \notin \mathsf{fv}(C)$ and C is thin, then $\exists x.C \supset C \supset \nu x.C$; and if $x \notin \mathsf{fv}(C)$ and C is monotone, then $\nu x.C \supset C \supset \exists x.C$).

The content quantifiers also have useful axioms. Appendix C.3 lists a selection.

5.4. **Axioms for Reachability.** We start from axioms for reachability. Note that our types include recursive types.

Proposition 5.9 (axioms for reachability)**.** *The following assertions are valid.*

(1) (1) $x \hookrightarrow x$; (2) $x \hookrightarrow y \wedge y \hookrightarrow z \ \supset\ x \hookrightarrow z$;
(2) (1) $y\#x^{\alpha}$ *with* $\alpha \in \{\mathsf{Unit}, \mathsf{Nat}, \mathsf{Bool}\}$; (2) $x\#y \ \Rightarrow\ x \neq y$; (3) $x\#w \wedge w \hookrightarrow u \ \supset\ x\#u$.
(3) (1) $\langle x_1, x_2\rangle \hookrightarrow y \ \equiv\ x_1 \hookrightarrow y \vee x_2 \hookrightarrow y$; (2) $\mathtt{inj}_i(x) \hookrightarrow y \ \equiv\ x \hookrightarrow y$; (3) $x \hookrightarrow y^{\mathsf{Ref}(\alpha)} \ \supset\ x \hookrightarrow !y$; (4) $x^{\mathsf{Ref}(\alpha)} \hookrightarrow y \wedge x \neq y \ \supset !x \hookrightarrow y$; (5) $[!x]y \hookrightarrow x \ \equiv\ y \hookrightarrow x$ *and* $[!x]x\#y \ \equiv\ x\#y$.

Proof. 1, 2 and 3.(1–4) are direct from the definition (e.g. for 3-(2) we observe $l \in \mathsf{fl}(\mathtt{inj}_i(V))$ iff $l \in \mathsf{fl}(V)$). For 3-(5), suppose $\mathcal{M} \models y \hookrightarrow x$, and take $\mathcal{M}'$ which only differs from $\mathcal{M}$ in the stored value at (the reference denoted by) x. Since $\mathcal{M} \models y \hookrightarrow x$ holds, there is a shortest sequence of connected references from y to x which, by definition, does not include x as its intermediate node. Hence this sequence also exists in $\mathcal{M}'$, i.e. $\mathcal{M}' \models y \hookrightarrow x$, proving $[!x]y \hookrightarrow x \ \equiv\ y \hookrightarrow x$. Similarly, we can prove $[!x]x\#y \ \equiv x\#y$. $\square$

3-(5) says that altering the content of x does not affect reachability *to* x. Note $[!x]y\#x \equiv y\#x$ is not valid at all. 3-(5) was already used for deriving $[\mathit{Inv}\text{-}\#]$ in §4.2 (notice that we cannot substitute $!x$ for y in $[!x]x\#y$ to avoid name capture [6]).

Let us say α is *finite* if it does not contains an arrow type or a type variable. We say $e \hookrightarrow e'$ is *finite* if e has a finite type.

Theorem 5.10 (elimination)**.** *Suppose all reachability predicates in C are finite. Then there exists C' such that $C \equiv C'$ and no reachability predicate occurs in C'.*

Proof. By Proposition 5.9. See Appendix C.5. □

The elimination of reachability predicates crucially uses type information in logical terms: as a simple example consider $x \hookrightarrow y$ where x has type $\mathsf{Ref}(\mathsf{Ref}(\mathsf{Nat}))$ and y has type $\mathsf{Ref}(\mathsf{Nat})$. Then we have $x \hookrightarrow y \equiv !!x = y$. The precise inductive elimination rules are given in Appendix C.5.

For analysing reachability with function types, it is useful to define the following "one-step" reachability predicate. Below e_2 is of a reference type.

$$\mathcal{M} \models e_1 \triangleright e_2 \quad \text{if} \quad [\![e_2]\!]_{\xi,\sigma} \in \mathsf{fl}([\![e_1]\!]_{\xi,\sigma}) \ \text{ for each } (\nu\tilde{l})(\xi,\sigma) \approx \mathcal{M} \tag{5.3}$$

The predicate $f \triangleright l'$ means l' occurs in any $\cong$-variant of the program f.

The following is straightforward from the definition.

Proposition 5.11 (Support)**.** $(\nu\tilde{l})(\xi,\sigma) \models x \triangleright l'$ *iff* $l' \in \bigcap\{\mathsf{fl}(V) \mid V \cong \xi(x)\}$.

The latter says that l' is in the support [12, 51, 59] of x.

We set $x \triangleright^n y$ for $n \geq 0$ by:

$$\begin{array}{rcl} x \triangleright^0 y & \equiv & x = y \\ x \triangleright^1 y & \equiv & x \triangleright y \\ x \triangleright^{n+1} y & \equiv & \exists z.(x \triangleright z \wedge !z \triangleright^n y) \quad (n \geq 1) \end{array}$$

By definition, we immediately observe:

Proposition 5.12. $x \hookrightarrow y \ \equiv\ \exists n.(x \triangleright^n y) \equiv (x = y \vee x \triangleright y \vee \exists z.(x \triangleright z \wedge z \neq y \wedge z \hookrightarrow y))$.

Proposition 5.12, combined with Theorem 5.10, suggests that if we can clarify one-step reachability at function types then we will be able to clarify the reachability relation as a whole. Unfortunately this relation is inherently intractable.

Proposition 5.13 (undecidability of $\triangleright$ and $\hookrightarrow$)**.** (1) $\mathcal{M} \models f^{\alpha\Rightarrow\beta} \triangleright x$ *is undecidable.* (2) $\mathcal{M} \models f^{\alpha\Rightarrow\beta} \hookrightarrow x$ *is undecidable.*

Proof. For (1), let $V \stackrel{\text{def}}{\equiv} \lambda().\mathtt{if}\ M = ()\ \mathtt{then}\ l\ \mathtt{else}\ \mathsf{Ref}(0)$ with a closed PCFv-term M of type Unit. Then $f : V, x : l \models f \triangleright x$ iff $M \Downarrow$, reducing the satisfiability to the halting problem of PCFv-terms. For (2), take the same V so that the type of l and x is $\mathsf{Ref}(\mathsf{Nat})$ in which case $\triangleright$ and $\hookrightarrow$ coincide. □

The same result holds for call-by-value $\beta\eta$-equality. Proposition 5.13 indicates inherent intractability of $\triangleright$ and $\hookrightarrow$.

However Proposition 5.13 does not imply that we cannot obtain useful axioms for (un)reachability at function types. Next, we discuss a collection of axioms with function types. First, the following axiom says that if x is unreachable from f, y and $\tilde{w}$, then the application of f to y with the effect set $\tilde{w}$ never exports x.

Proposition 5.14 (unreachable functions)**.** *For an arbitrary C, the following is valid with i and* X *fresh:*

$$\Box\{C \wedge x \# f y \tilde{w}\}\, f \bullet y = z \{C'\} @ \tilde{w} \ \supset\ \Box \forall \mathrm{X}, i^{\mathrm{X}}.\{C \wedge x \# f i y \tilde{w}\}\, f \bullet y = z \{C' \wedge x \# f i y z \tilde{w}\} @ \tilde{w}$$

Proof. See Appendix C.6. □

5.5. **Local Invariants.** We now introduce an axiom for local invariants. Let us first consider a function which writes to a local reference of base type. Even programs of this kind pose fundamental difficulties in reasoning, as shown in [34]. Take the following program:

$$\texttt{compHide} \stackrel{\text{def}}{=} \texttt{let}\ x = \texttt{ref}(7)\ \texttt{in}\ \lambda y.(y > !x) \tag{5.4}$$

The program behaves as a pure function $\lambda y.(y > 7)$. Clearly, the obvious local invariant $!x = 7$ is preserved. We demand this assertion to survive under arbitrary invocations of `compHide`: thus (naming the function u) we arrive at the following invariant:

$$C_0 \stackrel{\text{def}}{\equiv} !x = 7 \wedge \Box\forall y.\{!x = 7\}u \bullet y = z\{!x = 7\}@\emptyset \tag{5.5}$$

Assertion (5.5) says: (1) the invariant $!x = 7$ holds now; and that (2) once the invariant holds, it continues to hold for ever (note x can never be exported due to the type of y and z, so that only u will touch x). Using this assertion, `compHide` satisfies the following with i fresh:

$$\{\mathsf{T}\}\texttt{compHide} :_u \{\nu x.(x\#i^{\mathrm{X}} \wedge C_0 \wedge C_1)\} \tag{5.6}$$

$$C_1 \stackrel{\text{def}}{\equiv} \Box\forall y.\{!x = 7\}u \bullet y = z\{z = (y > 7)\}@\emptyset. \tag{5.7}$$

Thus, noting C_0 is only about the content of x (in fact it is syntactically stateless except x in the sense of Definition 3.19, we can conclude C_0 continues to hold automatically over any future computation by any programs. Hence we cancel C_0 together with x:

$$\{\mathsf{T}\}\texttt{compHide} :_u \{\Box\forall y.u \bullet y = z\{z = (y > 7)\}\} \tag{5.8}$$

which describes a purely functional behaviour.

Now we leave the example and move to the general case, stipulating the underlying reasoning principle as an axiom. Let y, z be fresh. We define:

$$\mathsf{Inv}(u, C_0, \tilde{x}) \stackrel{\text{def}}{\equiv} C_0 \wedge (\Box\forall yi.\{C_0\}u \bullet y \Downarrow \supset \Box\forall yi.\{C_0\}u \bullet y = z\{C_0 \wedge \tilde{x}\#z\}) \tag{5.9}$$

where $C_0 \supset \tilde{x}\#iy$. $\mathsf{Inv}(u, C_0, x)$ says that currently C_0 holds; and that if C_0 holds, applying u to y results in, if it ever converges, C_0 again and the returned z is disjoint from $\tilde{x}$. The axiom also uses:

$$x \hookrightarrow^{\bullet} \tilde{y} \stackrel{\text{def}}{\equiv} \forall z.(x \hookrightarrow z \supset z \in \{\tilde{y}\}) \tag{5.10}$$

Thus $x \hookrightarrow^{\bullet} \tilde{y}$ says that all references reachable from x are inside $\{\tilde{y}\}$. We write $\tilde{x} \hookrightarrow^{\bullet} \tilde{y}$ for the conjunction $\wedge_i x_i \hookrightarrow^{\bullet} \tilde{y}$. The axiom follows.

Proposition 5.15 (axiom for information hiding)**.** *Assume $C_0 \equiv C_0' \wedge \tilde{x}\#iy \wedge \tilde{g} \hookrightarrow^{\bullet} \tilde{x}$, C_0' is stateless except $\tilde{x}$, C is antimonotone, C' is monotone, i, m are fresh and $\{\tilde{x}, \tilde{g}\} \cap (\mathsf{fv}(C, C') \cup \{\tilde{w}\}) = \emptyset$. Then the following is valid:*

$$(\mathsf{AIH}) \qquad \forall \mathrm{X}.\forall i^{\mathrm{X}}.m \bullet () = u\{(\nu\tilde{x}.\exists\tilde{g}.E_1) \wedge E\} \supset \forall \mathrm{X}.\forall i^{\mathrm{X}}.m \bullet () = u\{E_2 \wedge E\}$$

with

- $E_1 \stackrel{\text{def}}{\equiv} \mathsf{Inv}(u, C_0, \tilde{x}) \wedge \Box\forall yi.\{C_0 \wedge C\}u \bullet y = z\{C'\}@\tilde{w}\tilde{x}$
- $E_2 \stackrel{\text{def}}{\equiv} \Box\forall y.\{C\}u \bullet y = z\{C'\}@\tilde{w}$ *and*
- E *is an arbitrary formula.*

Proof. See Appendix C.7. ∎

(AIH) is used with the refined consequence rule [*Cons-Eval*] (cf. Figure 1) to simplify from E_1 to E_2, eliminating hidings. Its validity is proved using Proposition 3.9. The axiom[10] says:

> *if a function u with fresh reference x_i is generated, and if it has a local invariant C_0 on the content of x_i, then we can cancel C_0 together with x_i.*

Note that:

- The statelessness of C_0 except $\tilde{x}$ ensures that satisfaction of C_0 is not affected by state change except at $\tilde{x}$; and
- The quantification $\exists\tilde{g}.E_1$ of $\tilde{g}$ in (AIH) allows the invariant to contain free variables, extending applicability of the axiom, for example in the presence of circular references as we shall use in §6 for `safeEven`. $\tilde{g} \hookrightarrow^{\bullet} \tilde{x}$ ensures that $\tilde{g}$ are contained in the $\tilde{x}$-hidden part of the model.

Coming back to `compHide`, we take, for (AIH):

(1) C_0' to be $!x = 7$ which is syntactically stateless except x;
(2) C_0 to be $C_0' \wedge x\#i$;
(3) $\tilde{s}$ and $\tilde{w}$ empty,
(4) both C and E to be T (which is anti-monotonic by Proposition 5.7, and
(5) C' to be $z = (y > 7)$ (which is monotonic by the same proposition),

thus arriving at the desired assertion.

(AIH) eliminates ν from the post-condition based on local invariants. The following axiom also eliminates νx, this time solely based on freshness and disjointness of x.

Proposition 5.16 (ν-elimination). *Let $x \notin \mathsf{fv}(C)$ and m, i, X be fresh. Then the following is valid:*

$$\forall \mathrm{X}, i^{\mathrm{X}}.\, m \bullet () = u\{\nu\tilde{x}.([!\tilde{x}]C \wedge \tilde{x}\#ui^{\mathrm{X}})\} \;\supset\; m \bullet () = u\{C\} \tag{5.11}$$

Proof. See Appendix C.8. □

This proposition says that if a hidden (and newly created) location x in the post-state is disjoint from any asserted data including the used function itself and those in the pre-state, then we can safely neglect it (in this sense it is a garbage collection rule when we are not concerned with newly created variables).

The following axiom stipulates how an invariant can be *transferred* by a function (caller) which uses another function (callee) when the latter only affects a set of references unreachable from the former.

Proposition 5.17 (invariant by application). *Assume C_0 is stateless except at $\tilde{x}$, $C_0 \supset \tilde{x}\#y$ and $y \notin \mathsf{fv}(C_0)$. Then the following is valid.*

$$(\Box\forall y.\{C_0\} f \bullet y = z\{C_0\}@\tilde{x} \wedge \Box\{C\} g \bullet f = z\{C'\}) \supset \Box\{C \wedge C_0 \wedge \tilde{x}\#g\} g \bullet f = z\{C_0 \wedge C'\}$$

Proof. See Appendix C.9. □

The axiom says that the result of applying a function g disjoint from each local reference x_i in $\tilde{x}$, to the argument function f which satisfies a local invariant exclusively at $\tilde{x}$, again preserves that local invariant.

Proposition 5.17 may be considered as a higher-order version of Proposition 5.14 and in fact is closely related in that both depend on localised effects of a function at references.

[10] We believe that the monotonicity of C' and anti-monotonicity of C are unnecessary in Proposition 5.15, though the present proof uses them.

6. Reasoning Examples

This section demonstrates the usage of the proposed logic through concrete reasoning examples.

6.1. **New Reference Declaration.** We first show a useful derived rule given by the combination of "let" and new reference generation.

$$[LetRef]\ \frac{\{C\}\ M :_m \{C_0\} \quad \{C_0[!x/m] \wedge x\#\tilde{e}\}\ N :_u \{C'\} \quad x \notin \mathsf{fpn}(\tilde{e})}{\{C\}\ \mathtt{let}\ x = \mathtt{ref}(M)\ \mathtt{in}\ N :_u \{\nu x.C'\}}$$

where C' is thin w.r.t. m. Above $\mathsf{fpn}(e)$ denotes the set of *free plain names* of e which are reference names in e that do not occur dereferenced, given in Definition 4.1. The meaning of $x\#\tilde{e}$ was given in Notation 2.1 in § 2.3. The rule reads:

> *Assume* (1) *executing M with precondition C leads to C_0, with the resulting value named m; and* (2) *running N from C_0 with m as the content of x together with the assumption x is unreachable from each e_i, leads to C' with the resulting value named u. Then running* $\mathtt{let}\ x = \mathtt{ref}(M)\ \mathtt{in}\ N$ *from C leads to C' whose x is fresh and hidden.*

The side condition $x \notin \mathsf{fpn}(e_i)$ is essential for consistency (e.g. without it, we could assume $x\#x$, i.e. F); and $\nu x.C'$ cannot be strengthened to $x\#i\ \wedge\ C'$ since N may store x in an existing reference. The use of general $\tilde{e}$ is also essential since the we can start from total disjointness (separation) and reach possibly partial disjointness in the conclusion. For this purpose we need to have explicit $x\#\tilde{e}$ initially, which may possibly be weakened in the post-condition C through the actions in N.

The rule directly gives a proof rule for new reference declaration [34, 48, 56], $\mathtt{new}\ x := M\ \mathtt{in}\ N$, which has the same operational behaviour as $\mathtt{let}\ x = \mathtt{ref}(M)\ \mathtt{in}\ N$.

We can derive [*LetRef*] as follows. Below i is fresh.

1. $\{C\}\ M :_m \{C_0\}$	(premise)
2. $\{C_0[!x/m] \wedge x\#\tilde{e}\}\ N :_u \{C'\}$ with $x \notin \mathsf{fpn}(\tilde{e})$	(premise)
3. $\{C\}\ \mathtt{ref}(M) :_x \{\nu y.(C_0[!x/m] \wedge x\#i \wedge x = y)\}$	(1,Ref)
4. $\{C\}\ \mathtt{ref}(M) :_x \{\nu y.(C_0[!x/m] \wedge x\#\tilde{e} \wedge x = y)\}$	(Subs n-times)
5. $\{C_0[!x/m] \wedge x\#\tilde{e} \wedge x = y\}\ N :_u \{C' \wedge x = y\}$	(2, Invariance)
6. $\{C\}\ \mathtt{let}\ x = \mathtt{ref}(M)\ \mathtt{in}\ N :_u \{\nu y.(C' \wedge x = y)\}$	(4,5,LetOpen)
7. $\{C\}\ \mathtt{let}\ x = \mathtt{ref}(M)\ \mathtt{in}\ N :_u \{\nu x.C'\}$	(Conseq)

[*LetOpen*] is the rule for let to open the scope:

$$[LetOpen]\ \frac{\{C\}\ M :_x \{\nu\tilde{y}.C_0\}@\tilde{e}_1 \quad \{C_0\}\ N :_u \{C'\}@\tilde{e}_2}{\{C\}\ \mathtt{let}\ x = M\ \mathtt{in}\ N :_u \{\nu\tilde{y}.C'\}@\tilde{e}_1\tilde{e}_2}$$

where C' is thin w.r.t. x. [*LetOpen*] and [*Subs*] (both rules being for located judgements) are found in Figure 6 in Appendix B, and their soundness is proved in Appendix B.3.

6.2. **Shared Stored Function.** We present a simple example of hiding-quantifiers and unreachability using `incShared` in (1.2) from § 1.

$$\texttt{incShared} \stackrel{\text{def}}{=} a := \texttt{Inc}; b := !a; c_1 := (!a)(); c_2 := (!b)(); (!c_1 + !c_2)$$

with $\texttt{Inc} \stackrel{\text{def}}{=} \texttt{let}\ x = \texttt{ref}(0)\ \texttt{in}\ \lambda().(x := !x+1;\ !x)$. Naming it u, the assertion $\nu x.\mathsf{inc}'(u,x,n)$ (defined below) captures the behaviour of `Inc`:

$$\begin{aligned} \mathsf{inc}(x,u) &\stackrel{\text{def}}{\equiv} \Box\forall j.\{!x = j\}u \bullet () = j+1\{!x = j+1\}@x. \\ \mathsf{inc}'(u,x,n) &\stackrel{\text{def}}{\equiv} !x = n \wedge \mathsf{inc}(x,u). \end{aligned}$$

The following derivation for `incShared` sheds light on how shared higher-order local state can be transparently reasoned in the present logic. For brevity we work with the implicit global assumption that a, b, c_1, c_2 are pairwise distinct and safely omit an anchor from judgements when the return value is of unit type.

1. $\{\top\}\ \texttt{Inc} :_u \{\nu x.\mathsf{inc}'(u,x,0)\}$	
2. $\{\top\}\ a := \texttt{Inc}\ \{\nu x.\mathsf{inc}'(!a,x,0)\}$	(1, Assign)
3. $\{\mathsf{inc}'(!a,x,0)\}\ b := !a\ \{\mathsf{inc}'(!a,x,0) \wedge \mathsf{inc}'(!b,x,0)\}$	(Assign)
4. $\{\mathsf{inc}'(!a,x,0)\}\ c_1 := (!a)()\ \{\mathsf{inc}'(!a,x,1) \wedge !c_1 = 1\}$	(Assign)
5. $\{\mathsf{inc}'(!b,x,1)\}\ c_2 := (!b)()\ \{\mathsf{inc}'(!b,x,2) \wedge !c_2 = 2\}$	(App etc.)
6. $\{!c_1 = 1 \wedge !c_2 = 2\}\ (!c_1) + (!c_2) :_u \{u = 3\}$	(Deref etc.)
7. $\{\top\}\ \texttt{incShared} :_u \{\nu x.u = 3\}$	(2–6, LetOpen)
8. $\{\top\}\ \texttt{incShared} :_u \{u = 3\}$	(Conseq)

Line 1 is by [*LetRef*]. Line 8 uses Proposition 5.8(2), $\nu x.C \supset C$.

To shed light on how the difference in sharing is captured in inferences, we list the inference for a program which assigns *distinct* copies of `Inc` to a and b,

$$\texttt{incUnShared} \stackrel{\text{def}}{=} a := \texttt{Inc}; b := \texttt{Inc}; c_1 := (!a)(); c_2 := (!b)(); (!c_1 + !c_2)$$

This program assigns to a and b two separate instances of `Inc`. This lack of sharing between a and b in `incUnShared` is captured by the following derivation:

$$
\begin{array}{l}
1.\{\mathsf{T}\}\ \mathtt{Inc} :_m \{\nu x.\mathsf{inc}(u,x,0)\} \\ \hline
2.\{\mathsf{T}\}\ a := \mathtt{Inc}\ \{\nu x.\mathsf{inc}(!a,x,0)\} \\ \hline
3.\{\mathsf{inc}(!a,x,0)\}\ b := \mathtt{Inc}\ \{\nu y.\mathsf{inc}'(0,0)\} \\ \hline
4.\{\mathsf{inc}'(0,0)\}\ z_1 := (!a)()\ \{\mathsf{inc}'(1,0) \wedge !z_1 = 1\} \\ \hline
5.\{\mathsf{inc}'(1,0)\}\ z_2 := (!b)()\ \{\mathsf{inc}'(1,1) \wedge !z_2 = 1\} \\ \hline
6.\{!z_1 = 1 \wedge !z_2 = 1\}\ (!z_1) + (!z_2) :_u \{u = 2\} \\ \hline
7.\{\mathsf{T}\}\ \mathtt{incUnShared} :_u \{\nu xy.u = 2\} \\ \hline
8.\{\mathsf{T}\}\ \mathtt{incUnShared} :_u \{u = 2\}
\end{array}
$$

Above $\mathsf{inc}'(n,m) \stackrel{\mathsf{def}}{\equiv} \mathsf{inc}(!a,x,n) \wedge \mathsf{inc}(!b,y,m) \wedge x \neq y$. Note $x \neq y$ is guaranteed by [*LetRef*]. This is in contrast to the derivation for `incShared`, where, in Line 3, x is automatically shared after "$b := !a$" which leads to scope extrusion.

6.3. **Memoised Factorial.** Next we treat the memoised factorial (1.4) (from [49]) in the introduction.

$$
\begin{array}{l}
\mathtt{memFact} \stackrel{\mathsf{def}}{=} \mathtt{let}\ a = \mathtt{ref}(0),\ \ b = \mathtt{ref}(1)\ \mathtt{in} \\
\qquad\qquad \lambda x.\mathtt{if}\ x = !a\ \mathtt{then}\ !b\ \mathtt{else}\ (a := x;\, b := \mathtt{fact}(x);\, !b)
\end{array}
$$

Above `fact` is the standard factorial function.

Our target assertion specifies the behaviour of a pure factorial.

$$
Fact(u) \stackrel{\mathsf{def}}{\equiv} \Box \forall x.u \bullet x = y\{y = x\mathbf{!}\}@\emptyset.
$$

The following inference starts from the `let`-body of `memFact`, which we name V. We set:

$$
\begin{array}{rcl}
E_{1a} & \stackrel{\mathsf{def}}{\equiv} & \Box \forall xi.\{C_0\}u \bullet x = y\{C_0 \wedge ab\#y\}@ab \\
E_{1b} & \stackrel{\mathsf{def}}{\equiv} & \Box \forall xi.\{C_0 \wedge C\}u \bullet x = y\{C'\}@ab
\end{array}
$$

and we set C_0 to be $ab\#ix \wedge !b = (!a)\mathbf{!}$, C to be T, and C' to be $y = x\mathbf{!}$. Note that $!b = (!a)\mathbf{!}$ is stateless except ab by Proposition 5.9(5); and that, by the type of x and y being `Nat` and Proposition 5.9 2-(1), we have $ab\#x \equiv ab\#y \equiv \mathsf{T}$.

We can now reason:

$$\begin{array}{lr}
1.\{\top\}\ 0 :_a \{a=0\}@\emptyset & \text{(Const)}\\ \hline
2.\{a=0\}\ 1 :_b \{b=a\mathbf{!}\}@\emptyset & \text{(Const)}\\ \hline
3.\{\top\}\ V :_u \{\Box\forall xi.\{C_0\}u\bullet x=y\{C_0\ \wedge\ C'\}\}@\emptyset & \text{(Abs)}\\ \hline
4.\{\top\}\ V :_u \{E_{1a}\wedge E_{1b}\}@\emptyset & \text{(3, Conseq)}\\ \hline
5.\{ab\#i\wedge !b=(!a)\mathbf{!}\}\ V :_u \{ab\#i\wedge !b=(!a)\mathbf{!}\wedge E_{1a}\wedge E_{1b}\}@\emptyset & \text{(4, Inv-\#, Inv-Val in § 4.4)}\\ \hline
6.\{\top\}\ \mathtt{memFact} :_u \{\nu ab.(C_0\wedge E_{1a}\wedge E_{1b})\}@\emptyset & \text{(1,2,4, LetRef in § 6.1)}\\ \hline
7.m\bullet()=u\{\nu ab.(C_0\wedge E_{1a}\wedge E_{1b})\}\supset m\bullet()=u\{Fact(u)\} & (\star)\\ \hline
8.\{\top\}\ \mathtt{memFact} :_u \{Fact(u)\}@\emptyset & \text{(6,7,ConsEval)}
\end{array}$$

Line 4 uses the axiom $\{C\}f\bullet x=y\{C_1\wedge C_2\}@\tilde{w}\supset\wedge_{i=1,2}\{C\}f\bullet x=y\{C_i\}@\tilde{w}$ (in [6]). Line 7 uses (AIH).

6.4. **Information Hiding (2): Stored Circular Procedures.** We next consider stored higher order functions which mimic stored procedures.

We start with a simple one, `circFact` from [25], which uses a self-recursive higher-order local store.

$$\begin{array}{lcl}
\mathtt{circFact} & \stackrel{\text{def}}{\equiv} & x := \lambda z.\mathtt{if}\ z=0\ \mathtt{then}\ 1\ \mathtt{else}\ z\times(!x)(z-1)\\
\mathtt{safeFact} & \stackrel{\text{def}}{\equiv} & \mathtt{let}\ x=\mathtt{ref}(\lambda y.y)\ \mathtt{in}\ (\mathtt{circFact};!x)
\end{array}$$

In [25], we have derived the following judgement.

$$\{\top\}\mathtt{circFact} :_u \{CircFact(u,x)\}@x \tag{6.1}$$

where

$$CircFact(u,x)\ \stackrel{\text{def}}{\equiv}\ \Box\forall n.\{!x=u\}!x\bullet n=z\{z=n!\wedge !x=u\}@\emptyset\ \wedge\ !x=u$$

which says:

> *After executing the program, x stores a procedure which would calculate a factorial if x stores that behaviour, and that x does store the behaviour.*

We now show `safeFact` named u satisfies $Fact(u)$. Below we use:

$$\begin{array}{lcl}
CF_a & \stackrel{\text{def}}{\equiv} & \Box\forall n.\{!x=u\}!x\bullet n=z\{!x=u\}@\emptyset\\
CF_b & \stackrel{\text{def}}{\equiv} & \Box\forall n.\{!x=u\}!x\bullet n=z\{z=n\mathbf{!}\}@\emptyset
\end{array}$$

(note that $x\#z \equiv \mathsf{T}$ and $x\#n \equiv \mathsf{T}$ by Proposition 5.9 (2)-1).

$$
\begin{array}{ll}
1.\{\mathsf{T}\}\lambda y.y :_m \{\mathsf{T}\}@\emptyset & \\
\hline
2.\{\mathsf{T}\}\mathtt{circFact}\,;\,!x :_u \{\mathit{CircFact}(u,x)\}@x & \\
\hline
3.\{\mathsf{T}\}\mathtt{circFact}\,;\,!x :_u \{!x = u \wedge \mathit{CF}_a \wedge \mathit{CF}_b\}@x & (2, \text{Conseq}) \\
\hline
4.\{x\#i\}\mathtt{circFact}\,;\,!x :_u \{x\#i \wedge !x = u \wedge \mathit{CF}_a \wedge \mathit{CF}_b\}@x & (3, \text{Inv-}\#) \\
\hline
5.\{\mathsf{T}\}\mathtt{safeFact} :_u \{\nu x.(C_0 \wedge \mathit{CF}_a \wedge \mathit{CF}_b)\}@\emptyset & (4, \text{LetRef}) \\
\hline
6.m\bullet() = u\{\nu x.(C_0 \wedge \mathit{CF}_a \wedge \mathit{CF}_b)\} \supset m\bullet() = u\{\mathit{Fact}(u)\} & (\star) \\
\hline
7.\{\mathsf{T}\}\mathtt{safeFact} :_u \{\mathit{Fact}(u)\}@\emptyset & (5, 6, \text{ConsEval})
\end{array}
$$

Line 1 is immediate. Line 2 is (6.1). Line 6, $(\star)$ is by (AIH), Proposition 5.15, setting $C_0 \stackrel{\text{def}}{\equiv} x\#i \wedge !x = u$, $C \stackrel{\text{def}}{\equiv} E \stackrel{\text{def}}{\equiv} \mathsf{T}$ and $C' \stackrel{\text{def}}{\equiv} y = x!$.

6.5. **Mutually Recursive Stored Functions.** Now we investigate the program from (1.6) in the introduction. The reasoning easily extends to programs which use multiple locally stored, and mutually recursive, procedures.

We first verify the following `mutualParity` (the `let`-body).

$$
\mathtt{mutualParity} \stackrel{\text{def}}{=} \begin{array}{l} x := \lambda n.\mathtt{if}\ n{=}0\ \mathtt{then\ f\ else\ not}((!y)(n-1)); \\ y := \lambda n.\mathtt{if}\ n{=}0\ \mathtt{then\ t\ else\ not}((!x)(n-1)) \end{array} \tag{6.2}
$$

Then we have:

$$
\{\mathsf{T}\}\mathtt{mutualParity} :_u \{\exists gh.\mathit{IsOddEven}(gh, !x!y, xy, n)\} \tag{6.3}
$$

where, with $\mathit{Even}(n) \equiv \exists x.(n{=}2\times x)$ and $\mathit{Odd}(n) \equiv \mathit{Even}(n{+}1)$:

$$
\mathit{IsOddEven}(gh, wu, xy, n) \stackrel{\text{def}}{\equiv} (\mathit{IsOdd}(w, gh, n, xy) \wedge \mathit{IsEven}(u, gh, n, xy) \wedge !x = g \wedge !y = h)
$$

$$
\mathit{IsOdd}(u, gh, n, xy) \stackrel{\text{def}}{\equiv} \Box\{!x = g \wedge !y = h\}u\bullet n{=}z\{z = \mathit{Odd}(n) \wedge !x = g \wedge !y = h\}@xy
$$

$$
\mathit{IsEven}(u, gh, n, xy) \stackrel{\text{def}}{\equiv} \Box\{!x = g \wedge !y = h\}u\bullet n{=}z\{z = \mathit{Even}(n) \wedge !x = g \wedge !y = h\}@xy
$$

The detailed derivations are given in Appendix D.1. Above $\mathit{IsOdd}(u, gh, n, xy)$ says that

> $!x$ *and* $!y$ *remain unchanged, and that* u *checks if its argument is odd.*

Similarly for $\mathit{IsEven}(u, gh, n, xy)$. Then above $\mathit{IsOddEven}(gh, wu, xy, n)$ says that

> x *stores a procedure which checks if its argument is odd if* y *stores a procedure which does the dual, and* x *does store the behaviour; and dually for* y.

Note that IsOdd and IsEven, the effect set is xy since x and y are free and assigned to the abstractions in `mutualParity`.

Our aim is to derive the judgement for `safeEven` given below:

$$
\mathtt{safeEven} \stackrel{\text{def}}{=} \mathtt{let}\ x = \mathtt{ref}(\lambda n.\mathtt{t}),\ y = \mathtt{ref}(\lambda n.\mathtt{t})\ \mathtt{in}\ (\mathtt{mutualParity}; !y) \tag{6.4}
$$

We start from (6.3) (the case for `safeOdd` is symmetric).

$$
\{\mathsf{T}\}\mathtt{safeEven} :_u \{\forall n.\Box u\bullet n = z\{z = \mathit{Even}(n)\}@\emptyset\}
$$

We first identify the local invariant:

$$C_0 \stackrel{\text{def}}{\equiv} \,!x = g \wedge \,!y = h \wedge IsEven(h, gh, n, xy) \wedge xy\#ijn \wedge gh \hookrightarrow^{\circ} xy$$

Note we have a free variable h. Since C_0 only talks about g, h and the content of x and y, we know $!x = g \wedge \,!y = h \wedge IsEven(h, gh, n, xy)$ is stateless except x, y; and $xy\#n \equiv xy\#z \equiv \mathsf{T}$ by Proposition 5.9 (2)-1.

Let us define:

$$\begin{array}{rcl} ValEven(u) & \stackrel{\text{def}}{\equiv} & \Box\forall n.\{\mathsf{T}\}u \bullet n = z\{z = Even(n)\}@\emptyset \\ Even_a & \stackrel{\text{def}}{\equiv} & \Box\forall n.\{C_0\}u \bullet n = z\{C_0\}@xy \\ Even_b & \stackrel{\text{def}}{\equiv} & \Box\forall n.\{C_0\}u \bullet n = z\{z = Even(n)\}@xy \end{array}$$

The derivation is given as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1.\{\mathsf{T}\}\lambda n.\mathtt{t} :_m \{\mathsf{T}\}@\emptyset}{2.\{\mathsf{T}\}\mathtt{mutualParity};\, !y :_u \{\exists gh.IsOddEven(gh, gu, xy, n)\}@xy}}{3.\{\mathsf{T}\}\mathtt{mutualParity};\, !y :_u \{\exists gh.(!x = g \wedge !y = h \wedge IsOdd(g, gh, n, xy) \wedge Even_a \wedge Even_b)\}@xy}}{4.\{xy\#ij\}\mathtt{mutualParity};\, !y :_u \{\exists gh.(C_0 \wedge Even_a \wedge Even_b)\}@xy}}{5.\{\mathsf{T}\}\mathtt{safeEven} :_u \{\nu xy.\exists gh.(C_0 \wedge Even_a \wedge Even_b)\}@\emptyset}}{6.\{\mathsf{T}\}m \bullet () = u\{\nu xy \exists gh.(C_0 \wedge Even_a \wedge Even_b)\} \supset \{\mathsf{T}\}m \bullet () = u\{ValEven(u)\} \quad \text{(by (AIH))}}}{7.\{\mathsf{T}\}\mathtt{safeEven} :_u \{ValEven(u)\}@\emptyset}$$

As we can see, the derivation follows the same pattern as that of `memoFact` and `safeFact`.

6.6. **Higher-Order Invariant.** We move to a program (from [59, p.104]) whose invariant behaviour depends on another function. The program instruments a program with simple profiling, counting the number of invocations.

$$\mathtt{profile} \quad \stackrel{\text{def}}{=} \quad \mathtt{let}\ x = \mathtt{ref}(0)\ \mathtt{in}\ \lambda y.(x := !x + 1; fy)$$

Since x is never exposed, this program should behave precisely as f. Thus our aim is to derive:

$$\{\Box\forall y.\{C\} f \bullet y = z\{C'\}@\tilde{w}\}\ \mathtt{profile} :_u \{\Box\forall y.\{C\}u \bullet y = z\{C'\}@\tilde{w}\} \tag{6.5}$$

with $x \notin \mathsf{fv}(C, C')$ (by the bound name condition) and arbitrary anti-monotonic C and monotonic C'.

This judgement says:

> *if f satisfies the specification $E \stackrel{\text{def}}{\equiv} \Box\forall y.\{C\} f \bullet y = z\{C'\}@\tilde{w}$, then* `profile` *satisfies the same specification E.*

To derive (6.5), we first set C_0, the invariant, to be $x\#fiy\tilde{w}$.

As with the previous derivations, we use two subderivations.

First we derive:

$$\begin{array}{llll} & E & \stackrel{\text{def}}{\equiv} & \Box\forall y.\{C\} f \bullet y = z\{C'\}@\tilde{w} \\ \supset & E_0 & \stackrel{\text{def}}{\equiv} & \Box\forall yi.\{C \wedge x\#fiy\tilde{w}\} f \bullet y = z\{C'\}@\tilde{w}x \qquad \text{Axiom (e8) in [25]} \\ \supset & E_1 & \stackrel{\text{def}}{\equiv} & \Box\forall yi.\{C \wedge x\#fiy\tilde{w}\} f \bullet y = z\{C' \wedge x\#zfiy\tilde{w}\}@\tilde{w}x \qquad \text{Axiom (e8) in [25]} \end{array}$$

where Axiom (e8) in [25] is given as:

$$(C \supset C_0 \ \wedge\ \{C_0\}x \bullet y = z\{C_0'\} \ \wedge\ C_0' \supset C) \quad \supset \{C\}x \bullet y = z\{C'\}$$

we use the first axiom in Proposition 5.3 (1). We also let

$$E_2 \stackrel{\text{def}}{\equiv} \Box \forall yi.\{[!x]C \ \wedge\ C_0\}\, f \bullet y = z\{C' \ \wedge\ C_0\}@\tilde{w}x$$

The inference follows.

$1.\{\mathsf{T}\}x :=!x+1\{\mathsf{T}\}@x$	(Assign)
$2.\{[!x]C \wedge E \wedge x\#\, fiy\tilde{w}\}\ x :=!x+1\ \{C \wedge E \wedge x\#\, fiy\tilde{w}\}@x$	(Inv-#, Conseq)
$3.\{C \wedge E \wedge C_0\}\ fy :_z\ \{C' \wedge C_0\}@\tilde{w}x$	(App, Conseq)
$4.\{[!x]C \wedge E \wedge C_0\}x := x+1; fy :_z\ \{C' \wedge C_0\}@x\tilde{w}$	(2, 3, Seq)
$5.\{E\}\ \lambda y.(x := x+1; fy) :_u\ \{E_2\}@\emptyset$	(4, Abs, Inv)
$6.\{E\}\ \lambda y.(x := x+1; fy) :_u\ \{\mathsf{Inv}(u, C_0, x)\}@\emptyset$	(Similar to 1-5 from E_2)
$7.\{E\}\mathtt{profile}\{\nu x.(\mathsf{Inv}(u, C_0, x)\ \wedge\ E_2)\}@\emptyset$	(5, 6, LetRef in § 6.1)
$8.m \bullet () = u\{\nu x.(\mathsf{Inv}(u, C_0, \tilde{x})\ \wedge\ E_2)\} \supset m \bullet () = u\{E\}$	($\star$)
$9.\{E\}\mathtt{profile} :_u \{E\}@\emptyset$	(7, 8, ConsEval)

Above in Line 2, we note E is tame (because of $\Box$) and equivalent to $[!x]E$, hence [*Inv*] becomes applicable. Line 8 is inferred by Proposition 5.15.

6.7. **Nested Local Invariant from [34,27].** The next example uses a function with local state as an argument to another function. Let $\Omega \stackrel{\text{def}}{=} \mu f.\lambda().(f())$. Below *even*$(n)$ tests for evenness of n.

$$\begin{array}{lll}\mathtt{MeyerSieber} & \stackrel{\text{def}}{=} & \mathtt{let}\ x = \mathtt{ref}(0)\ \mathtt{in}\ \mathtt{let}\ f = \lambda().x :=!x+2 \\ & & \qquad\qquad \mathtt{in}\ (g\, f\ ;\ \mathtt{if}\ \mathit{even}(!x)\ \mathtt{then}\ ()\ \mathtt{else}\ \Omega())\end{array}$$

Note $\Omega()$ immediately diverges. Since x is local, and because g will have no way to access x except by calling f, the local invariant that x stores an even number is maintained. Hence `MeyerSieber` satisfies the judgement:

$$\{E \wedge C\}\ \mathtt{MeyerSieber}\ \{C'\} \tag{6.6}$$

where, with $x, m \notin \mathsf{fv}(C, C')$:

$$E \stackrel{\text{def}}{\equiv} \forall f.(\Box\, f \bullet ()\{\mathsf{T}\}@\emptyset\ \supset\ \Box\{C\}g \bullet f\{C'\})$$

(anchors of type Unit are omitted following Notation 2.1(6)). The judgement (6.6) says that:

> *if feeding g with a total and effect-free f always satisfies* $\{C\}g \bullet f\{C'\}$*, then* `MeyerSieber` *starting from C also terminates with the final state* C'*.*

Note such f behaves as `skip`.

For the derivation of (6.6), from the axiom for reachability in Proposition 5.17, we can derive $E \supset E'$ where

$$E' \stackrel{\text{def}}{\equiv} \forall f.(\Box\, f \bullet ()\{\mathsf{T}\}@x\ \supset\ \Box\{[!x]C \wedge x\#g\}g \bullet f\{[!x]C'\})$$

Further $\lambda().x := !x + 2$ named f satisfies both:

$$A_1 \stackrel{\text{def}}{\equiv} \Box\{\mathsf{T}\}\, f \bullet ()\{\mathsf{T}\}@x \quad \text{and} \quad A_2 \stackrel{\text{def}}{\equiv} \Box\{Even(!x)\}\, f \bullet ()\{Even(!x)\}@x$$

Then from A_1 and E', we obtain $A_1' \stackrel{\text{def}}{\equiv} \Box\{[!x]C \wedge x\#g\}g \bullet f\{[!x]C'\}$.

Using Proposition 5.17, A_1' and A_2 we obtain:

$$\{Even(!x) \wedge [!x]C \wedge E \wedge x\#gi\}M\{[!x]C' \wedge x\#i\}$$

with $M \stackrel{\text{def}}{=} \texttt{let } f = \lambda().x := !x + 2 \texttt{ in } (gf \texttt{ ; if } even(!x) \texttt{ then } () \texttt{ else } \Omega())$.

We then apply a variant of [*LetRef*] (replacing $C_0[!x/m]$ in the premise of [*LetRef*] in §4.2 with $[!x]C_0 \wedge !x = m$) to obtain

$$\{E \wedge C\} \texttt{ MeyerSieber } \{\nu x.([!x]C' \wedge x\#i)\}$$

Finally by Proposition 5.16 (noting the returned value has a base type, cf. Proposition 5.9 2-(1)), we reach $\{E \wedge C\}$ `MeyerSieber` $\{C'\}$. The full derivation is given in Appendix D.2.

6.8. **Information Hiding (5): Object.** As final example, we treat information hiding for a program with state, a small object encoded in imperative higher-order functions, taken from [27] (cf. [10, 46, 47]). The following program generates a simple object each time it is invoked.

$$\texttt{cellGen} \quad \stackrel{\text{def}}{=} \quad \lambda z. \begin{pmatrix} \texttt{let } x_{0,1} = \texttt{ref}(z) \texttt{ in let } y = \texttt{ref}(0) \texttt{ in} \\ \begin{pmatrix} \lambda().\texttt{if } even(!y) \texttt{ then } !x_0 \texttt{ else } !x_1, \\ \lambda w.(y := !y + 1 \texttt{ ; } x_{0,1} := w) \end{pmatrix} \end{pmatrix}$$

The object has a getter and a setter method. Instead of having one local variable, it uses two with the same content, of which one is read at each odd-turn of the "read" requests, another at each even-turn. When writing, it writes the same value to both. Since having two variables in this way does not differ from having only one observationally, we expect the following judgement to hold `cellGen`:

$$\{\mathsf{T}\} \texttt{ cellGen} :_u \{CellGen(u)\} \tag{6.7}$$

where we set:

$$\begin{aligned} CellGen(u) \quad &\stackrel{\text{def}}{\equiv} \quad \Box\forall zi.u \bullet z = o\{\nu x.(Cell(o,x) \wedge !x = z \ \wedge \ o\#i \ \wedge \ x = o)\}@\emptyset \\ Cell(o,x) \quad &\stackrel{\text{def}}{\equiv} \quad \Box\forall v.\{!x = v\}\pi_1(o) \bullet () = z\{z = v \wedge !x = v\}@\emptyset \ \wedge \Box\forall w.\pi_2(o) \bullet w\{!x = w\}@x \end{aligned}$$

$Cell(o,x)$ says that $\pi_1(o)$, the getter of o, returns the content of a local variable x; and $\pi_2(o)$, the setter of o, writes the received value to x. Then $CellGen(u)$ says that, when u is invoked with a value, say z, an object is returned with its initial fresh local state initialised to z. Note both specifications only mention a single local variable. A straightforward derivation of (6.7) uses $!x_0 = !x_1$ as the invariant to erase x_1: then we α-converts x_0 to x to obtain the required assertion $Cell(o,x)$. See Appendix D.3 for full inferences.

7. EXTENSION, RELATED WORK AND FUTURE TOPICS

For lack of space, detailed comparisons with existing program logics and reasoning methods, in particular with Clarke's impossibility result, Spatial Logic [11] (which contain a hiding quantifier used in a concurrency setting), as well as other logics such as LCF, Dynamic logic, higher-order logic and specification logic are left to our past papers [6, 22, 24, 25]. Below we focus on directly related work that treats locality and freshness in higher-order languages.

7.1. **Three Completeness Results.** We discuss completeness properties of the proposed logic. A strong completeness property called *descriptive completeness* is studied in [23]. Descriptive completeness means that characteristic assertions are provable for each program (i.e. an assertion characterising a program's behaviour uniquely up to observational congruence). We have shown [23] that this property implies two other completeness properties in our base logic, *relative completeness* (which says that provability and validity of judgements coincide, i.e. completeness relative to an oracle which can decide the validity of formulae in the assertion language) and *observational completeness* (which says that validity precisely characterises the standard contextual equivalence).

For lack of space, we only state the latter, which we regard as a basic semantic property of the logic.

Write $\cong$ for the standard contextual congruence for programs [46]; further write $M_1 \cong_{\mathcal{L}} M_2$ to mean ($\models \{C\}M_1 :_u \{C'\}$ iff $\models \{C\}M_2 :_u \{C'\}$). We have:

Theorem 7.1 (observational completeness)**.** *For each* $\Gamma;\Delta \vdash M_i : \alpha$ *(*$i = 1,2$*), we have* $M_1 \cong_{\mathcal{L}} M_2$ *iff* $M_1 \cong M_2$.

The proofs of descriptive, observational and relative completeness follow [23] and are detailed in [5].

7.2. **Local Variables in Hoare Logic.** To our knowledge, Hoare and Wirth [19] are the first to present a rule for local variable declaration. In our notation, their rule is written as follows.

$$[\mathit{Hoare\text{-}Wirth}]\ \frac{\{C \wedge x \neq \tilde{y}\}\, P\, \{C'\} \quad x \notin \mathsf{fv}(C') \cup \{\tilde{y}\}}{\{C[e/!x]\}\ \mathtt{new}\ x := e\ \mathtt{in}\ P\ \{C'\}}$$

Because this rule assumes references are never exported beyond their original scope, there is no need to have x in C'. Since aliasing is not permitted in [19] either, we can also dispense with $x \neq \tilde{y}$ in the premise. [*LetRef*] in § 6.2 differs from [*Hoare-Wirth*] in that the former can treat aliased references, higher-order procedures and new references generation extruded beyond their original scope. [*Hoare-Wirth*] is derivable from [*LetRef*], [*Assign*] and ν-elimination in Prop. 5.16.

Among the studies of verification methods for ML-like languages [2, 38], *Extended ML* [57] is a formal development framework for Standard ML. A specification is given by combining module signatures and algebraic axioms. Correctness of an implementation w.r.t. a specification is verified by incremental syntactic transformations. *Larch/ML* [61] is a design proposal of a Larch-based interface language for ML. Integration of typing and interface specification is the main focus of the proposal in [61]. These two works do not (aim to) offer a program logic with compositional proof rules; nor do either of these works treat specifications for functions with dynamically generated references.

7.3. **Related Work and Future Topics.**

Reasoning Principles for Functions with Local State. There is a long tradition of studying equivalences over higher-order programs with local state. Meyer and Sieber [34] present examples and reasoning principles based on denotational semantics. Mason, Talcott and others [26, 31, 32] investigate equational axioms for an untyped version of the language treated in the present paper, including local invariance. Pitts and Stark [48, 49, 59] present powerful operational reasoning principles for the same ML-fragment considered here, including reasoning principle for local invariance at higher-order types [49]. Our axioms for information hiding in § 5, which capture a basic pattern of programming with local state, are closely related with these reasoning principles. Our logic differs in that its aim is to offer a method for describing and validating properties of programs beyond program equivalence. Equational and logical approaches are complimentary: Theorem 7.1 offers a basis for integration. For example, we may consider deriving a property of the optimised version M' of M: if we can easily verify $\{C\}M :_u \{C'\}$ and if we know $M \cong M'$, we can conclude $\{C\}M' :_u \{C'\}$, which is useful if M is better structured than M'.

Separation Logic. The approach by Reynolds et al. [56] represents fresh data generation by relative spatial disjointness from the original datum, using a sub-structural separating conjunction. This method captures a significant part of program properties. The proposed logic represents freshness as temporal disjointness through generic (un)reachability from arbitrary data in the initial state. The presented approach enables uniform treatment of known data types in verification, including product, sum, reference, closure, etc., through the use of anchors, which matches the observational semantics precisely: we have examined this point through several examples, including objects from [27], circular lists from [29], and tree-, dag- and graph-copy from [9]. These results will be reported in future expositions. Reynolds [56] criticises the use of reachability for describing data structures, taking in-place reversal of a linear list as an example. Following § 6, tractable reasoning is possible for such examples using reachability combined with [*Inv*] and located assertions, see [62].

Birkedal et al. [8] present a "separation logic typing" for a variant of Idealised Algol where types are constructed from formulae of disjunction-free separation logic. The typing system uses subtyping calculated via categorical semantics, the focus of their study. The work [7] extends separation logic with higher-order predicates (higher-order frame rule), and demonstrates how the extension helps modular reasoning about priority queues. Both works consider neither exportable fresh reference generation nor higher-order/stored procedures in full generality, so it would be difficult to represent assertions and validate the examples in § 6. Examining the use of higher-order predicate abstraction in the present logic is an interesting future topic.

Other Hoare Logics. Names have been used in Hoare logic since early work by Kowaltowski [28], and are found in the work by von Oheimb [60], Leavens and Baker [30] and Abadi and Leino [3], for treating parameter passing and return values. These works do not treat higher-order procedures and data types, which are uniformly captured in the present logic along with parameters and return values through the use of names. This generality comes from the fact that a large class of program behaviour is faithfully represented as name passing processes which interact at names: our assertion language offers a concise way to describe such interactive behaviour in a logical framework.

Nanevski et al. [42, 43] study Hoare Type Theory (HTT) which combines dependent types and Hoare triples with anchors based on a monadic understanding of computation. HTT aims to provide an effective general framework which unifies standard static checking techniques with logical verification. Their system emphasises the clean separation between static validation and assertions. In their later work [42], the integration of programs and specifications in HTT is further pursued by introducing local state. Because of their basis in type theory, one interesting aspect is that their

"Hoare Triple" of the form "$\{P\}x : A\{Q\}$" is in fact a *type* and that A can contain an arbitrary complex specification. Note that the use of type theory does prohibit potentially useful assertions about circular data structures and references (this is called a "smallness" condition). The use of monad in their logic poses a question whether if we equip the underlying programming language with monad what reasoning principles we may obtain as a refinement of the present program logic.

Reus and Streicher [54] present a Hoare logic for a simple language with higher-order stored procedures, extended in [53], with primitives for the dynamic allocation and de-allocation of references. Soundness is proved with denotational methods, but completeness is not proved. Their assertions contain quoted programs, which is necessary to handle recursion via stored functions. Their language does not allow procedure parameters and general reference creation.

No work mentioned in this section studies local invariance in the context of program logics.

Dynamic and Evaluation Logics. Dynamic Logic [16], introduced by Pratt [52] and studied by Harel and others [15], uses programs and predicates on them as part of formulae, facilitating detailed specification of various programs properties such as (non-)termination, or more intensional features. As far as we know, higher-order procedures and local state have not been treated in Dynamic Logic, even though we believe part of the proposed method to treat higher-order functions would work consistently in their framework.

Evaluation Logic, introduced by Pitts [50] and studied by Moggi [39, 40], is a typed logic for higher-order programs based on the metalanguage for computational monads which permits statements about the evaluation of programs to values using evaluation modalities. Recently Mossakowski et. al [41] studied a generic framework for reasoning about purity [44] and effects based on a monad-based dynamic logic which is similar to Evaluation Logic. Evaluation logic is closely related to the present logic in that it is based on the decomposition of semantic points into values and computation and that it captures applications as part of the logic even though the approach of Evaluation Logic is based on denotations. Evaluation Logic has uniformity in that it does not use separate judgements such as Hoare triples. Evaluation Logic also includes expressions involving applications as part of terms. Thus its assertion language already includes judgements for programs.

The logic studied in the present paper distinguishes formulae for evaluation in the logical language (evaluation formulae) from judgements for programs (pre/post conditions together with an anchor). This is motivated by our wish to have the assertion language separate (independent) from programs, which we believe to fit such engineering purposes as design-by-contract (where one wishes to have abstract description of behaviour *before* we construct programs). This aspect of the present logic is closely related with its compositionality: we wish to build assertions for a program from those for its subprograms, and if one of its subprograms, say M, allows the same assertion as another program, say M', then we can *replace* M by M' and still obtain the same assertion for the whole program. Separating the assertion language from programs is also vital for verification of multi-language programs. We believe that it is a meaningful topic to explore a uniform treatment of both assertions for evaluations and judgements for programs, while keeping the key features of the present logic.

Meta-Logical Study on Freshness. Freshness of names has recently been studied from the viewpoint of formalising binding relations in programming languages and computational calculi. Pitts and Gabbay [12, 51] extend first-order logic with constructs to reason about freshness of names based on the theory of permutations. The key syntactic additions are the (inter-definable) "fresh" quantifier И and the freshness predicate #, mediated by a swapping (finite permutation) predicate. Miller and Tiu [35] are motivated by the significance of generic (or eigen-) variables and quantifiers at the level of both formulae and sequents, and split universal quantification in two, introduce a self-dual freshness quantifier ∇ and develop the corresponding sequent calculus of Generic Judgements. While these logics are not program logics, their logical machinery may well be usable in the present context. As noted in Proposition 5.12, reasoning about $\hookrightarrow$ or # is tantamount to reasoning about $\rhd$, which denotes the support (the semantic notion of freely occurring locations) of a datum/program. A characterisation of support by the swapping operation may lead to deeper understanding of reachability axiomatisations.

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Appendix A. Typing Rules

The typing rules are standard [46], and listed in Figure 2 for reference (we list only two first-order operations). We take the equi-isomorphic approach [46] for recursive types. In the first rule of Figure 2, $\mathtt{c}^{C}$ indicates a constant $\mathtt{c}$ has a base type C.

We also list the typing rules for terms and formulae in Figure 3.

$$[\mathit{Var}]\ \frac{-}{\Gamma, x:\alpha \vdash x:\alpha} \quad [\mathit{Label}]\ \frac{-}{\Gamma\cdot l:\alpha \vdash l:\alpha} \quad [\mathit{Constant}]\ \frac{-}{\Gamma \vdash \mathtt{c}:C}$$

$$[\mathit{Add}]\ \frac{\Gamma \vdash M_{1,2}:\mathsf{Nat}}{\Gamma \vdash M_1 + M_2 : \mathsf{Nat}} \quad [\mathit{Eq}]\ \frac{\Gamma \vdash M_{1,2}:\mathsf{Nat}}{\Gamma \vdash M_1 = M_2 : \mathsf{Bool}}$$

$$[\mathit{If}]\ \frac{\Gamma \vdash M:\mathsf{Bool} \quad \Gamma \vdash N_i:\alpha_i\ (i=1,2)}{\Gamma \vdash \mathtt{if}\ M\ \mathtt{then}\ N_1\ \mathtt{else}\ N_2 : \alpha}$$

$$[\mathit{Abs}]\ \frac{\Gamma, x:\alpha \vdash M:\beta}{\Gamma \vdash \lambda x^{\alpha}.M : \alpha \Rightarrow \beta} \quad [\mathit{App}]\ \frac{\Gamma \vdash M:\alpha\Rightarrow\beta \quad \Gamma \vdash N:\alpha}{\Gamma \vdash MN:\beta}$$

$$[\mathit{Rec}]\ \frac{\Gamma, x:\alpha\Rightarrow\beta \vdash \lambda y^{\alpha}.M:\alpha\Rightarrow\beta}{\Gamma \vdash \mu x^{\alpha\Rightarrow\beta}.\lambda y^{\alpha}.M:\alpha\Rightarrow\beta} \quad [\mathit{Iso}]\ \frac{\Gamma \vdash M:\alpha \quad \alpha\approx\beta}{\Gamma \vdash M:\beta}$$

$$[\mathit{Deref}]\ \frac{\Gamma \vdash M:\mathsf{Ref}(\alpha)}{\Gamma \vdash !M:\alpha} \quad [\mathit{Assign}]\ \frac{\Gamma \vdash M:\mathsf{Ref}(\alpha) \quad \Gamma \vdash N:\alpha}{\Gamma \vdash M := N:\mathsf{Unit}}$$

$$[\mathit{Ref}]\ \frac{\Gamma \vdash V:\alpha}{\Gamma \vdash \mathtt{ref}(V):\mathsf{Ref}(\alpha)} \quad [\mathit{New}]\ \frac{\Gamma \vdash M:\alpha \quad \Gamma, x:\mathsf{Ref}(\alpha) \vdash N:\beta}{\Gamma \vdash \mathtt{new}\ x := M\ \mathtt{in}\ N:\beta}$$

$$[\mathit{Inj}]\ \frac{\Gamma \vdash M:\alpha_i}{\Gamma \vdash \mathtt{in}_i(M):\alpha_1+\alpha_2} \quad [\mathit{Case}]\ \frac{\Gamma \vdash M:\alpha_1+\alpha_2 \quad \Gamma, x_i:\alpha_i \vdash N_i:\beta}{\Gamma \vdash \mathtt{case}\ M\ \mathtt{of}\ \{\mathtt{in}_i(x_i^{\alpha_i}).N_i\}_{i\in\{1,2\}}:\beta}$$

$$[\mathit{Pair}]\ \frac{\Gamma \vdash M_i:\alpha_i\ (i=1,2)}{\Gamma \vdash \langle M_1, M_2\rangle:\alpha_1\times\alpha_2} \quad [\mathit{Proj}]\ \frac{\Gamma \vdash M:\alpha_1\times\alpha_2}{\Gamma \vdash \pi_i(M):\alpha_i\ (i=1,2)}$$

Figure 2: Typing Rules

$$\frac{-}{\Gamma \vdash x:\Gamma(x)} \quad \frac{-}{\Gamma \vdash \mathsf{n}:\mathsf{Nat}} \quad \frac{-}{\Gamma \vdash \mathsf{t},\mathsf{f}:\mathsf{Bool}} \quad \frac{-}{\Gamma \vdash l:\Gamma(l)} \quad \frac{\Gamma \vdash e:\mathsf{Bool}}{\Gamma \vdash \neg e:\mathsf{Bool}}$$

$$\frac{\Gamma \vdash e_i:\alpha_i}{\Gamma \vdash \langle e_1,\ e_2\rangle:\alpha_1\times\alpha_2} \quad \frac{\Gamma \vdash e:\alpha_i}{\Gamma \vdash \mathsf{inj}_i^{\alpha_1+\alpha_2}(e):\alpha_1+\alpha_2} \quad \frac{\Gamma \vdash e:\mathsf{Ref}(\alpha)}{\Gamma \vdash !e:\alpha}$$

$$\frac{\Gamma \vdash e_i:\alpha_i}{\Gamma \vdash e_1 = e_2} \quad \frac{\Gamma \vdash C}{\Gamma \vdash \neg C} \quad \frac{\Gamma \vdash C_{1,2}}{\Gamma \vdash C_1 \star C_2}\star\in\{\wedge,\vee,\supset\} \quad \frac{\Gamma\cdot x:\alpha \vdash C}{\Gamma \vdash \mathcal{Q}x^{\alpha}.C}\ \mathcal{Q}\in\{\forall,\exists\}$$

$$\frac{\Gamma\cdot x:\mathsf{Ref}(\alpha) \vdash C}{\Gamma \vdash \mathcal{Q}x.C}\ \mathcal{Q}\in\{\nu,\overline{\nu}\} \quad \frac{\Gamma \vdash C}{\Gamma \vdash \mathcal{Q}\mathrm{X}.C}\ \mathcal{Q}\in\{\forall,\exists\} \quad \frac{\Gamma \vdash e:\mathsf{Ref}(\alpha) \quad \Gamma \vdash C}{\Gamma \vdash [!e]C} \quad \frac{\Gamma \vdash e:\mathsf{Ref}(\alpha) \quad \Gamma \vdash C}{\Gamma \vdash \langle !e\rangle C}$$

$$\frac{\Gamma \vdash e_1:\alpha\Rightarrow\beta \quad \Gamma \vdash e_2:\alpha \quad \Gamma\cdot z:\beta \vdash C}{\Gamma \vdash e_1\bullet e_2 = z\{C\}} \quad \frac{\Gamma \vdash C}{\Gamma \vdash \Box C} \quad \frac{\Gamma \vdash C}{\Gamma \vdash \Diamond C}$$

$$\frac{\Gamma \vdash e:\alpha \quad \Gamma \vdash e':\mathsf{Ref}(\beta)}{\Gamma \vdash e \hookrightarrow e'} \quad \frac{\Gamma \vdash e:\mathsf{Ref}(\alpha) \quad \Gamma \vdash e':\beta}{\Gamma \vdash e \# e'}$$

Figure 3: Typing rules for terms and formulae

$$[Var]\ \frac{-}{\{C[x/u]\}\ x :_u \{C\}@\emptyset} \qquad [Const]\ \frac{-}{\{C[\mathsf{c}/u]\}\ \mathtt{c} :_u \{C\}@\emptyset}$$

$$[Add]\ \frac{\{C\} M_1 :_{m_1} \{C_0\}@\tilde{e}_1 \quad \{C_0\} M_2 :_{m_2} \{C'[m_1+m_2/u]\}@\tilde{e}_2}{\{C\} M_1 + M_2 :_u \{C'\}@\tilde{e}_1\tilde{e}_2}$$

$$[In_1]\ \frac{\{C\}\ M :_v \{C'[\mathsf{inj}_1(v)/u]\}@\tilde{e}}{\{C\}\ \mathtt{inj}_1(M) :_u \{C'\}@\tilde{e}}$$

$$[Case]\ \frac{\{C^{-\tilde{x}}\}\ M :_m \{C_0^{-\tilde{x}}\}@\tilde{e}_1 \quad \{C_0[\mathsf{inj}_i(x_i)/m]\}\ M_i :_u \{C'^{\,-\tilde{x}}\}@\tilde{e}_2}{\{C\}\ \mathtt{case}\ M\ \mathtt{of}\ \{\mathtt{in}_i(x_i).M_i\}_{i\in\{1,2\}} :_u \{C'\}@\tilde{e}_1\tilde{e}_2}$$

$$[Proj_1]\ \frac{\{C\}\ M :_m \{C'[\pi_1(m)/u]\}@\tilde{e}}{\{C\}\ \pi_1(M) :_u \{C'\}@\tilde{e}}$$

$$[Pair]\ \frac{\{C\}\ M_1 :_{m_1} \{C_0\}@\tilde{e}_1 \quad \{C_0\}\ M_2 :_{m_2} \{C'[\langle m_1, m_2\rangle/u]\}@\tilde{e}_2}{\{C\}\ \langle M_1, M_2\rangle :_u \{C'\}@\tilde{e}_1\tilde{e}_2}$$

$$[Abs]\frac{\{C \wedge A^{-x\tilde{i}}\}\ M :_m \{C'\}@\tilde{e}}{\{A\}\ \lambda x.M :_u \{\Box\forall x\tilde{i}.(\{C\}u \bullet x = m\{C'\})\}@\emptyset} \qquad [Rec\text{-}Ren]\ \frac{\{A^{-x}\}\ \lambda y.M :_u \{B\}@\tilde{e}}{\{A^{-x}\}\ \mu x.\lambda y.M :_u \{B[u/x]\}@\tilde{e}}$$

$$[App]\frac{\{C\} M :_m \{C_0\}@\tilde{e} \quad \{C_0\} N :_n \{m \bullet n = u\{C'\}\tilde{e}_2\}@\tilde{e}_1}{\{C\}\ MN :_u \{C'\}@\tilde{e}\tilde{e}_1\tilde{e}_2}$$

$$[If]\ \frac{\{C\}\ M :_b \{C_0\}@\tilde{e}_1 \quad \{C_0[\mathsf{t}/b]\}\ M_1 :_u \{C'\}@\tilde{e}_2 \quad \{C_0[\mathsf{f}/b]\}\ M_2 :_u \{C'\}@\tilde{e}_2}{\{C\}\ \mathtt{if}\ M\ \mathtt{then}\ M_1\ \mathtt{else}\ M_2 :_u \{C'\}@\tilde{e}_1\tilde{e}_2}$$

$$[Deref]\ \frac{\{C\}\ M :_m \{C'[!m/u]\}@\tilde{e}}{\{C\}\ !M :_u \{C'\}@\tilde{e}}$$

$$[Assign]\ \frac{\{C\}\ M :_m \{C_0\}@\tilde{e}_1 \quad \{C_0\}\ N :_n \{C'\{\!|n/\,!m|\!\}\}@\tilde{e}_2 \quad C_0\ \supset\ m = e'}{\{C\}\ M := N\ \{C'\}@\tilde{e}_1\tilde{e}_2 e'}$$

$$[Ref]\ \frac{\{C\}\ M :_m \{C'\}@\tilde{e}}{\{C\}\ \mathtt{ref}(M) :_u \{\nu x.(C'[!u/m] \wedge u \# i^{\mathrm{X}} \wedge u = x)\}@\tilde{e}}$$

We require C' is thin w.r.t. m in [*Case*] and [*Deref*], and C' is thin with respect to m, n in [*App*, *Assign*].

Figure 4: Derived Compositional Rules for Located Assertions

APPENDIX B. PROOF RULES

$$[Inv]\frac{\{C\}\ M :_m \{C'\}@\tilde{w}}{\{C \wedge [!\tilde{w}]C_0\}\ M :_m \{C' \wedge [!\tilde{w}]C_0\}@\tilde{w}}$$

$$[Inv\text{-}Val]\ \frac{\{C\}\ V :_m \{C'\}@\emptyset}{\{C \wedge C_0\}\ V :_m \{C' \wedge C_0\}@\emptyset}$$

$$[Inv\text{-}\#]\ \frac{\{C\}\ M :_m \{C'\}@x \quad \text{no dereference occurs in } \tilde{e}}{\{C \wedge x\#\tilde{e}\}\ M :_m \{C' \wedge x\#\tilde{e}\}@x}$$

$$[Cons]\ \frac{C \supset C_0 \quad \{C_0\}\ M :_u \{C_0'\}@\tilde{e} \quad C_0' \supset C'}{\{C\}\ M :_u \{C'\}@\tilde{e}}$$

$$[Cons\text{-}Eval]\ \frac{\begin{array}{c}\{C_0\}\ M :_m \{C_0'\}@\tilde{e} \quad x \text{ fresh; } \ \tilde{i} \text{ auxiliary} \\ \forall\tilde{i}.\{C_0\}x\bullet() = m\{C_0'\} \supset \forall\tilde{i}.\{C\}x\bullet() = m\{C'\}\end{array}}{\{C\}\ M :_m \{C'\}@\tilde{e}}$$

$$[\wedge\text{-}\supset]\ \frac{\{C \wedge A\}\ V :_u \{C'\}@\emptyset}{\{C\}\ V :_u \{A \supset C'\}@\emptyset} \qquad [\supset\text{-}\wedge]\ \frac{\{C\}\ M :_u \{A \supset C'\}@\tilde{e}}{\{C \wedge A\}\ M :_u \{C'\}@\tilde{e}}$$

$$[\vee\text{-}Pre]\ \frac{\{C_1\}\ M :_u \{C\}@\tilde{e} \quad \{C_2\}\ M :_u \{C\}@\tilde{e}}{\{C_1 \vee C_2\}\ M :_u \{C\}@\tilde{e}} \qquad [\wedge\text{-}Post]\ \frac{\{C\}\ M :_u \{C_1\}@\tilde{e} \quad \{C\}\ M :_u \{C_2\}@\tilde{e}}{\{C\}\ M :_u \{C_1 \wedge C_2\}@\tilde{e}}$$

$$[Aux_{\exists}]\ \frac{\{C\}\ M :_u \{C'^{-i}\}@\tilde{e}}{\{\exists i.C\}\ M :_u \{C'\}@\tilde{e}}$$

$$[Aux_{\forall}V]\ \frac{\{C^{-i}\}\ V :_u \{C'\}@\tilde{e}}{\{C\}\ V :_u \{\forall i^{\alpha}.C'\}@\tilde{e}} \qquad [Aux_{\forall}]\ \frac{\{C^{-i}\}\ M :_u \{C'\}@\tilde{e} \quad \alpha \text{ is of a base type.}}{\{C\}\ M :_u \{\forall i^{\alpha}.C'\}@\tilde{e}}$$

$$[Aux_{inst}]\ \frac{\{C(i^{\alpha})\}\ M :_u \{C'(i^{\alpha})\}@\tilde{e} \quad \alpha \text{ atomic}}{\{C(\mathsf{c}^{\alpha})\}\ M :_u \{C'(\mathsf{c}^{\alpha})\}@\tilde{e}} \qquad [Aux_{abst}]\ \frac{\forall \mathsf{c}^{\alpha}.\ \{C(\mathsf{c}^{\alpha})\}\ M :_u \{C'(\mathsf{c}^{\alpha})\}@\tilde{e}}{\{C(i^{\alpha})\}\ M :_u \{C'(i^{\alpha})\}@\tilde{e}}$$

$$[Weak]\ \frac{\{C\}\ M :_m \{C'\}@\tilde{e}}{\{C\}\ M :_m \{C'\}@\tilde{e}e'} \qquad [Thinning]\ \frac{\{C \wedge !e' = i\}\ M :_m \{C' \wedge !e' = i\}@\tilde{e}e' \quad i \text{ fresh}}{\{C\}\ M :_m \{C'\}@\tilde{e}}$$

Figure 5: Structural Rules for Located Judgements.

B.1. **Proofs of Soundness.** We prove the soundness theorem. We use the following lemma.

Lemma B.1 (Substitution and Thinning)**.**

(1) $\mathcal{M} \models C \wedge u = V$ *iff* $\mathcal{M}[u : V] \models C$.

(2) *Suppose* $m, m_1, m_2 \notin \mathsf{fv}(\mathcal{M}, C) \cup \{u, v\}$. *Then:*

  (a) *If* $(\nu\tilde{l})\mathcal{M}[m : V][u : \mathtt{inj}_i(m)] \models C$*, then* $(\nu\tilde{l})\mathcal{M}[u : \mathtt{inj}_i(V)] \models C$.

  (b) *If* $(\nu\tilde{l})\mathcal{M}[m : V][u : \pi_1(m)] \models C$*, then* $(\nu\tilde{l})\mathcal{M}[u : \pi_1(V)] \models C$.

  (c) *If* $(\nu\tilde{l})\mathcal{M}[m_1 : V_1][m_2 : V_2][u : \langle m_1,\ m_2\rangle] \models C$*, then* $(\nu\tilde{l})\mathcal{M}[u : \langle V_1,\ V_2\rangle] \models C$.

  (d) *Suppose* $l \notin \mathsf{fl}(\mathcal{M})$*. Then* $(\nu\tilde{l}l)\mathcal{M}[m : l][u : V][l \mapsto V] \models C$ *implies* $(\nu\tilde{l})\mathcal{M}[u : V] \models C$

  (e) *Suppose* $l \notin \mathsf{fl}(\mathcal{M})$ *and* $\mathsf{fv}(V) \cup \mathsf{fl}(V) = \emptyset$*. Then* $(\nu l)\mathcal{M}[m : l][l \mapsto V] \models C$ *implies* $\mathcal{M} \models C$.

  (f) *Suppose* $l \notin \mathsf{fl}(\mathcal{M})$ *and* $\mathsf{fv}(V) \cup \mathsf{fl}(V) = \emptyset$*. Then* $\mathcal{M}[m : l][l \mapsto V] \models C$ *implies* $\mathcal{M} \models C$.

(3) $\mathcal{M} \models \exists m.(\langle !x\rangle(C \wedge !x = m) \wedge m = e)$ *iff* $\mathcal{M}[x \mapsto [\![e]\!]_{\xi,\sigma}] \models C$

$$[New]\ \frac{\{C\}\ M :_m \{C_0\}@\tilde{e}_1 \quad \{C_0[!x/m] \wedge x\#\tilde{e}\}\ N :_u \{C'\}@\tilde{e}_2 x \quad x \notin \mathsf{fpn}(\tilde{e})}{\{C\}\ \mathtt{let}\ x = \mathtt{ref}(M)\ \mathtt{in}\ N :_u \{\nu x.C'\}@\tilde{e}_1\tilde{e}_2}$$

$$[Rec]\ \frac{\{A^{\text{-}xi} \wedge \forall j \lneq i.B(j)[x/u]\}\ \lambda y.M :_u \{B(i)^{\text{-}x}\}@\tilde{e}}{\{A\}\ \mu x.\lambda y.M :_u \{\forall i.B(i)\}@\tilde{e}}$$

$$[Let]\ \frac{\{C\}\ M :_x \{C_0\}@\tilde{e} \quad \{C_0\}\ N :_u \{C'\}@\tilde{e}'}{\{C\}\ \mathtt{let}\ x = M\ \mathtt{in}\ N :_u \{C'\}@\tilde{e}\tilde{e}'} \qquad [LetOpen]\ \frac{\{C\}\ M :_x \{\nu\tilde{y}.C_0\}@\tilde{e}_1 \quad \{C_0\}\ N :_u \{C'\}@\tilde{e}_2}{\{C\}\ \mathtt{let}\ x = M\ \mathtt{in}\ N :_u \{\nu\tilde{y}.C'\}@\tilde{e}_1\tilde{e}_2}$$

$$[Simple]\frac{-}{\{C[e/u]\}e :_u \{C\}@\tilde{e}} \qquad [IfH]\ \frac{\{C \wedge e\}\ M_1\ \{C'\}@\tilde{e} \quad \{C \wedge \neg e\}\ M_2\ \{C'\}@\tilde{e}}{\{C\}\ \mathtt{if}\ e\ \mathtt{then}\ M_1\ \mathtt{else}\ M_2\ \{C'\}@\tilde{e}}$$

$$[AppS]\frac{C \supset \{C\}\, e \bullet (e_1..e_n) = u\{C'\}@\tilde{e}'}{\{C\}\ e(e_1..e_n) :_u\ \{C'\}\ @\,\tilde{e}'} \qquad [Subs]\ \frac{\{C\}\ M :_u \{C'\}@\tilde{e}' \quad u \notin \mathsf{fpn}(e)}{\{C[e/i]\}\ M :_u \{C'[e/i]\}@\tilde{e}'}$$

$$[Seq]\ \frac{\{C\}\ M\ \{C_0\}@\tilde{e} \quad \{C_0\}\ N\ \{C'\}@\tilde{e}'}{\{C\}\ M;N\ \{C'\}@\tilde{e}\tilde{e}'} \qquad [Seq\text{-}Inv]\ \frac{\{C_1\}\ M\ \{C_1'\}@\tilde{e_1} \quad \{C_2\}\ N\ \{C_2'\}@\tilde{e_2}}{\{C_1 \wedge [!\tilde{e_1}]C_2\}\ M;N\ \{C_2' \wedge \langle !\tilde{e_2}\rangle C_1'\}@\tilde{e_1}\tilde{e_2}}$$

C' is thin w.r.t. m in [*New* and x in [*Let*, *LetOpen*]. C_1' and C_2 are tame in [*Seq-Inv*].

Figure 6: Other Located Proof Rules.

Proof. For (1), we derive:

$$\begin{array}{rcl}
\mathcal{M} \models C \wedge u = V & \equiv & \mathcal{M} \models C \wedge \mathcal{M} \models u = V \\
 & \equiv & \mathcal{M} \models C \wedge \mathcal{M}[u:V] \approx \mathcal{M} \\
 & \equiv & \mathcal{M}[u:V] \models C
\end{array}$$

(2) is mechanical by induction on C. We only show some interesting cases. Others are similar. For (2-a), let $\mathcal{M}_1 \equiv (\nu\tilde{l})\mathcal{M}[m:V][u:\mathtt{inj}_i(m)]$ and $\mathcal{M}_2 \equiv (\nu\tilde{l})\mathcal{M}[u:\mathtt{inj}_i(V)]$.

Assume $C = e_1 = e_2$. Then, with w fresh and $m \notin \mathsf{fv}(e_1, e_2)$, we have $\mathcal{M}_1[w:e_1] \approx \mathcal{M}_1[w:e_2]$ iff $\mathcal{M}_2[w:e_1] \approx \mathcal{M}_2[w:e_2]$. Hence $\mathcal{M}_1 \models e_1 = e_2$ iff $\mathcal{M}_2 \models e_1 = e_2$.

Assume $C = \forall x.C'$. Then we have:

$$\begin{array}{rcll}
\mathcal{M}_1 \models \forall x.C' & \equiv & \forall L \in \mathcal{F}.\mathcal{M}_1[x:L] \models C' & \\
 & \equiv & \forall L' \in \mathcal{F}.\mathcal{M}_2[x:L'] \models C' & \text{such that } m \notin \mathsf{fv}(L') \\
 & \equiv & \mathcal{M}_2 \models \forall x.C' &
\end{array}$$

Assume $C = \nu x.C'$. Then we have:

$$\begin{array}{rcll}
\mathcal{M}_1 \models \nu x.C' & \equiv & \exists \mathcal{M}_0.(\mathcal{M}_1 \approx (\nu l)\mathcal{M}_0 \wedge \mathcal{M}_0[x:l] \models C') & \\
 & \equiv & \exists \mathcal{M}_0'.(\mathcal{M}_2 \approx (\nu l)\mathcal{M}_0' \wedge \mathcal{M}_0'[x:l] \models C') & \text{such that } \mathcal{M}_0' \equiv \mathcal{M}_0/m[V/m] \\
 & \equiv & \mathcal{M}_2 \models \forall x.C' &
\end{array}$$

Assume $C = x \bullet y = z\{C'\}$. Then we derive:

$$\begin{array}{rcl}
\mathcal{M}_1 \models x \bullet y = z\{C'\} & \equiv & \exists \mathcal{M}_1'.(\mathcal{M}_1[z:xy] \Downarrow \mathcal{M}_1' \wedge \mathcal{M}_1' \models C') \\
 & & \qquad\qquad \text{with } \mathcal{M}_1' \equiv (\nu\tilde{l})\mathcal{M}'[m:V][u:\mathtt{inj}_i(m)] \\
 & \equiv & \exists \mathcal{M}_2'.(\mathcal{M}_2[z:xy] \Downarrow \mathcal{M}_2' \wedge \mathcal{M}_2' \models C') \\
 & & \qquad\qquad \text{with } \mathcal{M}_2' \equiv (\nu\tilde{l})\mathcal{M}'[u:\mathtt{inj}_i(V)] \text{ and (IH)} \\
 & \equiv & \mathcal{M}_2 \models x \bullet y = z\{C'\}
\end{array}$$

Others (b-f) are similar. (3) is from [6]. □

Below we write:

$$\mathcal{M} \Downarrow \mathcal{M}' \models C' \ \text{ for } \ \mathcal{M} \Downarrow \mathcal{M}' \wedge \mathcal{M}' \models C'$$

We start with [*Var*].

$$\begin{array}{lll} \mathcal{M} \models C[x/u] & \Rightarrow & \mathcal{M} \models C \wedge u = x \\ & \Rightarrow & \mathcal{M}[u\!:\!x] \models C \qquad \text{Lemma B.1(1)} \end{array}$$

Similarly for [*Const*] using Lemma B.1(1). Next, [*Add*] is proved as follows:

$$\begin{array}{llll} \mathcal{M} \models C & \Rightarrow & \mathcal{M}[m_1\!:\!M_1] \Downarrow \mathcal{M}_1 \models C_0 & \text{IH} \\ & \Rightarrow & \mathcal{M}_1[m_2\!:\!M_2] \Downarrow \mathcal{M}_2 \models C'[m_1+m_2/u] & \text{IH} \\ & \Rightarrow & \mathcal{M}[m_1\!:\!M_1][m_2\!:\!M_2][u\!:\!m_1+m_2] \Downarrow \mathcal{M}' \models C' & \\ & \Rightarrow & \mathcal{M}[u\!:\!M_1+M_2] \Downarrow \mathcal{M}'/m_1m_2 \models C' & \text{Proposition 3.13 (1)} \end{array}$$

[Inj_1] is proved as:

$$\begin{array}{llll} \mathcal{M} \models C & \Rightarrow & \mathcal{M}[m\!:\!M] \Downarrow (\nu\tilde{l})\mathcal{M}'[m:V] \models C'[\mathtt{inj}_1(m)/u] & \text{IH} \\ & \Rightarrow & \mathcal{M}[m\!:\!M][u\!:\!\mathtt{inj}_1(m)] \Downarrow (\nu\tilde{l})\mathcal{M}'[m:V][u:\mathtt{inj}_1(V)] \models C' & \text{Lemma B.1(1)} \\ & \Rightarrow & (\nu\tilde{l})\mathcal{M}'[u:\mathtt{inj}_1(V)] \models C' & \text{Lemma B.1(2-a)} \\ & \Rightarrow & \mathcal{M}[u\!:\!\mathtt{inj}_1(M)] \models C' & \end{array}$$

[*Proj*] and [*Pair*] are similarly proved using Lemma B.1(2-b,c) respectively.

For [*Case*], we reason:

$$\begin{array}{ll} \mathcal{M} \models C \Rightarrow & \mathcal{M}[m\!:\!M] \Downarrow (\nu\tilde{l}')\mathcal{M}_0[m:\mathtt{inj}_i(V)] \models C_0 \\ & \quad \text{if } \mathcal{M} = (\nu\tilde{l})(\xi,\sigma),\ (\nu\tilde{l})(M\xi,\sigma) \Downarrow (\nu\tilde{l}')(\mathtt{inj}_i(V),\sigma'), \text{ and } \mathcal{M}_0 = (\xi,\sigma') \\ \Rightarrow & (\nu\tilde{l}')\mathcal{M}_0[m\!:\!\mathtt{inj}_i(V)] \models C_0 \wedge m = \mathtt{inj}_i(x_i) \\ \Rightarrow & (\nu\tilde{l}')\mathcal{M}_0[m\!:\!\mathtt{inj}_i(x_i)][u\!:\!M_i] \Downarrow (\nu\tilde{l}'')\mathcal{M}'[m\!:\!\mathtt{inj}_i(V)][u\!:\!W] \models C' \\ \Rightarrow & (\nu\tilde{l}'')\mathcal{M}'[u\!:\!W] \models C' \end{array}$$

The last line is by the thinness of C' with respect to m.

Now we reason for [*Abs*]. We note, if A is stateless (cf. Definition 3.14) and $\mathcal{M} \models A$, then: $\mathcal{M}[u\!:\!M] \Downarrow \mathcal{M}'$ with u fresh implies $\mathcal{M}' \models A$.

$$\begin{array}{ll} & \mathcal{M} \models A \supset \mathcal{M}[u\!:\!\lambda x.M] \models \Box\forall x\tilde{i}.\{C\}u \bullet x = m\{C'\} \\ \equiv & \mathcal{M} \models A \supset \mathcal{M}[u\!:\!\lambda x.M][x\!:\!N_x][\tilde{i}\!:\!\tilde{N}][k\!:\!N] \Downarrow \mathcal{M}' \ \wedge\ \mathcal{M} \approx \mathcal{M}'/x\tilde{i} \ \wedge\ \mathcal{M}' \models \{C\}u \bullet x = m\{C'\} \\ \equiv & \mathcal{M} \models A \supset ((\mathcal{M}[u\!:\!\lambda x.M][x\!:\!N_x][\tilde{i}\!:\!\tilde{N}][k\!:\!N] \Downarrow \mathcal{M}' \ \wedge\ \mathcal{M} \approx \mathcal{M}'/x\tilde{i} \ \wedge\ \mathcal{M}' \models C) \\ & \qquad \supset \mathcal{M}'[m:ux] \Downarrow \mathcal{M}'' \ \wedge\ \mathcal{M}'' \models C') \\ \equiv & \mathcal{M} \models A \supset ((\mathcal{M}[u\!:\!\lambda x.M][x\!:\!N_x][\tilde{i}\!:\!\tilde{N}][k\!:\!N] \Downarrow \mathcal{M}' \ \wedge\ \mathcal{M} \approx \mathcal{M}'/x\tilde{i} \ \wedge\ \mathcal{M}' \models C \wedge A) \\ & \qquad \supset \mathcal{M}'[m:ux] \Downarrow \mathcal{M}'' \ \wedge\ \mathcal{M}'' \models C') \\ \subset & \mathcal{M}' \models A \wedge C \supset (\mathcal{M}'[m\!:\!M] \Downarrow \mathcal{M}'' \wedge \mathcal{M}'' \models C') \end{array}$$

[*App*] is reasoned as follows. Below k fresh.

$$\begin{array}{lll} \mathcal{M} \models C & \Rightarrow & \mathcal{M}[m\!:\!M] \Downarrow \mathcal{M}_0 \models C_0 \\ & \Rightarrow & \mathcal{M}[n\!:\!N] \Downarrow \mathcal{M}_1 \models C_1 \wedge m \bullet n = n\{C'\} \\ & \Rightarrow & \mathcal{M}[m\!:\!M][n\!:\!N][u\!:\!mn] \Downarrow \mathcal{M}' \models C' \\ & \Rightarrow & \mathcal{M}[m\!:\!M][n\!:\!N][u\!:\!MN] \Downarrow \mathcal{M}' \models C' \\ & \Rightarrow & \mathcal{M}[u\!:\!MN] \Downarrow \mathcal{M}'/mn \models C' \end{array}$$

The last line is derived by the thinness of C' with respect to m,n.

For [*Deref*], we infer:

$$\begin{array}{lll}\mathcal{M} \models C & \Rightarrow & \mathcal{M}[m\!:\!M] \Downarrow \mathcal{M}' \models C'[!m/u] \\ & \Rightarrow & \mathcal{M}[m\!:\!!M] \Downarrow \mathcal{M}'/m \models C'\end{array}$$

For [*Assign*] Assume u is fresh.

$$\begin{array}{lll}\mathcal{M} \models C & \Rightarrow & \mathcal{M}[m\!:\!M] \Downarrow \mathcal{M}_0 \models C_0 \\ & \Rightarrow & \mathcal{M}_0[n\!:\!N] \Downarrow \mathcal{M}' \models C'\{\!|n/!m|\!\} \\ & \Rightarrow & \mathcal{M}'[m \mapsto n] \Downarrow \mathcal{M}'' \models C' \quad \text{Lemma B.1(3)} \\ & \Rightarrow & \mathcal{M}[u\!:\!M := N] \Downarrow \mathcal{M}''/mn[u\!:\!()] \models C' \ \wedge \ u = ()\end{array}$$

For [*Rec-Ren*],

$$\begin{array}{lll}\mathcal{M} \models A & \Rightarrow & \mathcal{M}[u\!:\!\lambda x.M] \models B \\ & \Rightarrow & \mathcal{M}[f\!:\!\mu f.\lambda x.M][u\!:\!\lambda x.M] \models A \\ & \Rightarrow & \mathcal{M}[f\!:\!\mu f.\lambda x.M][u\!:\!\mu f.\lambda x.M] \models A \\ & \Rightarrow & \mathcal{M}[u\!:\!\mu f.\lambda x.M] \models f = u \supset A \\ & \Rightarrow & \mathcal{M}[u\!:\!\mu f.\lambda x.M] \models A[u/f] \quad \text{Lemma B.1(1)}\end{array}$$

[*If*] is similar with [*Add*] using Proposition 1.

[*Ref*] appeared in the main text (the second last line uses Lemma B.1(2-d) to delete m).

We complete all cases. □

B.2. **Soundness of the Invariant Rule.** Among the structural rules, we prove the soundness of the main invariance rule, [*Inv*] in Figure 5.

Lemma B.2. *Suppose C is tame and $\mathcal{M} \models C$. Suppose $\mathcal{M} \overset{u_1..u_n}{\rightsquigarrow} \mathcal{M}'$ and $\mathcal{M} \approx \mathcal{M}/u_1..u_n$. Then $\mathcal{M}' \models C$.*

Proof. By mechanical induction on C noting it only contains evaluation formulae under $\Box$. □

Lemma B.3. *Suppose $\mathcal{M} \models [!\tilde{w}]C$ and C is tame. Then for each M and $\mathcal{M}'$ if $\mathcal{M}[u\!:\!M] \Downarrow \mathcal{M}'$ and $\mathcal{M}[z : \mathtt{let}\ \tilde{x} = !\tilde{w}\ \mathtt{in}\ \mathtt{let}\ y = M\ \mathtt{in}\ \tilde{w} := \tilde{x}] \Downarrow \mathcal{M}''$ s.t. $\mathcal{M}''/z \approx \mathcal{M}$ then we have $\mathcal{M}' \models C$.*

Proof. For simplicity we assume $\tilde{w}$ is a singleton (the general case is the same). Let $\mathcal{M} \models [!w]C$ and C be tame. Suppose $\mathcal{M}[u\!:\!M] \Downarrow \mathcal{M}'$ such that only the content of w is affected. We let with appropriate closed V_0:

$$\mathcal{M}[x :!w][y : \mathtt{ref}(V_0)][u : \mathtt{let}\ m = M\ \mathtt{in}\ (y :=!w; w := x; m)] \Downarrow \mathcal{M}'' \qquad \mathcal{M} \approx \mathcal{M}''/xyu \tag{B.1}$$

Hence by Lemma B.2 we have:

$$\mathcal{M}'' \models [!w]C \tag{B.2}$$

Further note

$$\mathcal{M}''[w \mapsto !y] \Downarrow \mathcal{M}''' \qquad \mathcal{M}' \approx \mathcal{M}'''/xy \tag{B.3}$$

By (B.2) and (B.3) we obtain $\mathcal{M}''' \models C$. By Lemma B.2 and this, we have $\mathcal{M}' \models C$, as required. □

We now prove:

Proposition B.4. *The following rule is sound.*

$$[\mathit{Inv}]\frac{\{C\}\ M :_m \{C'\}@\tilde{w} \qquad C_0 \text{ is tame}}{\{C \wedge [!\tilde{w}]C_0\}\ M :_m \{C' \wedge [!\tilde{w}]C_0\}@\tilde{w}}$$

Proof. Assume $\{C\}\ M :_u \{C'\}@\tilde{w}$. Then by definition, for each $\mathcal{M}$ such that $\mathcal{M} \models C$ we have:

$$\mathcal{M}[u\!:\!M] \Downarrow \mathcal{M}' \models C' \tag{B.4}$$

$$\mathcal{M}[z : \mathtt{let}\ \tilde{x} = !\tilde{w}\ \mathtt{in}\ \mathtt{let}\ y = ee'\ \mathtt{in}\ \tilde{w} := \tilde{x}] \Downarrow \mathcal{M}'' \text{ s.t. } \mathcal{M}''/z \approx \mathcal{M} \tag{B.5}$$

Then:

$$\begin{array}{lll} \mathcal{M} \vdash C \ \wedge\ [!\tilde{w}]C_0 & & \\ \quad \Rightarrow & \mathcal{M} \vdash [!\tilde{w}][!\tilde{w}]C_0 \qquad \text{(by the axiom } [!\tilde{w}][!\tilde{w}]C_0 \equiv [!\tilde{w}]C_0\text{)} & \\ \quad \Rightarrow & \forall \mathcal{M}', M.((\mathcal{M}[u\!:\!M] \Downarrow \mathcal{M}' \wedge & \\ & \quad \mathcal{M}[z : \mathtt{let}\ \tilde{x} = !\tilde{w}\ \mathtt{in}\ \mathtt{let}\ y = ee'\ \mathtt{in}\ \tilde{w} := \tilde{x}] \Downarrow \mathcal{M}'' \approx \mathcal{M}[z : ()] \supset \mathcal{M}' \models [!\tilde{w}]C_0) & \\ \quad \Rightarrow & \mathcal{M}' \models C' & \text{((B.4,B.5) above)} \\ \quad \Rightarrow & \mathcal{M}' \models C' \ \wedge\ [!\tilde{w}]C_0 & \end{array}$$

Hence we have $\{C \wedge [!\tilde{w}]C_0\}\ M :_m \{C' \wedge [!\tilde{w}]C_0\}@\tilde{w}$, as required. □

B.3. **Soundness of [*LetOpen*] and [*Subs*].** We prove soundness of [*LetOpen*] and [*Subs*] used in § 6.1. For [*LetOpen*] (we prove the case that $\tilde{y}$ is a singleton), we derive:

$$\begin{array}{rcll} \mathcal{M} \models C & \Rightarrow & \mathcal{M}[x\!:\!M] \Downarrow \mathcal{M}' \models \nu y.C_0 & \text{Assumption} \\ & \equiv & \mathcal{M}[x\!:\!M] \Downarrow \mathcal{M}' \wedge \exists \mathcal{M}_0.(\mathcal{M}' \approx (\nu l)\mathcal{M}_0 \wedge \mathcal{M}_0 \models C_0) & \end{array}$$

Also we have:

$$\mathcal{M}_0 \models C_0 \quad \Rightarrow \quad \mathcal{M}[u\!:\!N] \Downarrow \mathcal{M}_0' \models C' \quad \text{Assumption}$$

Combining these two, we have:

$$\begin{array}{rcl} \mathcal{M} \models C & \Rightarrow & \mathcal{M}[x\!:\!M] \Downarrow \mathcal{M}' \wedge \exists \mathcal{M}_0.(\mathcal{M}' \approx (\nu l)\mathcal{M}_0 \wedge \mathcal{M}_0[u : N] \Downarrow \mathcal{M}_0' \wedge \mathcal{M}_0' \models C') \\ & \Rightarrow & \mathcal{M}[u\!:\!\mathtt{let}\ x = M\ \mathtt{in}\ N] \Downarrow \mathcal{M}'' \wedge \mathcal{M}'' \models C' \text{ with } \mathcal{M}''/x = \mathcal{M}_0' \end{array}$$

The last line is by thinness.

For [*Subs*] (we prove the case that $\tilde{e}$ is a singleton), we have:

$$\begin{array}{rcll} \mathcal{M} \models C & \Rightarrow & \mathcal{M}[u : M] \Downarrow \mathcal{M}' \wedge \mathcal{M}' \models C & \\ & \Rightarrow & \forall \mathcal{M}_0.(\mathcal{M}_0 \models i = e \wedge \mathcal{M}_0[u : M] \Downarrow \mathcal{M}' \ \Rightarrow\ \mathcal{M}' \models i = e) & (u \notin \mathsf{fpn}(e)) \\ & \Rightarrow & \forall \mathcal{M}_0.(\mathcal{M}_0 \models (C \wedge i = e) \wedge \mathcal{M}_0[u : M] \Downarrow \mathcal{M}' \models (C' \wedge i = e)) & \end{array}$$

Appendix C. Soundness of the Axioms

This appendix lists the omitted proofs from Section 5. We first prove the basic lemma and propositions. In § C.3, we show the axioms for the content quantifications. In § C.7, we prove (AIH)-axioms.

C.1. **Proofs of Lemma 5.1.** For (1), both directions are simultaneously established by induction on C, proving for both C and its negation. If C is $e_1 = e_2$, we have, letting $\mathcal{M} \stackrel{\text{def}}{=} (\nu\tilde{l})(\xi,\sigma)$, $\delta \stackrel{\text{def}}{=} [u/v;v/u]$ and $\xi' \stackrel{\text{def}}{=} \xi\delta$:

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathcal{M} \models e_1 = e_2} \\
\Rightarrow & \mathcal{M}[x:e_1] \approx \mathcal{M}[x:e_2] \\
\Rightarrow & (\nu\tilde{l})(\xi\cdot x:[\![e_1]\!]_{(\xi,\sigma)},\sigma) \cong_{\mathsf{id}} (\nu\tilde{l})(\xi\cdot x:[\![e_2]\!]_{(\xi,\sigma)},\sigma) \\
\Rightarrow & (\nu\tilde{l})(\xi'\cdot x:[\![e_1\delta]\!]_{(\xi',\sigma)},\sigma) \cong_{\mathsf{id}} (\nu\tilde{l})(\xi'\cdot x:[\![e_2\delta]\!]_{(\xi',\sigma)},\sigma) \quad (*) \\
\Rightarrow & \mathcal{M}\delta[x:e_1\delta] \approx \mathcal{M}\rho[x:e_2\delta] \\
\Rightarrow & \mathcal{M}\delta \models (e_1 = e_2)\delta
\end{array}$$

Above $(*)$ used $[\![e_i]\!]_{(\xi,\sigma)} \stackrel{\text{def}}{=} [\![e_i\delta]\!]_{(\xi',\sigma)}$. Dually for its negation. The rest is easy by induction. (2) is by precisely the same reasoning. (3) is immediate from (1) and (2). (4) is similar, for which we again show a base case.

$$\begin{array}{lll}
\multicolumn{3}{l}{\mathcal{M}' \models e_1 = e_2} \\
\Leftrightarrow & \mathcal{M}[x:e_1] \approx \mathcal{M}[x:e_2] & \text{(By Definition)} \\
\Leftrightarrow & \mathcal{M}[x:e_1][u:e] \approx \mathcal{M}[x:e_2][u:e] & \text{(congruency of } \approx) \\
\Leftrightarrow & \mathcal{M}[u:e][x:e_1] \approx \mathcal{M}[u:e][x:e_2] & \text{(By (3))}
\end{array}$$

Dually for the negation. For (5), the "only if" direction:

$$\begin{array}{lll}
\multicolumn{3}{l}{\mathcal{M} \models e_1 = e_2} \\
\Leftrightarrow & \mathcal{M}[u:e_1] \approx \mathcal{M}[u:e_2] & \text{(By Definition)} \\
\Leftrightarrow & \mathcal{M}[u:e_1][v:e_2] \approx \mathcal{M}[u:e_2][v:e_2] \wedge & \\
 & \mathcal{M}[u:e_2][v:e_2] \approx \mathcal{M}[u:e_2][v:e_1] & \text{(By (3))} \\
\Rightarrow & \mathcal{M}[u:e_1][v:e_2] \approx \mathcal{M}[u:e_2][v:e_1]. &
\end{array}$$

Operationally, the encoding of models simply removes all references to u,v and replaces them by positional information: hence all relevant difference is induced, if ever, by behavioural differences between e_1 and e_2, which however cannot exist by assumption. The "if" direction is immediate by projection.

(6) is best argued using concrete models. For the former, let $\mathcal{M} = (\nu\tilde{l})(\xi,\sigma)$ and let $\xi(x) = W$. We infer:

$$\begin{array}{rcl}
\mathcal{M}[u{:}x][v{:}e] & \stackrel{\text{def}}{=} & (\nu\tilde{l})(\xi\cdot u:W\cdot v:e\xi,\ \sigma) \\
 & \stackrel{\text{def}}{=} & (\nu\tilde{l})(\xi\cdot u:W\cdot v:(e[u/x])\xi,\ \sigma)
\end{array}$$

For the latter, let $\mathcal{M} = (\nu\tilde{l})(\xi,\sigma)$ and $W = [\![e]\!]_{\xi,\sigma}$ (the standard interpretation of e by ξ and σ). We then have

$$\begin{array}{rcl}
\mathcal{M}[u{:}e][v{:}e'] & \approx & (\nu\tilde{l})(\xi\cdot u:W\cdot v:[\![e']\!]_{\xi,\sigma},\ \sigma) \\
 & \stackrel{\text{def}}{=} & (\nu\tilde{l})(\xi\cdot u:W\cdot v:[\![e'[e/u]]\!]_{\xi,\sigma},\ \sigma)
\end{array}$$

The last line is because the interpretation is homomorphic. $\square$

C.2. **Proof of Proposition 5.3.**

Proposition 5.3.

(1) $\Box(C_1 \supset C_2) \supset \Box C_1 \supset \Box C_2$*;* $\Box C \supset C$*;* $\Box\Box C \equiv \Box C$*;* $C \supset \Diamond C$*. Hence* $\Box C \supset \Diamond C$*.*

(2) (permutation and decomposition)

(a) $\Box e_1 = e_2 \equiv e_1 = e_2$ *and* $\Box e_1 \neq e_2 \equiv e_1 \neq e_2$ *if* e_i *does not contain dereference.*

(b) $\Box(C_1 \wedge C_2) \equiv \Box C_1 \wedge \Box C_2$.
(c) $\Box C_1 \vee \Box C_2 \supset \Box(C_1 \vee C_2)$.
(d) $\Box \forall x.C \supset \forall x.\Box C$ *and* $\Box \forall x.\Box C \equiv \Box \forall x.C$.
(e) $\exists x.\Box C \supset \Box \exists x.C$.
(f) $\Box \overline{\nu} x.C \equiv \overline{\nu} x.\Box C$; *and* $\nu x.\Box C \supset \Box \nu x.C$.
(g) $\Box \exists \mathrm{X}.C \equiv \exists \mathrm{X}.\Box C$ *and* $\Box \forall \mathrm{X}.C \equiv \forall \mathrm{X}.\Box C$.
(h) $\Box [!x]C \equiv [!x]\Box C \equiv \Box C$ *and* $\langle !x \rangle \Box C \equiv \Box C \supset \Box \langle !x \rangle C$.

(1) is obvious by definition. For (2-a), suppose $\mathcal{M} \models e_1 = e_2$. Then by definition $\approx$, for all $\mathcal{M}'$ such that $\mathcal{M} \rightsquigarrow \mathcal{M}'$, we have $\mathcal{M}'[u : e_1] \approx \mathcal{M}'[u : e_2]$. Hence $\mathcal{M} \models \Box e_1 = e_2$, as required. Similarly for $e_1 \neq e_2$. For (2-b), we have:

$$\begin{array}{rl}\mathcal{M} \models \Box(C_1 \wedge C_2) \equiv & \forall \mathcal{M}'.(\mathcal{M} \rightsquigarrow \mathcal{M}' \supset \mathcal{M}' \models C_1 \wedge \mathcal{M}' \models C_2) \\ \equiv & \forall \mathcal{M}'.(\mathcal{M} \rightsquigarrow \mathcal{M}' \supset \mathcal{M}' \models C_i) \quad (i = 1, 2) \\ \equiv & \mathcal{M} \models \Box C_1 \wedge \Box C_2\end{array}$$

For (2-c), we derive:

$$\begin{array}{rl}\mathcal{M} \models \Box C_1 \vee \Box C_2 \equiv & \forall \mathcal{M}'.(\mathcal{M} \rightsquigarrow \mathcal{M}' \supset \mathcal{M}' \models C_i) \quad (i = 1 \vee i = 2) \\ \Rightarrow & \forall \mathcal{M}'.(\mathcal{M} \rightsquigarrow \mathcal{M}' \supset \mathcal{M}' \models C_1 \vee C_2) \\ \equiv & \mathcal{M} \models \Box(C_1 \vee C_2)\end{array}$$

For (2-d(1)), we derive, with u fresh:

$$\begin{array}{ll}\multicolumn{2}{l}{\mathcal{M} \models \Box \forall x.C} \\ \equiv & \forall \mathcal{M}'.(\mathcal{M}[u : N] \Downarrow \mathcal{M}' \supset \forall L \in \mathcal{F}.(\mathcal{M}'[x : L] \Downarrow \mathcal{M}'' \supset \mathcal{M}'' \models C)) \\ \Rightarrow & \forall \mathcal{M}'_0, L' \in \mathcal{F}, \forall N.(\mathcal{M}[x : L'][u : N[L'/x]] \Downarrow \mathcal{M}'_0 \supset \mathcal{M}'_0 \models C) \text{ such that } u \notin \mathsf{fv}(L') \\ \Rightarrow & \forall \mathcal{M}'_0, L' \in \mathcal{F}, \forall N.(\mathcal{M}[x : L'][u : N] \Downarrow \mathcal{M}'_0 \supset \mathcal{M}'_0 \models C) \text{ such that } u \notin \mathsf{fv}(L') \\ \Rightarrow & \mathcal{M} \models \forall x.\Box C\end{array}$$

Note that $x \notin \mathsf{fv}(N)$ in the second line. To derive the third line, we use the fact for all $L \in \mathcal{F}$ such that $u \notin \mathsf{fv}(L)$ and all N, if $\mathcal{M}[x : L][u : N[L/x]] \Downarrow \mathcal{M}'$, then $\mathcal{M}[x : L][u : N] \Downarrow \mathcal{M}'$.

For (2-d(2)), by $\Box C \supset C$, we have $\Box \forall x.\Box C \supset \Box \forall x.C$. The other direction is proved with $\Box\Box C \equiv \Box C$, as $\Box \forall x.C \equiv \Box\Box \forall x.C \supset \Box \forall x.\Box C$.

For (f-1), we derive, with u fresh:

$$\begin{array}{ll}\multicolumn{2}{l}{\mathcal{M} \models \overline{\nu} x.\Box C} \\ \equiv & \forall \mathcal{M}'.((\nu l)\mathcal{M}' \approx \mathcal{M} \supset \forall \mathcal{M}'', N.(\mathcal{M}'[x : l][u : N] \Downarrow \mathcal{M}'' \stackrel{\text{def}}{=} (\nu \tilde{l})\mathcal{M}'''[x : l][u : V] \supset \mathcal{M}'' \models C)) \\ \equiv & \forall \mathcal{M}_0, N.(\mathcal{M}[u : N] \Downarrow \mathcal{M}_0 \supset \forall \mathcal{M}'_0.(\mathcal{M}_0 \approx (\nu l)\mathcal{M}'_0 \supset \mathcal{M}'_0[x : l] \models C)) \\ & \text{such that } l, x \notin \mathsf{fv}(N) \cup \mathsf{fl}(N) \\ & \text{with } \mathcal{M}_0 \stackrel{\text{def}}{=} (\nu \tilde{l})(\nu l)\mathcal{M}'''[u : V], \quad \mathcal{M}'_0 \stackrel{\text{def}}{=} (\nu \tilde{l})\mathcal{M}''[u : V] \\ \equiv & \mathcal{M} \models \Box \overline{\nu} x.C\end{array}$$

For (f-2), we derive, with u fresh:

$$\begin{array}{ll}\multicolumn{2}{l}{\mathcal{M} \models \nu x.\Box C} \\ \equiv & \exists \mathcal{M}'.((\nu l)\mathcal{M}' \approx \mathcal{M} \wedge \forall \mathcal{M}''.(\mathcal{M}'[x : l][u : N] \Downarrow \mathcal{M}'' \stackrel{\text{def}}{=} (\nu \tilde{l})\mathcal{M}'''[x : l][u : V] \supset \mathcal{M}'' \models C)) \\ \Rightarrow & \forall \mathcal{M}_0.(\mathcal{M}[u : N] \Downarrow \mathcal{M}_0 \supset \exists \mathcal{M}'_0.(\mathcal{M}_0 \approx (\nu l)\mathcal{M}'_0 \wedge \mathcal{M}'_0[x : l] \models C)) \\ & \text{with } \mathcal{M}_0 \stackrel{\text{def}}{=} (\nu \tilde{l})(\nu l)\mathcal{M}'''[u : V], \quad \mathcal{M}'_0 \stackrel{\text{def}}{=} (\nu \tilde{l})\mathcal{M}''[u : V] \\ \equiv & \mathcal{M} \models \Box \nu x.C\end{array}$$

(g) is trivial. For (h), we prove the first equation. With u fresh, we have:

$$\begin{array}{rcl}\mathcal{M} \models \Box[!x]C & \equiv & \forall N.(\mathcal{M}[u:N] \Downarrow \mathcal{M}' \supset \forall L \in \mathcal{F}.\mathcal{M}'[x \mapsto L] \models C) \\ & \Rightarrow & \forall L \in \mathcal{F}, \forall N.\mathcal{M}[u:((x:=L);N)] \Downarrow \mathcal{M}' \supset \mathcal{M}' \models C \\ & \equiv & \mathcal{M} \models [!x]\Box C \\ & \Rightarrow & \mathcal{M} \models \Box C\end{array}$$

The last line is by the axiom $[!x]C \supset C$. For other direction, with u fresh,

$$\begin{array}{rcl}\mathcal{M} \models \Box C & \equiv & \forall N.(\mathcal{M}[u:N] \Downarrow \mathcal{M}' \supset \mathcal{M}' \models C) \\ & \Rightarrow & \forall L \in \mathcal{F}, \forall N.(\mathcal{M}[u:(x:=L;N)] \Downarrow \mathcal{M}' \supset \mathcal{M}' \models C) \\ & \Rightarrow & \mathcal{M} \models [!x]\Box C \\ & \equiv & \forall L, L' \in \mathcal{F}, \forall N.(\mathcal{M}[x \mapsto L][u:(N;x:=L')] \Downarrow \mathcal{M}' \supset \mathcal{M}' \models C) \\ & \equiv & \forall L' \in \mathcal{F}, \forall N.(\mathcal{M}[x \mapsto !x][u:(N;x:=L')] \Downarrow \mathcal{M}' \supset \mathcal{M}' \models C) \\ & \equiv & \forall L' \in \mathcal{F}, \forall N.(\mathcal{M}[u:(N;x:=L')] \Downarrow \mathcal{M}' \supset \mathcal{M}' \models C) \\ & \Rightarrow & \mathcal{M} \models \Box[!x]C\end{array}$$

In Line 5, we use the fact $!x$ is a functional term. In Line 6, we note that $\mathcal{M}[x \mapsto !x] \equiv \mathcal{M}$. The equation for $\langle !x \rangle C$ is similar. This concludes the proofs.

C.3. **Axioms for Content Quantification.** The axiomatisation of content quantification in [6] uses the well-known axioms [33, §2.3] for standard quantifiers. Despite the presence of local state, most of the axioms stay valid.

Proposition C.1 (Axioms for Content Quantifications)**.** *Recall A denotes the stateless formula.*

(1) $[!x]A \equiv A$
(2) $[!x]!y = z \equiv x \neq y \wedge !y = z$
(3) $[!x]([!x]C_1 \supset C_2) \supset ([!x]C_1 \supset [!x]C_2)$.
(4) $[!x][!x]C \equiv [!x]C$
(5) $[!x][!y]C \equiv [!y][!x]C$
(6) $[!x](C_1 \wedge C_2) \equiv [!x]C_1 \wedge [!x]C_2$
(7) $[!x]C_1 \vee [!x]C_2 \supset [!x](C_1 \vee C_2)$

Proof. For (1), assume $\mathcal{M} \models \Box A$. By definition, for all N, if $\mathcal{M}[u:N] \Downarrow \mathcal{M}'$, then $\mathcal{M}' \models A$. This implies: for all V and $L \in \mathcal{F}$, if $\mathcal{M}[u:x:=V;L] \Downarrow \mathcal{M}'$, then $\mathcal{M}' \models A$, which means $\mathcal{M} \models [!x]A$. Others are proved as in [6, Appendix C]. ☐

C.4. **Proof of Proposition 5.8.**

Proposition Axioms for $\forall$, $\exists$ and ν. *Below we assume there is no capture of variables in types and formulae.*

(1) (introduction) $C \supset \nu x.C$ *if* $x \notin \mathsf{fv}(C)$
(2) (elimination) $\nu x.C \equiv C$ *if* $x \notin \mathsf{fv}(C)$ *and* C *is monotone.*
(3) *For any C we have $C \supset \exists x.C$. Given C such that $x \notin \mathsf{fv}(C)$ and C is thin with respect to x, we have $\exists x.C \supset C$.*
(4) *For any C we have $\forall x.C \supset C$. For C such that $x \notin \mathsf{fv}(C)$ and C is thin with respect to x, we have $C \supset \forall x.C$.*
(5) $\nu x.(C_1 \wedge C_2) \supset \nu x.C_1 \wedge \nu x.C_2$.
(6) $\nu x.(C_1 \vee C_2) \equiv \nu x.C_1 \vee \nu x.C_2$.
(7) $\nu y.\forall x.C \supset \forall x.\nu y.C$

(8) $\exists x.\nu y.C \supset \nu y.\exists x.C$ *and* $\exists x^{\alpha}.\nu y.C \equiv \nu y.\exists x^{\alpha}.C$ *with* $\alpha \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$.
(9) $\nu y.\overline{\nu} x.C \supset \overline{\nu} x.\nu y.C$*; and* $\nu y.\nu x.C \equiv \nu x.\nu y.C$.
(10) $\nu y.\exists \mathrm{X}.C \equiv \exists \mathrm{X}.\nu y.C$*; and* $\nu y.\forall \mathrm{X}.C \supset \forall \mathrm{X}.\nu y.C$.
(11) $\nu y.[!x]C \supset [!x]\nu y.C$ *and* $\nu y.\langle !x\rangle C \supset \langle !x\rangle \nu y.C$

(1) is by definition. For (2), we have:

$$\begin{array}{lcll} \mathcal{M} \models \nu x.C & \Rightarrow & \exists \mathcal{M}', l.((\nu l)\mathcal{M}' \cong \mathcal{M} \ \wedge \ \mathcal{M}'[x:l] \models C) & \\ & \Rightarrow & \exists \mathcal{M}', l.((\nu l)\mathcal{M}' \cong \mathcal{M} \ \wedge \ \mathcal{M}' \models C) & \text{Lemma 5.1 (4)} \\ & \Rightarrow & (\nu l)\mathcal{M}' \models C & C \text{ is monotone} \end{array}$$

For (7), we derive:

$$\begin{array}{lcl} \mathcal{M} \models \nu y.\forall x.C & \equiv & \exists \mathcal{M}_0, \forall L \in \mathcal{F}.(\mathcal{M} \approx (\nu l)\mathcal{M}_0 \wedge (\mathcal{M}_0[x:L] \Downarrow \mathcal{M}'_0 \supset \mathcal{M}'_0 \models C)) \\ & \Rightarrow & \forall L \in \mathcal{F}, \exists \mathcal{M}_0.(\mathcal{M}[x:L] \approx ((\nu l)\mathcal{M}_0)[x:L] \wedge \mathcal{M}_0[x:L] \models C) \text{ such that } l \notin \mathsf{fl}(L) \\ & & \mathcal{M} \models \forall x.\nu y.C \end{array}$$

For (8-1), we derive:

$$\begin{array}{lcl} \mathcal{M} \models \exists x.\nu y.C & \equiv & \exists L \in \mathcal{F}.(\mathcal{M}[x:L] \Downarrow \mathcal{M}' \wedge \exists \mathcal{M}_0.(\mathcal{M}' \approx (\nu l)\mathcal{M}_0 \wedge \mathcal{M}_0[y:l] \models C) \\ & \Rightarrow & \exists L \in \mathcal{F}, \mathcal{M}_0.(\mathcal{M} \approx (\mathcal{M}[x:L])/x \approx ((\nu l)\mathcal{M}_0)/x \wedge \mathcal{M}_0[y:l] \models C) \\ & \equiv & \exists L \in \mathcal{F}, \mathcal{M}'_0.(\mathcal{M} \approx (\nu l)\mathcal{M}'_0 \wedge \mathcal{M}'_0[y:l][x:L] \models C) \quad \text{with } \mathcal{M}'_0 = \mathcal{M}/x \\ & \equiv & \mathcal{M} \models \nu y.\exists x.C \end{array}$$

Note that the other direction does not generally hold. Consider $\mathcal{M} \models \nu y.\exists x.C$. This is equivalent to:

$$\exists L \in \mathcal{F}, \mathcal{M}'_0.(\mathcal{M} \approx (\nu l)\mathcal{M}_0 \wedge \mathcal{M}_0[y:l][x:L] \models C)$$

Since L might contain the new reference l hidden in $\mathcal{M}$, $\mathcal{M}[x:L[l/y]]$ is undefined (hence we cannot permute $[y:l]$ and $[x:L[l/y]]$).

For (8-2), we only have to prove $\nu y.\exists x^{\alpha}.C \supset \exists x^{\alpha}.\nu y.C$ with $\alpha \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$. We derive:

$$\begin{array}{ll} \mathcal{M} \models \nu y.\exists x.C & \equiv \exists \mathcal{M}_0, \mathsf{c}.(\mathcal{M} \approx (\nu l)\mathcal{M}_0 \wedge \mathcal{M}_0[y:l][x:\mathsf{c}] \models C) \\ & \equiv \exists \mathsf{c}, \mathcal{M}_0.(\mathcal{M}[x:\mathsf{c}] \approx ((\nu l)\mathcal{M}_0)[x:\mathsf{c}] \wedge \mathcal{M}_0[x:\mathsf{c}][y:l] \models C) \quad \equiv \mathcal{M} \models \exists x.\nu y.C \end{array}$$

For (9-2), we have:

$$\begin{array}{lcl} \mathcal{M} \vdash \nu x.\nu y.C & \equiv & \exists \mathcal{M}'.(\mathcal{M} \approx (\nu l)\mathcal{M}' \wedge \mathcal{M}'[x:l] \models \nu y.C) \\ & \equiv & \exists \mathcal{M}', \mathcal{M}''.(\mathcal{M} \approx (\nu l)\mathcal{M}' \wedge \mathcal{M}'[x:l] \approx (\nu l')\mathcal{M}'' \wedge \mathcal{M}''[y:l'] \models C) \\ & \equiv & \exists \mathcal{M}', \mathcal{M}''.(\mathcal{M} \approx (\nu l)\mathcal{M}' \approx (\nu l l')\mathcal{M}'' \wedge \mathcal{M}''[x:l] \approx (\nu l')\mathcal{M}'' \wedge \mathcal{M}''[y:l'] \models C) \\ & \equiv & \mathcal{M} \vdash \nu y.\nu x.C \end{array}$$

For (11), we derive:

$$\begin{array}{lcl} \mathcal{M} \vdash \nu y.[!x]C & \equiv & \exists \mathcal{M}_0.(\mathcal{M} \approx (\nu l)\mathcal{M}_0 \wedge \forall L \in \mathcal{F}.(\mathcal{M}_0[y:l][x \mapsto L] \models C)) \\ & \equiv & \forall L \in \mathcal{F}.\exists \mathcal{M}_0.(\mathcal{M} \approx (\nu l)\mathcal{M}_0 \wedge \mathcal{M}_0[y:l][x \mapsto L] \models C)) \\ & & \qquad \text{such that } l, y \notin \mathsf{fv}(L) \cup \mathsf{fl}(L) \\ & \equiv & \forall L \in \mathcal{F}.\exists \mathcal{M}_0.(\mathcal{M}[x \mapsto L] \approx (\nu l)(\mathcal{M}_0[x \mapsto L]) \wedge \mathcal{M}_0[x \mapsto L][y:l] \models C)) \\ & \equiv & \mathcal{M} \vdash [!x]\nu y.C \end{array}$$

The remaining claims are similar.

C.5. **Proof of Theorem 5.10.**

Theorem 5.10. *Suppose all reachability predicates in C are finite. Then there exists C' such that $C \equiv C'$ and no reachability predicate occurs in C'.*

As the first step, we define a simple inductive method for defining reachability from a datum of a finite type.

Definition C.2. (i-step reachability) Let α be a finite type. Then the i-step reachability predicate $\mathsf{reach}(x^{\alpha},\ y^{\mathsf{Ref}(\beta)},\ i^{\mathsf{Nat}})$ (read: *"a reference y is reachable from x in at most i-steps"*) is inductively given as follows (below we assume y is typed $\mathsf{Ref}(\beta)$, $C \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$, and omit types when evident).

$$
\begin{array}{rcl}
\mathsf{reach}(x^{\alpha},\ y,\ 0) & \equiv & x = y \\
\mathsf{reach}(x^{C},\ y,\ n+1) & \equiv & \mathsf{F} \\
\mathsf{reach}(x^{\alpha_1 \times \alpha_2},\ y,\ n+1) & \equiv & \vee_i \mathsf{reach}(\pi_i(x),\ y,\ n) \ \vee\ \mathsf{reach}(x,\ y,\ n) \\
\mathsf{reach}(x^{\alpha_1 + \alpha_2},\ y,\ n+1) & \equiv & \exists x'.(x' = \mathtt{inj}_1(x) \ \wedge\ \mathsf{reach}(x',\ y,\ n)) \ \vee \\
 & & \exists x'.(x' = \mathtt{inj}_2(x) \ \wedge\ \mathsf{reach}(x',\ y,\ n)) \ \vee \\
 & & \mathsf{reach}(x,\ y,\ n) \\
\mathsf{reach}(x^{\mathsf{Ref}(\alpha)},\ y,\ n+1) & \equiv & \mathsf{reach}(!x,\ y,\ n) \ \vee\ \mathsf{reach}(x,\ y,\ n)
\end{array}
$$

With C being a base type, $\mathsf{reach}(x^{C},\ y,\ 0) \equiv x = y \equiv \mathsf{F}$ (since a reference y cannot be equal to a datum of a base type).

A key lemma follows.

Proposition C.3. *If α is finite, then the logical equivalence $x^{\alpha} \hookrightarrow y \equiv \exists i.\mathsf{reach}(x^{\alpha},\ y,\ i)$ is valid, i.e. is true in any model.*

Proof. For the "if" direction, we show, by induction on i, $\mathsf{reach}(x^{\alpha},\ y,\ i) \supset x^{\alpha} \hookrightarrow y$. For the base case, we have $i = 0$, in which case $\mathsf{reach}(x^{\alpha},\ y,\ 0) \supset x = y \supset x \hookrightarrow y$.

For induction, let the statement holds up to n. We only show the case of a product. Other cases are similar.

$$
\begin{array}{rcl}
\mathsf{reach}(x^{\alpha_1 \times \alpha_2},\ y,\ n+1) & \Rightarrow & \vee_i \mathsf{reach}(\pi_i(x),\ y,\ n) \ \vee\ \mathsf{reach}(x,\ y,\ n) \\
 & \Rightarrow & \vee_i \pi_i(x) \hookrightarrow y \ \vee\ x \hookrightarrow y
\end{array}
$$

But if $\pi_1(x) \hookrightarrow y$ then $x \hookrightarrow y$ by the definition of reachability. Similarly when $\pi_2(x) \hookrightarrow y$, hence done.

For the converse, we show the contrapositive, showing:

$$\mathcal{M} \models \neg\exists i.\mathsf{reach}(x^{\alpha},\ y,\ i) \quad \Rightarrow \quad \mathcal{M} \models \neg x^{\alpha} \hookrightarrow y$$

If we have $\mathcal{M} \models \neg\exists i.\mathsf{reach}(x^{\alpha},\ y,\ i)$ with α finite, then the reference y is not among references reachable from x (if it is, then either $x = y$ or y is the content of a reference reachable from x because of the finiteness of α, so that we can find some i such that $\mathcal{M} \models \mathsf{reach}(x^{\alpha},\ y,\ i)$), hence done. $\square$

Now let us define the predicate $x^{\alpha} \hookrightarrow^{\circ} y^{\mathsf{Ref}(\beta)}$ with α finite, by the axioms given in Proposition 5.9 which we reproduce below ($C \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Nat}\}$).

$$
\begin{array}{rcl}
x^{C} \hookrightarrow^{\circ} y^{\mathsf{Ref}(\beta)} & \equiv & \mathsf{F} \\
x^{\alpha_1 \times \alpha_2} \hookrightarrow^{\circ} y^{\mathsf{Ref}(\beta)} & \equiv & \exists x_{1,2}.(x = \langle x_1, x_2 \rangle \wedge \bigvee_{i=1,2} x_i \hookrightarrow^{\circ} y) \\
x^{\alpha_1 + \alpha_2} \hookrightarrow^{\circ} y^{\mathsf{Ref}(\beta)} & \equiv & \exists x'.((\bigvee_{i=1,2} x = \mathtt{inj}_i(x')) \ \wedge\ x' \hookrightarrow^{\circ} y) \\
x^{\mathsf{Ref}(\alpha)} \hookrightarrow^{\circ} y^{\mathsf{Ref}(\beta)} & \equiv & x = y \vee\ !x \hookrightarrow^{\circ} y
\end{array}
$$

The inductive definition is possible due to finiteness. We now show:

Proposition C.4. *If α is finite, then the logical equivalence, $x^{\alpha} \hookrightarrow^{\circ} y^{\mathsf{Ref}(\beta)} \equiv \exists i.\mathsf{reach}(x^{\alpha},\, y^{\mathsf{Ref}(\beta)},\, i)$, is valid.*

Proof. $\mathsf{reach}(x^{\alpha},\, y^{\mathsf{Ref}(\beta)},\, i) \supset x^{\alpha} \hookrightarrow^{\circ} y^{\mathsf{Ref}(\beta)}$ is by induction on i. The converse is by induction on α. Both are mechanical and omitted. □

Corollary C.5. *If α is finite, then the logical equivalence $x^{\alpha} \hookrightarrow y^{\mathsf{Ref}(\beta)} \equiv x^{\alpha} \hookrightarrow^{\circ} y^{\mathsf{Ref}(\beta)}$ is valid, i.e. $\hookrightarrow$ is completely characterised by the axioms for $\hookrightarrow^{\circ}$ given above.*

Proof. Immediate from Propositions C.3 and C.4. □

C.6. **Proof of Proposition 5.14.**

Proposition 5.14. *For an arbitrary C, the following is valid with i, X fresh:*

$$\Box\{C \wedge x\# fy\tilde{w}\}\, f \bullet y = z\{C'\}@\tilde{w} \;\supset\; \Box\forall \mathrm{X}, i^{\mathrm{X}}.\{C \wedge x\# fiy\tilde{w}\}\, f \bullet y = z\{C' \wedge x\# fiyz\tilde{w}\}@\tilde{w}$$

Proof. The proof traces the transition of state using the elementary fact that the set of names and labels in a term always gets smaller as reduction goes by. Suppose we have

$$\mathcal{M} \models \Box\{x\# fyw \wedge C\}\, f \bullet y = z\{C'\}@w$$

The definition of the evaluation formula says:

$$(\mathcal{M} \rightsquigarrow \mathcal{M}_0 \;\wedge\; \mathcal{M}_0 \models x\# fywi \wedge C) \;\supset\; \exists \mathcal{M}'.(\mathcal{M}[z : fy] \Downarrow \mathcal{M}' \wedge \mathcal{M}' \models C').$$

We prove such $\mathcal{M}'$ always satisfies $\mathcal{M}' \models x\# fiyzw$. Assume

$$\mathcal{M}_0 \approx (\nu \tilde{l})(\xi, \sigma_0 \uplus \sigma_x)$$

with $\xi(x) = l$, $\xi(y) = V_y$, $\xi(f) = V_f$ and $\xi(w) = l_w$ such that

$$\mathsf{lc}(\mathsf{fl}(V_f, V_y, l_w), \sigma_0 \uplus \sigma_x) = \mathsf{fl}(\sigma_0) = \mathsf{dom}(\sigma_0)$$

and $l_x \in \mathsf{dom}(\sigma_x)$. By this partition, during evaluation of $z : fy$, σ_x is unchanged, i.e.

$$(\nu \tilde{l})(\xi \cdot z : fy, \sigma_0 \uplus \sigma_x) \longrightarrow\!\!\!\!\!\to (\nu \tilde{l})(\xi \cdot z : V_f V_y, \sigma_0 \uplus \sigma_x) \longrightarrow\!\!\!\!\!\to (\nu \tilde{l}')(\xi \cdot z : V_z, \sigma_0' \uplus \sigma_x)$$

Then obviously there exists σ_1 such that $\sigma_1 \subset \sigma_0'$ and

$$\mathsf{lc}(\mathsf{fl}(V_z, l_w), \sigma_0' \uplus \sigma_x) = \mathsf{fl}(\sigma_1) = \mathsf{dom}(\sigma_1)$$

Hence by Proposition 3.9, we have $\mathcal{M}_0 \models x\# fyiwz$, completing the proof. □

C.7. **Proof of Propositions 5.15.**

Proposition 5.15. *Assume* $C_0 \equiv C_0' \wedge \tilde{x}\# iy \wedge \tilde{g} \hookrightarrow^{\bullet} \tilde{x}$*,* C_0' *is stateless except* $\tilde{x}$*,* C *is anti-monotone,* C' *is monotone,* i, m *are fresh and* $\{\tilde{x},\tilde{g}\} \cap (\mathsf{fv}(C,C') \cup \{\tilde{w}\}) = \emptyset$*. Then the following is valid:*

$$\text{(AIH)} \qquad \forall \mathrm{X}.\forall i^{\mathrm{X}}.m \bullet () = u\{(\nu\tilde{x}.\exists\tilde{g}.E_1) \wedge E\} \supset \forall \mathrm{X}.\forall i^{\mathrm{X}}.m \bullet () = u\{E_2 \wedge E\}$$

with

- $E_1 \stackrel{\text{def}}{\equiv} \mathsf{Inv}(u,C_0,\tilde{x}) \wedge \Box\forall yi.\{C_0 \wedge C\}u \bullet y = z\{C'\}@\tilde{w}\tilde{x}$ *and*
- $E_2 \stackrel{\text{def}}{\equiv} \Box\forall y.\{C\}u \bullet y = z\{C'\}@\tilde{w}$*.*

Proof. W.l.o.g. we assume all vectors are unary, setting $\tilde{r} = r$, $\tilde{w} = w$, $\tilde{x} = x$ and $\tilde{g} = g$. The proof proceeds as follows, starting from the current model $\mathcal{M}_0$.

Stage 1. We take $\mathcal{M}$ such that:

$$\mathcal{M}_0 \stackrel{m}{\leadsto} \mathcal{M}$$

We then take off the hiding, name it x and the result is called $\mathcal{M}_*$

$$(\nu l)(\mathcal{M}_*/x) \approx \mathcal{M}.$$

Stage 2. We further let $\mathcal{M}$ evolve so that:

$$\mathcal{M} \stackrel{u}{\leadsto} \mathcal{M}'$$

We then again take off the corresponding hiding, name it x and the result is called $\mathcal{M}'_*$

$$(\nu l)(\mathcal{M}'_*/x) \approx \mathcal{M}'$$

Stage 3. We show if $\mathcal{M}_*$ satisfies C_0 then again $\mathcal{M}'_*$ satisfies C_0 again:

$$\mathcal{M}_* \models C_0 \qquad \supset \qquad \mathcal{M}'_* \models C_0$$

using $\mathsf{Inv}(u,C_0,\tilde{x})$ as well as the unreachability of x from u.

By reaching Stage 3, we know if $\mathcal{M} \models C$ then it is also the case $\mathcal{M}_* \models C_0 \wedge C$ hence we can use the assumption (together with monotonicity of C'):

$$\forall yi.\{C_0 \wedge C\}u \bullet y = z\{C'\}@\tilde{w}\tilde{x}$$

hence we know we arrive at C' as a result.

We now implement these steps. We set:

$$E \quad \equiv \quad \mathsf{T}. \tag{C.1}$$

The trivialisation of E (taken as truth) is just for simplicity and does not affect the argument. Now fix an arbitrary $\mathcal{M}_0$ and suppose we reach:

$$\mathcal{M}_0 \stackrel{m}{\leadsto} \mathcal{M} \tag{C.2}$$

This gives the status of the post-condition of the whole formula (to be precise this is through the encoding in (4.9) in § 4.5 to relate $m \bullet ()$ and the transition above). Assuming the hidden x in the formula in E_1 is about a (fresh) l we can set:

$$\mathcal{M} \stackrel{\text{def}}{=} (\nu l)(\nu\tilde{l'})(\xi, \sigma\cdot[l \mapsto V]) \models \nu x.\exists g.E_1 \tag{C.3}$$

as well as by revealing l:

$$\mathcal{M}_* \stackrel{\text{def}}{=} (\nu\tilde{l'})(\xi\cdot x : l\cdot g : U, \sigma\cdot[l \mapsto V]) \models E_1 \tag{C.4}$$

Note by assumption we have:

$$l \notin \mathsf{fl}(\xi,\sigma). \tag{C.5}$$

Further U does not contain any hidden or free locations from $\mathcal{M}$ by $g \hookrightarrow^{\bullet} \tilde{x}$.

Now we consider the right-hand side of E_1, $\Box\forall yi.\{C_0 \wedge C\}u \bullet y = z\{C'\}@\tilde{w}\tilde{x}$ by taking for fresh N:

$$\mathcal{M}[f : N] \Downarrow \mathcal{M}' \tag{C.6}$$

Corresponding to the relationship between $\mathcal{M}$ and $\mathcal{M}_*$ we set:

$$\mathcal{M}_*[f : N] \Downarrow \mathcal{M}'_* \tag{C.7}$$

Note we have

$$(\nu l)(\mathcal{M}'_*/xg) \approx \mathcal{M}' \tag{C.8}$$

We now show:

$$\mathcal{M}_* \models C_0 \quad \supset \quad \mathcal{M}'_* \models C_0 \tag{C.9}$$

that is C_0 is invariant under the evaluation (effects) of N. Assume

$$\mathcal{M}_* \models C_0 \tag{C.10}$$

First observe

$$\mathcal{M}_* \models C_0 \ \wedge \ x\#yrw \tag{C.11}$$

Now in the standard way N can be approximated by a finite term, that is a term which does not contain recursion except divergent programs. We take N as such an approximation without loss of generality. Such N can be written as a sequence of let expressions including assignments. Without loss of generality we focus on a "let" expression which is either a function call or an assignment. Then at each evaluation we have either:

- The let has the form $\texttt{let}\ x = uV\ \texttt{in}\ M'$ that is it invokes u;
- The let has the form $\texttt{let}\ x = WV\ \texttt{in}\ M'$ where W is not u.
- The let has the form $w' := V; M'$.

We observe:

- In the first case u is directly invoked: thus by the invariance $\mathsf{Inv}(u, C_0, \tilde{x})$, C_0 continues to hold. Note w' is not x since N has no access to x except through u.
- In the second case of the let (i.e. u is not called), since x is disjoint from all visible data, by Proposition 5.14 we know x (hence the content of x) is never touched by the execution of the function body after the invocation, until again u is called (if ever): since C_0 is insensitive to state change except at x (by being stateless except x), it continues to hold again in this case.
- In the third case again x is not touched hence C_0 continues to hold.

Thus we have:

$$\mathcal{M}'_* \models C_0 \tag{C.12}$$

Now suppose we have

$$\mathcal{M} \models C \tag{C.13}$$

By anti-monotonicity of C we have

$$\mathcal{M}_*/xg \models C \tag{C.14}$$

By Lemma 5.1 (4), we can arbitrarily weaken a disjoint extension (at x and g) so that:

$$\mathcal{M}_* \models C \tag{C.15}$$

Thus we know:

$$\mathcal{M}'_* \models C_0 \ \wedge \ C \tag{C.16}$$

Now we can apply:

$$\mathcal{M}' \models \nu x.\exists g.\forall y.\{C_0 \wedge C\}u \bullet y = z\{C'\}@\tilde{w}x \tag{C.17}$$

by which we know:

$$\mathcal{M}'_*[z : uy] \Downarrow \mathcal{M}''_* \models C' \tag{C.18}$$

Accordingly let

$$\mathcal{M}'[z : uy] \Downarrow \mathcal{M}'' \approx (\nu l)(\mathcal{M}''_*/x) \tag{C.19}$$

for which we know, by (C.18) and (C.19) together with monotonicity of C':

$$\mathcal{M}'' \models C' \tag{C.20}$$

Hence we know:

$$\mathcal{M} \models \{C\}u \bullet y = z\{C'\}@w \tag{C.21}$$

which is the required assertion. □

C.8. **Proof of Proposition 5.16.**

Proposition 5.16. *Let $x \notin \mathsf{fv}(C)$ and m, i, X be fresh. Then the following is valid:*

$$\forall \mathrm{X}, i^{\mathrm{X}}.m \bullet () = u\{\nu\tilde{x}.([!\tilde{x}]C \wedge \tilde{x}\#ui^{\mathrm{X}})\} \ \supset \ m \bullet () = u\{C\}$$

Proof. For simplicity, set $\tilde{x}$ to be a singleton x. Assume

$$\mathcal{M}[u : m()] \Downarrow \mathcal{M}'$$

By assumption we can set

$$\mathcal{M}' \approx (\nu l)(\nu\tilde{l}')(\xi \cdot u : V, \sigma \cdot l \mapsto W)$$

such that

$$(\nu\tilde{l}')(\xi \cdot u : V \cdot x : l, \sigma \cdot l \mapsto W) \models [!x]C$$

where l is not reachable from anywhere else in the model. By Lemma B.1 we obtain $(\nu\tilde{l}')(\xi \cdot u : V, \sigma) \models C$, that is $\mathcal{M}' \models C$, as required. □

C.9. **Proof of Proposition 5.17.**

Assume C_0 is stateless except $\tilde{x}$ and suppose:

$$\mathcal{M} \models \mathsf{Inv}(f, C_0, \tilde{x}) \ \wedge \ \{\mathsf{T}\}g \bullet f = z\{\mathsf{T}\}. \tag{C.22}$$

Further assume $\mathcal{M} \rightsquigarrow \mathcal{M}_0$ and

$$\mathcal{M}_0 \models C_0 \wedge \tilde{x}\#g\tilde{r} \ \text{ and } \ \mathcal{M}_0[z : fg] \Downarrow \mathcal{M}'. \tag{C.23}$$

By $\mathsf{Inv}(f, C_0, \tilde{x})$ we know that once C_0 holds and f is invoked, it continues to hold. By $\{\mathsf{T}\}g \bullet f = z\{\mathsf{T}\}$, we know the application gf always terminates. Now this application invokes f zero or more times. First time it can only apply f to some $\tilde{x}$-unreachable datum. Similarly for the second time, since the context cannot obtain $\tilde{x}$-reachable datum (given g itself is $\tilde{x}$-unreachable). By induction the same holds up to the last invocation. In each invocation, C_0 is invariant. Further, other computations in fg never touch the content of $\tilde{x}$, hence because of C_0 being stateless except $\tilde{x}$, we know C_0 is again invariant in such computations. Thus we conclude that C_0 still holds in the post-condition, and that the return value being $\tilde{x}$-unreachable, i.e. $\tilde{x}\#z$, as required. □

APPENDIX D. DERIVATIONS FOR EXAMPLES IN SECTION 6

This appendix lists the derivations omitted in Section 6.

1.	$\{(n \geq 1 \supset IsEven'(!y, gh, n-1, xy)) \ \wedge \ n = 0\}$ `f` $:_z \{z = Odd(n) \wedge !x = g \wedge !y = h\}@\emptyset$	(Const)
2.	$\{(n \geq 1 \supset IsEven'(!y, gh, n-1, xy)) \ \wedge \ n \geq 1\}$ `not`$((!y)(n-1)) :_z \{z = Odd(n) \wedge !x = g \wedge !y = h\}@\emptyset$	(Simple, App)
3.	$\{n \geq 1 \supset IsEven'(!y, gh, n-1, xy)\}$ `if` $n = 0$ `then f else not`$((!y)(n-1)) :_m \{z = Odd(n) \wedge !x = g \wedge !y = h\}@\emptyset$	(IfH)
4.	$\{\mathsf{T}\}\ \lambda n.$`if` $n = 0$ `then f else not`$((!y)(n-1)) :_u$ $\{\ \Box \forall gh, n \geq 1.\{IsEven'(h, gh, n-1, xy)\}u \bullet n = z\{z = Odd(n) \wedge !x = g \wedge !y = h\}@\emptyset\}@\emptyset$	(Abs, $\forall$, Conseq)
5.	$\{\mathsf{T}\}\ M_x :_u \{\ \forall gh, n \geq 1.(IsEven(h, gh, n-1, xy) \supset IsOdd(u, gh, n, xy))\}@\emptyset$	(Conseq)
6.	$\{\mathsf{T}\}\ x := M_x\{\ \forall gh, n \geq 1.(IsEven(h, gh, n-1, xy) \supset IsOdd(!x, gh, n, xy)) \wedge !x = g\}@x$	(Assign)
7.	$\{\mathsf{T}\}\ y := M_y\{\ \forall gh, n \geq 1.(IsOdd(g, gh, n-1, xy) \supset IsEven(!y, gh, n, xy)) \wedge !y = h\}@y$	
8.	$\{\mathsf{T}\}$ `mutualParity` $\{\forall gh.n \geq 1.((IsEven(h, gh, n-1, xy) \wedge IsOdd(g, gh, n-1, xy)) \supset$ $(IsEven(!y, gh, n, xy) \wedge IsOdd(!x, gh, n, xy) \wedge !x = g \wedge !y = h)\ \}@xy$	($\wedge$-Post)
9.	$\{\mathsf{T}\}$ `mutualParity` $\{\forall n \geq 1gh.((IsEven(h, gh, n-1, xy) \wedge IsOdd(g, gh, n-1, xy) \wedge !x = g \wedge !y = h) \supset$ $(IsEven(!y, gh, n, xy) \wedge IsOdd(!x, gh, n, xy) \wedge !x = g \wedge !y = h)\}@xy$	(Conseq)
10.	$\{\mathsf{T}\}$ `mutualParity` $\{\forall n \geq 1gh.((IsEven(!y, gh, n-1, xy) \wedge IsOdd(!x, gh, n-1, xy) \wedge !x = g \wedge !y = h) \supset$ $(IsEven(!y, gh, n, xy) \wedge IsOdd(!x, gh, n, xy) \wedge !x = g \wedge !y = h)\}@xy$	(Conseq)
11.	$\{\mathsf{T}\}$ `mutualParity` $\{\forall n \geq 1.(\exists gh.(IsEven(!x, gh, n-1, xy) \wedge IsOdd(!y, gh, n-1, xy) \wedge !x = g \wedge !y = h) \supset$ $\exists gh.(IsEven(!y, gh, n, xy) \wedge IsOdd(!x, gh, n, xy) \wedge !x = g \wedge !y = h)\}@xy$	(Conseq)
12.	$\{\mathsf{T}\}$ `mutualParity`$\{\exists gh.IsOddEven(gh, !x!y, xy, n)\}@xy$	

Figure 7: `mutualParity` derivations

D.1. **Derivation for `mutualParity`.** Let us define:

$$M_x \stackrel{\text{def}}{=} \lambda n.\texttt{if } y = 0 \texttt{ then f else not}((!y)(n-1))$$
$$M_y \stackrel{\text{def}}{=} \lambda n.\texttt{if } y = 0 \texttt{ then t else not}((!x)(n-1))$$

We also use:

$$IsOdd'(u, gh, n, xy) \stackrel{\text{def}}{\equiv} IsOdd(u, gh, n, xy) \wedge \ !x = g \wedge !y = h$$
$$IsEven'(u, gh, n, xy) \stackrel{\text{def}}{\equiv} IsEven(u, gh, n, xy) \wedge \ !x = g \wedge !y = h$$

Figure 7 lists the derivation for `MutualParity`. In Line 4, h in the evaluation formula can be replaced by $!y$ and vice versa because of $!y = h$ and the universal quantification of h.

$$\forall h.(!y = h \wedge \{C\}h \bullet n = z\{C'\}) \quad \equiv \quad \forall h.(!y = h \wedge \{C\}(!y) \bullet n = z\{C'\})$$

In Line 5, we use the following axiom for the evaluation formula from [25]:

$$\{C \wedge A\}\ e_1 \bullet e_2 = z\{C'\} \quad \equiv \quad A \supset \{C\} e_1 \bullet e_2 = z\{C'\}$$

where A is stateless and we set $A = IsEven(h, gh, n-1, xy)$. Line 9 is derived as Line 4 by replacing h and g by $!y$ and $!x$, respectively. Line 11 is the standard logical implication $(\forall x.(C_1 \supset C_2) \supset (\exists x.C_1 \supset \exists x.C_2))$.

D.2. **Derivation for Meyer-Sieber.** For the derivation of (6.6) we use:

$$E \stackrel{\mathsf{def}}{\equiv} \forall f.(\Box\{\mathsf{T}\} f \bullet ()\{\mathsf{T}\}@\emptyset \supset \Box\{C\} g \bullet f\{C'\})$$

We use the following $[LetRef]$ which is derived by $[Ref]$ where C' is replaced by $[!x]C'$.

$$[LetRef]\ \frac{\{C\}\ M :_m \{C_0\} \quad \{[!x]C_0 \wedge !x = m \wedge x\#\tilde{e}\}\ N :_u \{C'\} \quad x \notin \mathsf{fpn}(\tilde{e})}{\{C\}\ \mathtt{let}\ x = \mathtt{ref}(M)\ \mathtt{in}\ N :_u \{\nu x.C'\}}$$

with C' think w.r.t. m. The derivation follows. Below $M_{1,2}$ is the body of the first/second lets, respectively.

1.$\{Even(!x) \wedge [!x]C'\}$ $\mathtt{if}\ even(!x)\ \mathtt{then}\ ()\ \mathtt{else}\ \Omega()$ $\{[!x]C'\}@\emptyset$	(If)
2.$\{[!x]C\}\ gf\ \{[!x]C'\}$	(cf. § 6.7)
3.$\{Even(!x) \wedge [!x]C\}\ gf\ \{Even(!x) \wedge [!x]C'\}$	(2, Inv)
4.$\{E \wedge [!x]C \wedge Even(!x) \wedge x\#gi\}\mathtt{let}\ f = \ldots\ \mathtt{in}\ (gf; \ldots)\{[!x]C' \wedge x\#i\}$	(3, Seq, Let)
5.$\{E \wedge C\}$ $\mathtt{MeyerSieber}$ $\{\nu x.([!x]C' \wedge x\#i)\}$	(4, LetRef)
6.$\{E \wedge C\}$ $\mathtt{MeyerSieber}$ $\{C'\}$	(9, Prop. 5.16)

D.3. **Derivation for Object.** We need the following generalisation. The procedure u in (AIH) is of function type $\alpha \Rightarrow \beta$: when values of other types such as $\alpha \times \beta$ or $\alpha + \beta$ are returned, we can make use of a generalisation. For simplicity we restrict our attention to the case when types do not contain recursive or reference types.

$$\begin{aligned}
\mathsf{Inv}(u^{\alpha\times\beta}, C_0, \tilde{x}) &\stackrel{\mathsf{def}}{\equiv} \wedge_{i=1,2}\mathsf{Inv}(\pi_i(u), C_0, \tilde{x}) \\
\mathsf{Inv}(u^{\alpha+\beta}, C_0, \tilde{x}) &\stackrel{\mathsf{def}}{\equiv} \wedge_{i=1,2}\forall y_i.(u = \mathsf{inj}_i(y_i) \supset \mathsf{Inv}(y_i, C_0, \tilde{x})) \\
\mathsf{Inv}(u^{\alpha}, C_0, \tilde{x}) &\stackrel{\mathsf{def}}{\equiv} \mathsf{T} \qquad (\alpha \in \{\mathsf{Unit}, \mathsf{Nat}, \mathsf{Bool}\})
\end{aligned}$$

Using this extension, we can generalise (AIH) so that the cancelling of C_0 is possible for all components of u. For example, if u is a pair of functions, those two functions need to satisfy the same condition as in (AIH). This is what we shall use for `cellGen`. We call the resulting generalised axiom $(\mathsf{AIH_c})$.

Let `cell` be the internal λ-abstraction of `cellGen`. First, it is easy to obtain:

$$\{\mathsf{T}\}\ \mathtt{cell} :_o \{I_0 \wedge G_1 \wedge G_2 \wedge E'\} \tag{D.1}$$

where, with $I_0 \stackrel{\mathsf{def}}{\equiv} !x_0 = !x_1 \ \wedge\ x_0 \# iw$ (noting $x\#v \equiv \mathsf{T}$) and $E' \stackrel{\mathsf{def}}{\equiv} !x_0 = z$.

$$\begin{array}{rcl} G_1 & \stackrel{\mathsf{def}}{\equiv} & \Box\{I_0\}\pi_1(o) \bullet () = v\{v = !x_0 \wedge I_0\}@\emptyset \\ G_2 & \stackrel{\mathsf{def}}{\equiv} & \Box\forall w.\{I_0\}\pi_1(o) \bullet w\{!x_0 = w \wedge I_0\}@x_0x_1 \end{array}$$

which will become, after taking off the invariant I_0:

$$\begin{array}{rcl} G'_1 & \stackrel{\mathsf{def}}{\equiv} & \Box\pi_1(o) \bullet () = v\{v = !x_1\}@\emptyset \\ G'_2 & \stackrel{\mathsf{def}}{\equiv} & \Box\forall w.\pi_1(o) \bullet w\{!x_0 = w\}@x_0. \end{array}$$

Note I_0 is stateless except for x_0. In G_1, notice the empty effect set means $!x_1$ does not change from the pre to the post-condition. We now present the inference. Below we set $\mathtt{cell}' \stackrel{\mathrm{def}}{=} \mathtt{let}\ y = \mathtt{ref}(0)\ \mathtt{in}\ \mathtt{cell}$ and i, k fresh.

1.$\{\mathsf{T}\}$ $\mathtt{cell} :_o \ \{I_0 \wedge G_1 \wedge G_2 \wedge E'\}$	
2.$\{\mathsf{T}\}$ $\mathtt{cell}' :_o \ \{!x_0 = !x_1 \wedge G_1 \wedge G_2 \wedge E'\}$	(LetRef)
3.$\{\mathsf{T}\}$ $\mathtt{let}\ x_1 = z\ \mathtt{in}\ \mathtt{cell}' :_o \{\nu x_1.(I_0 \wedge G_1 \wedge G_2) \wedge E'\}$	(LetRef)
4.$\{\mathsf{T}\}$ $\mathtt{let}\ x_1 = z\ \mathtt{in}\ \mathtt{cell}' :_o \ \{G'_1 \wedge G'_2 \wedge E'\}$	($\mathrm{AIH}_{\mathsf{c}}$, ConsEval)
5.$\{\mathsf{T}\}$ $\mathtt{let}\ x_{0,1} = z\ \mathtt{in}\ \mathtt{cell}' :_o \ \{\nu x.(\ x\#k \wedge Cell(o,x) \wedge !x = z\ )\}$	(LetRef)
6.$\{\mathsf{T}\}$ $\mathtt{cellGen} :_u \ \{CellGen(u)\}$.	(Abs)