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ABOUT THE RANGE PROPERTY FOR $\mathcal{H}$

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Abstract. Recently, A. Polonsky (see [Pol12]) has shown that the range property fails for $\mathcal{H}$. We give here some conditions on a closed term that imply that its range has an infinite cardinality.

Introduction

0.1. **Our motivations.**

Let $\mathcal{T}$ be a λ-theory. The *range property* for $\mathcal{T}$ states that if $\lambda x.F$ is a closed λ-term, then its range (considering $\lambda x.F$ as a map from $\mathcal{M}$ to $\mathcal{M}$ where $\mathcal{M}$ is the algebra of closed λ-terms modulo the equality defined by $\mathcal{T}$) has cardinality either 1 or $Card(\mathcal{M})$. It has been proved by Barendregt in [Bar93] that the range property holds for all recursively enumerable theories. For the theory $\mathcal{H}$ equating all unsolvable terms, the validity of the range property has been an open problem for a long time. Very recently A. Polonsky has shown (see [Pol12]) that it fails.

In an old attempt to prove the range property for $\mathcal{H}$, Barendregt (see [Bar08]) suggested a possible way to get the result. The idea was, roughly, as follows. First observe that, if the range of a term is not a singleton, it will reduce to a term of the form $\lambda x.F[x]$. Assuming that, for some A, $F[x := A] \neq_{\mathcal{H}} F[x := \Omega]$, he proposed another term A' (the term $J_\nu \circ A$ of Conjecture 3.2 in [Bar08]) having a free variable ν that could never be erased or used in a reduction of $F[x := A']$. He claimed that, by the properties of the variable ν, the terms $F[x := A_n]$ should be different where $A_n = A'[\nu := c_n]$ and c_n is, for example, the Church numeral for n. It has been rather quickly understood by various researchers that the term proposed in [Bar08] actually *had not* the desired property and, of course, Polonsky's result shows this method could not work. Moreover, even if the term A' had the desired property it is, actually, not true that the terms $F[x := A_n]$ would be different: see section 0.3 below.

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0.2. **Our results.**

Even though the failure of the range property for $\mathcal{H}$ is now known, we believe that having conditions which imply that the range of a term is infinite is, "by itself" interesting. We also think that the idea proposed by Barendregt remains interesting and, in this paper, we consider the following problem.

Say that F has the *Barendregt's persistence property* if for each A such that $F[x := A] \neq_{\mathcal{H}} F[x := \Omega]$ we can find a term A' that has a free variable ν that could never be erased or applied in a reduction of $F[x := A']$ (see definition 2.18).

We give here some conditions on terms that imply the Barendregt's persistence property. Our main result is Theorem 2.20. It can be stated as follows. Let F be a term having a unique free variable x and A be a closed term such that $F[x := A] \neq_{\mathcal{H}} F[x := \Omega]$. We introduce a sequence $(F_k)_{k\in\mathbb{N}}$ of reducts of F that can simulate (see Lemma 2.6) all the reductions of F. By considering, for each k, the different occurrences of x in F_k we introduce a tree $\mathcal{T}$ and a special branch in it (see Theorem 2.9). Denoting by $x_{(k)}$ the corresponding occurrence of x in F_k, Theorem 2.20 states that:

(1) If, for $k \in \mathbb{N}$, the number of arguments of $x_{(k)}$ in F_k is bounded, F has the Barendregt's persistence property.

(2) Otherwise and assuming the branch in $\mathcal{T}$ is recursive there are two cases.
  - (2.a) If some of the arguments of $x_{(k)}$ in F_k come in head position infinitely often during the head reduction of $F[x := A]$, then F has the Barendregt's persistence property.
  - (2.b) Otherwise, it is possible that F does not have the Barendregt's persistence property.

In case (1) we give two arguments. The first one is quite easy and uses this very particular situation. It gives a term A' where ν cannot be erased but we have not shown that it is never applied (it is not applied only in the branch defined in section 2). The second one (which is more complicated) is a complete proof that F has the Barendregt's persistence property. It can also be seen as an introduction to the more elaborate case 2.(a). For case 2.(b) we give examples of terms F for which there is no term having this property.

0.3. **A final remark.**

Note that, actually, even if a term A' *having* the Barendregt's persistence property can be found this would not give an infinite range for $\lambda x.F$. This is due to the fact that the property of A' does not imply that $F[x = A_n] \neq_{\mathcal{H}} F[x = A_m]$ for $n \neq m$ where $A_k = A'[\nu = c_k]$ and c_k is the Church integer for k. This is an old result of Plotkin (see [Plo74]). Thus, having an infinite range will need another assumption. We will also consider this other assumption and thus give simple criteria that imply that the range of $\lambda x.F$ is infinite.

The paper is organized as follows. Section 1 gives the necessary definitions. Section 2 considers the possible situations for F and states our main result. Section 3 and 4 give the proof in the cases where we actually can find an A'. Section 5 gives some complements.

1. Preliminaries

Notation 1.1.

(1) We denote by Λ° the set of closed λ-terms.
(2) We denote by c_k the Church integer for k and by Suc a closed term for the successor function. As usual we denote by I (resp. K, Ω) the term $\lambda x.x$ (resp. $\lambda x \lambda y.x$, $(\delta\ \delta)$ where $\delta = \lambda x.(x\ x)$).
(3) $\#(t)$ is the code of t i.e. an integer coding the way t is built.
(4) We denote by $\triangleright$ the β-reduction, by $\triangleright_\Omega$ the Ω-reduction (i.e. $t \triangleright_\Omega \Omega$ if t is unsolvable), by $\triangleright_h$ the head β-reduction and by $\triangleright_{\beta\Omega}$ the union of $\triangleright$ and $\triangleright_\Omega$.
(5) If R is a notion of reduction, we denote by R^* its reflexive and transitive closure.
(6) Let x be a free variable of a term t. We denote by $x \in \beta\Omega(t)$ the fact that x does occur in any t' such that $t \triangleright^*_{\beta\Omega} t'$.
(7) As usual $(u\ t_1\ t_2\ ...\ t_n)$ denotes $(...((u\ t_1)\ t_2)\ ...\ t_n)$. $(u^n\ v)$ will denote $(u\ (u\ ...\ (u\ v)...))$ and $(v\ u^{\sim n})$ will denote $(v\ u\ ...\ u)$ with n occurrences of u.

Notation 1.2.

(1) Unknown sequences (possibly empty) of abstractions or terms will be denoted with an arrow. For example $\lambda \overrightarrow{z}$ or $\overrightarrow{w}$. However, to improve readability, capital letters will also be used to denote sequences. The notable exceptions are F, A, J taken from Barendregt's paper or standard notations for terms as I, K, Ω. When the meaning is not clear from the context, we will explicitly say something as "the sequence $\lambda\overrightarrow{z}$ of abstractions".
(2) For example, to mean that a term t can be written as some abstractions followed by the application of the variable x to some terms, we will say that $t = \lambda\overrightarrow{z}.(x\ \overrightarrow{w})$ or $t = \lambda Z.(x\ W)$.
(3) If R, S are sequences of terms of the same length, $R \triangleright^* S$ means that each term of the sequence R reduces to the corresponding term of the sequence S.

Lemma 1.3.

(1) *If $u \triangleright^*_{\beta\Omega} v$, then $u \triangleright^* w \triangleright^*_\Omega v$ for some w.*
(2) *$\triangleright^*_{\beta\Omega}$ satisfies the Church-Rosser property : if $t \triangleright^*_{\beta\Omega} t_1$ and $t \triangleright^*_{\beta\Omega} t_2$, then $t_1 \triangleright^*_{\beta\Omega} t_3$ and $t_2 \triangleright^*_{\beta\Omega} t_3$ for some t_3.*

Proof. See, for example, [Bar85]. □

Theorem 1.4. *If $t \triangleright^* t_1$ and $t \triangleright^* t_2$, then $t_1 \triangleright^* t_3$ and $t_2 \triangleright^* t_3$ for some t_3. Moreover $\#(t_3)$ can be computed from $\#(t_1)$, $\#(t_2)$ and a code for the reductions $t \triangleright^* t_1$, $t \triangleright^* t_2$.*

Proof. See [Bar85]. □

Definition 1.5.

(1) We denote by $\simeq$ the equality modulo $\triangleright^*_{\beta\Omega}$ i.e. $u \simeq v$ iff there is w such that $u \triangleright^*_{\beta\Omega} w$ and $v \triangleright^*_{\beta\Omega} w$ and we denote by $[t]_\mathcal{H}$ the class of t modulo $\simeq$.
(2) For $\lambda x.F \in \Lambda^\circ$, the range of $\lambda x.F$ in $\mathcal{H}$ is the set $\Im(\lambda x.F) = \{[F[x := u]]_\mathcal{H}\ /\ u \in \Lambda^\circ\}$.
(3) A closed term $\lambda x.F$ has the range property for $\mathcal{H}$ if the set $\Im(\lambda x.F)$ is either infinite or has a unique element.

Theorem 1.6. *There is a term $\lambda x.F \in \Lambda^\circ$ that has not the range property for $\mathcal{H}$.*

Proof. See [Pol12]. □

Definition 1.7. Let U, V be finite sequences of terms.

(1) We denote by $U :: V$ the list obtained by putting U in front of V.
(2) $U \sqsubseteq V$ means that some initial subsequence of V is obtained from U by substitutions and reductions.

Lemma 1.8.

(1) *$U \sqsubseteq V$ iff there is a substitution σ such that $\sigma(U)$ reduces to an initial segment of V.*
(2) *The relation $\sqsubseteq$ is transitive.*

Proof. Easy. $\square$

Notation 1.9. We will have to use the following notion. *A sub-term u of a term v comes in head position during the head reduction of v.* This means the following: v can be written as $C[u]$ where C is some context with exactly one hole. During the head reduction of C the hole comes in head position i.e C reduces to $\lambda \overrightarrow{x}.([]\ \overrightarrow{w})$ for some $\overrightarrow{w}$. The only problem in making this definition precise is that, during the reduction of C to $\lambda \overrightarrow{x}.([]\ \overrightarrow{w})$ the potentially free variables of u may be substituted and we have to deal with that. The notations and tools developed in, for example, [Dav01], [DN95] allows to do that precisely. Since this is intuitively quite clear and we do not need any technical result on this definition, we will not go further.

2. The different cases

Let $t = \lambda x.F[x]$ be a closed term. First observe that

(1) If $x \notin \beta\Omega(F)$, then the range of t is a singleton.
(2) If $x \in \beta\Omega(F)$ and F is normalizable, then the range of t is trivially infinite since then the set $\{[F[x := \lambda x_1 ... \lambda x_n c_k]]_{\mathcal{H}}\ /\ k \in \mathbb{N}\}$ is infinite where n is the size of the normal form of F.
(3) More generally, if t has a finite Böhm tree then it satisfies the range property.

From now on, we thus fix terms F and A. We assume that:

- F is not normalizable and has a unique free variable denoted as x such that $x \in \beta\Omega(F)$. In the rest of the paper we will write $t[A]$ instead of $t[x := A]$.
- $F[A] \not\simeq F[\Omega]$

Notation 2.1.

(1) The different occurrences of a free variable x in a term will be denoted as $x_{[i]}$ for various indexes i.
(2) Let t be a term and $x_{[i]}$ be an occurrence of the (free) variable x in t. We denote by $Arg(x_{[i]}, t)$ (this is called the scope of $x_{[i]}$ in [Bar85]) the maximal list of arguments of $x_{[i]}$ in t i.e the list V such that $([]\ V)$ is the applicative context of $x_{[i]}$ in t.

Since $Arg(x_{[i]}, t)$ may contain variables that are bounded in t and since a term is defined modulo α-equivalence, this notion is not, strictly speaking, well defined. This is not problematic and we do not try to give a more formal definition.

Lemma 2.2. *Assume $u \triangleright^* v$ and $x_{[i]}$ (resp. $x_{[j]}$) is an occurrence of x in u (resp. v) such that $x_{[j]}$ is a residue of $x_{[i]}$. Then $Arg(x_{[i]}, u) \sqsubseteq Arg(x_{[j]}, v)$.*

Proof. Since the relation $\sqsubseteq$ is transitive it is enough to show the result when $u \triangleright v$. This is easily done by considering the position of the reduced redex. $\square$

Definition 2.3. Let $x_{[i]}$ be an occurrence of x in some term t. We say that $x_{[i]}$ is pure in t if there is no other occurrence $x_{[j]}$ of x in t such that $x_{[i]}$ occurs in one of the elements of the list $Arg(x_{[j]}, t)$.

For example, let $t = (x\ (x\ y))$. It can be written as $(x_{[1]}\ (x_{[2]}\ y))$ where the occurrence $x_{[1]}$ is pure but $x_{[2]}$ is not pure and $Arg(x_{[1]}, t) = (x_{[2]}\ y)$.

Note that, if $x_{[1]}, ..., x_{[n]}$ are all the pure occurrences of x in t, there is a context C with holes $[]_1, ..., []_n$ such that $t = C[[]_i = (x_{[i]}\ Arg(x_{[i]}, t)) : i = 1...n]$ and x does not occur in C.

Lemma 2.4. *Let t, t' be some terms such that t reduces to t'. Assume that $x_{[i']}$ is a residue in t' of $x_{[i]}$ in t and $x_{[i']}$ is pure in t'. Then $x_{[i]}$ is pure in t.*

Proof. Immediate. □

The next technical result is akin to Barendregt's lemma discussed by de Vrijer in Barendregt's festschrift (see [Vri07]). It has a curious history discussed there. First proved by van Dalen in [Daa80], it appears as an exercise in Barendregt's book at the end of chapter 14. Its truth may look strange. Note that the reductions coming from the term B are done *in the holes of the reduct* G of F.

Lemma 2.5. *Let B be a closed term. Assume $F[B] \triangleright^* t$. Then there is a G such that*

- $F \triangleright^* G = D[[]_i = w_i\ :\ i \in \mathcal{I}]$ *where D is a context with holes $[]_i$ (indexed by the set $\mathcal{I}$ of all the pure occurrences $x_{[i]}$ of x in G) and $w_i = (x_{[i]}\ Arg(x_{[i]}, G))$.*
- $t = D[[]_i = w'_i\ :\ i \in \mathcal{I}]$ *where $w_i[B] \triangleright^* w'_i$.*

Proof. By induction on $\langle lg(F[B] \triangleright^* t), cxty(F)\rangle$ where $lg(F[B] \triangleright^* t)$ is the length of a *standard* reduction of $F[B]$ to t and $cxty(F)$ is the complexity of F, i.e the number of symbols in F.

- If $F = \lambda y.F'$, then $t = \lambda y.t'$ where $F'[B] \triangleright^* t'$. Since $lg(F[B] \triangleright^* t) = lg(F'[B] \triangleright^* t')$ and $cxty(F') < cxty(F)$, we conclude by applying the induction hypothesis on the reduction $F'[B] \triangleright^* t'$.

- If $F = (y\ F_1...F_n)$, then $t = (y\ t_1...t_n)$ where $F_i[B] \triangleright^* t_i$. Since $lg(F_i[B] \triangleright^* t_i) \leq lg(F[B] \triangleright^* t)$ and $cxty(F_i) < cxty(F)$, we conclude by applying the induction hypothesis on the reductions $F_i[B] \triangleright^* t_i$.

- If $F = (\lambda y.U\ V\ F_1...F_n)$ where the head redex is not reduced during the reduction $F[B] \triangleright^* t$, then $t = (\lambda yu\ v\ f_1...f_n)$ where $U[B] \triangleright^* u$, $V[B] \triangleright^* v$ and $F_i[B] \triangleright^* f_i$. Since $lg(U[B] \triangleright^* u) \leq lg(F[B] \triangleright^* t)$, $lg(V[B] \triangleright^* v) \leq lg(F[B] \triangleright^* t)$, $lg(F_i[B] \triangleright^* f_i) \leq lg(F[B] \triangleright^* t)$, $cxty(U) < cxty(F)$, $cxty(V) < cxty(F)$ and $cxty(F_i) < cxty(F)$, we conclude by applying the induction hypothesis on the reductions $U[B] \triangleright^* u$, $V[B] \triangleright^* v$, $F_i[B] \triangleright^* f_i$.

- If $F = (\lambda y.U\ V\ \overrightarrow{F})$ and the first step of the standard reduction reduces the head redex, then $F[B] \triangleright = (U[y := V]\ \overrightarrow{F})[B] \triangleright^* t$. Let $F' = (U[y := V]\ \overrightarrow{F})$. Since $lg(F'[B] \triangleright^* t) < lg(F[B] \triangleright^* t)$, we conclude by applying the induction hypothesis on the reduction $F'[B] \triangleright^* t$.

- If $F = (x\ \overrightarrow{F})$, then $G = F$ and D is the term made of a single context $[]$ and $w = F$. □

The next lemma concerns the reduction of F under a recursive cofinal strategy. The canonical one is the Gross–Knuth strategy, where one takes, at each step, the full development of the previous one.

Lemma 2.6. *There is a sequence $(F_k)_{k \in \mathbb{N}}$ such that*

(1) *$F_0 = F$ and, for each k, $F_k \triangleright^* F_{k+1}$.*
(2) *If $F \triangleright^* G$, then $G \triangleright^* F_k$ for some k.*
(3) *The function $k \hookrightarrow \#(F_k)$ is recursive.*

Proof. By Theorem 1.4, choose F_{k+1} as a common reduct of F_k and all the reducts of F in less than k steps. □

Definition 2.7. Let $x_{[i]}$ be an occurrence of x in some F_k. We say that $x_{[i]}$ is good in F_k if it satisfies the following properties:

- $x_{[i]}$ is pure in F_k.
- $u[A]$ is solvable for every sub-term u of F_k such that $x_{[i]}$ occurs in u.

  Note that this implies that $(x_{[i]}\ Arg(x_{[i]}, F_k))[A]$ is solvable.

Observe that every pure occurrence of x in F_{k+1} is a residue of a pure occurrence of x in F_k. This allows the following definition.

Definition 2.8.

(1) Let $\mathcal{T}$ be the following tree. The level k in $\mathcal{T}$ is the set of pure occurrences of x in F_k. An occurrence $x_{[i]}$ of x in F_{k+1} is the son of an occurrence $x_{[j]}$ of x in F_k if $x_{[i]}$ is a residue of $x_{[j]}$.

(2) A branch in $\mathcal{T}$ is good if, for each k, the occurrence $x_{[k]}$ of x in F_k chosen by the branch is good in F_k.

Theorem 2.9. *There is an infinite branch in $\mathcal{T}$ that is good.*

Proof. By Konig's Lemma it is enough to show: (1) for each k, there is an occurrence of x in F_k that is good and (2) if the son of an occurrence $x_{[i]}$ of x is good then so is $x_{[i]}$.

(1) Assume first that there is no good occurrences of x in F_k. This means that, for all pure occurrences $x_{[i]}$ of x in F_k, either $(x_{[i]}\ Arg(x_{[i]}, F_k))[A]$ is unsolvable or this occurrence appears inside a sub-term u of F_k such that $u[A]$ is unsolvable. But, if $(x_{[i]}\ Arg(x_{[i]}, F_k))[A]$ is unsolvable then so is $(x_{[i]}\ Arg(x_{[i]}, F_k))[\Omega]$ and, if $u[A]$ is unsolvable, then so is $u[\Omega]$ (proof : consider the head reduction of $u[A]$; either A comes in head position or not ; in both cases the result is clear). This implies that $F[A] \simeq F[\Omega]$.

(2) follows immediately from the fact that a residue of an unsolvable term also is unsolvable and that, if $(x\ U)[A]$ is unsolvable and $U \sqsubseteq V$ then so is $(x\ V)[A]$. □

Example 2.10. Let G be a λ-term such that $G \triangleright^* \lambda u \lambda v.(v\ (G\ (x\ u)\ v))$ and $F = (G\ I)$. If we take $\lambda v.(v^k(G\ (x\ (x\ ...(x\ I)))\ v))$ for F_k, what is the good occurrence of x in F_k which appears in a good branch in $\mathcal{T}$? It is none of those in $(x\ (x\ ...(x\ I))...)$, it is the one in G !

From now on, we fix an infinite branch in $\mathcal{T}$ that is good

Notation 2.11. We denote by $x_{(k)}$ the occurrence of x in F_k chosen by the branch. Let $U_k = Arg(x_{(k)}, F_k)$.

Lemma 2.12. *There is a sequence $(\sigma_k)_{k\in\mathbb{N}}$ of substitutions and there are sequences $(S_k)_{k\in\mathbb{N}}$, $(R_k)_{k\in\mathbb{N}}$ of finite sequences of terms such that, for each k, R_k is obtained from $\sigma_k(U_k)$ by some reductions and $U_{k+1} = R_k :: S_k$.*

Proof. This follows immediately from Lemma 2.2. □

Definition 2.13. We define the sequence V_k by : $V_0 = U_0$ and $V_{k+1} = \sigma_k(V_k) :: S_k$.

Lemma 2.14.
(1) *For each k, $V_k \triangleright^* U_k$.*
(2) *For each $k' > k$, there is a substitution $\sigma_{k'k}$ such that $\sigma_{k'k}(S_k)$ is a sub-sequence of $V_{k'}$.*

Proof. This follows immediately from Lemma 2.12. If $k' = k+1$, $\sigma_{k'k} = id$. Otherwise $\sigma_{k'k} = \sigma_{k'-1} \circ \sigma_{k'-2} \circ ... \circ \sigma_{k+1}$ □

Definition 2.15. We define, by induction on k, the sequence ρ_k of reductions and the terms t_k as follows.
(1) ρ_0 is the head reduction of $(A\ V_0[A])$ to its head normal form.
(2) $t_k = \lambda \overrightarrow{z_k}.\ (y_k\ \overrightarrow{w_k})$ is the result of ρ_k.
(3) ρ_{k+1} is the head reduction of $(\sigma_k(t_k)\ S_k[A])$ to its head normal form.

Lemma 2.16. *The term t_k is the head normal form of $(A\ V_k[A])$.*

Proof. Easy. □

Notation and comments
(1) Denote by ρ the infinite sequence of reductions $\rho_0, \rho_1, ..., \rho_k, ...$. Note that it is not the reduction of one unique term. ρ_0 computes the head normal form t_0 of $(A\ V_0[A])$. The role of σ_0 is to substitute in the result the substitution that changes U_0 into the first part of U_1. Note that, by Lemma 2.12, this first part may also have been reduced but here we forget this reduction. Then we use ρ_1 to get the head normal form t_1 of $(\sigma_0(t_0)\ S_1[A])$ and keep going like that.
(2) Note that, by Lemma 2.14 and 2.16, t_k is *some* head normal form for $(A\ U_k[A])$ but it is not the *canonical* one i.e. the one obtained by reducing, at each step, the head redex.

Definition 2.17. Say that S_k comes in head position during ρ if, for some $k' > k$, an element of the list $\sigma_{k'k}(S_k)[A]$ comes in head position during the head reduction of $(A\ V_{k'}[A])$.

Definition 2.18. (1) Let t be a term with a free variable ν.
 (a) Say that ν is never applied in a reduct of t if no reduct t' of t contains a sub-term of the form $(\nu\ u)$.
 (b) Say that ν is persisting in t if $\nu \in \beta\Omega(t)$ and ν is never applied in any reduct of t.
(2) We say that the term F has the Barendregt's persistence property if we can find a term A' that has a free variable ν that is persisting in $F[A']$.

Comment and Example 2.19.
(1) The condition "ν is never applied" in the previous definition implies that, letting $A_n = A'[\nu = c_n]$, a reduct of $F[A_n]$ is, essentially, a reduct of $F[A']$.
(2) Here is an example. Let G be a λ-term such that $G \triangleright^* \lambda u \lambda v.(v\ (G\ (u\ I)\ v))$ and $F = (G\ x)$. We can take $\lambda z.(z^k\ (G\ (x\ I^{\sim k})\ z))$ for F_k. It follows easily that $F[I] \not\simeq F[\Omega]$. We have $U_k = I^{\sim k}$, $S_k = I$, $t_k = I$ and $\sigma_k = id$.
 Let J be a λ-term such that $J \triangleright^* \lambda u \lambda v \lambda y.(v\ (J\ u\ y))$ and $I' = (J\ \nu\ I)$.
 If $F[I'] \triangleright^* t$, then, by Lemma 2.5, $t \triangleright^* \lambda z.(z^n\ (G\ I''\ z))$ for some n where I'' is a reduct of $(I'\ I^{\sim n})$. It is easily checked that ν is persisting in $F[I']$. Since no c_n occurs as a sub-term of a reduct of $F[I']$, it is not difficult to show that, if $n \neq m$, then $F[I_n] \not\simeq F[I_m]$ where $I_n = I'[\nu := c_n]$ and thus $\Im(\lambda x.F)$ is infinite.

Theorem 2.20.

(1) *Assume first that the length of the U_k are bounded. Then, F has the Barendregt's persistence property.*
(2) *Assume next that the length of the U_k are not bounded and the branch we have chosen in $\mathcal{T}$ is* **recursive**.
    (a) *If the set of those k such that S_k comes in head position during ρ is infinite, then F has the Barendregt's persistence property.*
    (b) *Otherwise it is possible that F does not have the Barendregt's persistence pro-perty.*

Proof.

(1) It follows immediately from Lemma 2.2 that there are $l, k_0 > 0$ such that for all $k \geq k_0, lg(U_k) = l$. The fact that F has the Barendregt's persistence property is proved in section 3.
(2) (a) The fact that F has the Barendregt's persistence property is proved in section 4.
    (b) There are actually two cases and the reasons why we cannot find a term A' are quite different.
        (i) For all k there is $k' > k$ such that $y_{k'} \in dom(\sigma_{k'})$. The fact that the head variable of t_k may change infinitely often does not allow to use the technic of sections 3 or 4. A. Polonsky has given a term F that corresponds to this situation and such that a variable ν can never be persisting in a term A' of the form $\lambda x_1 ... x_n.(x_i\ w_1 ...\ w_m)$. See example 1 below.
        (ii) For some k_1, $y_k \notin dom(\sigma_k)$ for all $k \geq k_1$. Since there are infinitely many $k \geq k_1$ such that S_k is non empty this implies that, after some steps, t_k does not begin by λ. Thus, there is k_2 and y, such that, for all $k \geq k_2$, $t_k = (y\ \overrightarrow{w_k})$. Using the technic of sections 3 or 4 allows to put a term J in front of some (fixed) element of the sequence $\overrightarrow{w_k}$ but this is not enough to keep ν. We adapt the example of A. Polonsky to give a term F that corresponds to this situation and such that a variable ν can never be persisting in a term A' of the form $\lambda x_1 ... \lambda x_n.(A\ w_1 ...\ w_n)$ where $w_j \simeq \lambda y_1 ... \lambda y_{r_j}.(x_j\ w_1^j ... w_{r_j}^j)$. See example 2 below. □

Comments 2.21.

(1) For case 1. we will give two proofs. The first one is quite simple. The second one is much more elaborate and even though it, actually, does not work for all the possible situations, we give it because it is an introduction to the more complex section 4. In the first proof, we simply use the fact that the length of the U_k are bounded to find a term A' that has nothing to do with A. In the second proof and in section 4, the term A' that we give has the Bargendregt's property and behaves like A (using the idea of [Bar08]) in the sense that it looks like an infinite η-expansion of A.
(2) When we say, in case 2.(b) of the theorem, that it is possible that F does not have the Barendregt's persistence property we are a bit cheating. We only show (except in example 1) that there is no A' with a persisting ν satisfying an extra condition. This condition is that A' looks like A i.e. the first levels of the Böhm tree of A' must be, up-to some η-equivalence, the same as the ones of A.
(3) It is known that there are recursive (by this we mean that we can compute their levels) and infinite trees such that each level is finite and that have no recursive infinite branch. We have not tried to transform such a tree in a lambda term such that the corresponding $\mathcal{T}$ has no branch that is good and recursive but we guess this is possible.

Example 2.22. This example is due to A. Polonsky. Let G, H be λ-terms such that $G \triangleright^* \lambda y \lambda z.(z\ G\ \lambda u.(y\ (K\ u))\ z)$ and $H \triangleright^* \lambda u \lambda v \lambda w.(w\ (H\ u\ (v\ u)\ w))$.
Let $F = (G\ \lambda y.(H\ y\ x))$.
We have $F \triangleright^* \lambda z.(z^n\ (G\ \lambda y \lambda w.(w^m\ (H\ (K^n\ y)\ (x\ (K^n\ y)^{\sim m})\ w))\ z))$.
Thus, if $B \simeq \lambda y_1 ... \lambda y_r.(y_i\ w_1 ... w_l)$ where ν is possibly free in the w_j,
$F[B] \simeq \lambda z.(z^l\ (G\ \lambda y \lambda w.(w^r\ (H\ (K^l\ y)\ y\ w))\ z))$ and ν is not persisting in $F[B]$ (since it can be erased). This means that for all closed term A such that $F[A] \not\simeq F[\Omega]$, and for all solvable term A', ν is not persisting in $F[A']$.

Example 2.23. This example is an adaptation of the previous one. Let G, H be λ-terms such that $G \triangleright^* \lambda y \lambda z.(z\ (G\ \lambda u.(y\ (K\ u))\ z))$ and $H \triangleright^* \lambda u \lambda v \lambda w.(w\ (H\ u\ (v\ u)\ w))$.
Let $F = \lambda y.(G\ \lambda v.(H\ v\ (x\ y)))$.
We have $F \triangleright^* \lambda y \lambda z.(z^n\ (G\ \lambda v \lambda w.(w^m\ (H\ (K^n\ v)\ (x\ y\ (K^n\ v)^{\sim m})\ w))\ z))$ and it is clear that $F[I] \not\simeq F[\Omega]$.
Let $I' \simeq \lambda x \lambda x_1 ... \lambda x_r.(x\ w_1 ... w_r)$ where, for $1 \leq j \leq r$, $w_j \simeq \lambda y_1 ... \lambda y_{r_j}(x_j\ w_1^j ... w_{r_j}^j)$.
For $n \geq max_{1 \leq j \leq r}(r_j)$, $F[I'] \simeq \lambda y \lambda z.(z^n\ (G\ \lambda v \lambda w.(w^m\ (H\ (K^n\ v)\ (y\ w'_1 ... w'_r)\ w))\ z))$ for some w'_j where ν does not occur and thus ν is not persisting in $F[I']$.

Note, however, that ν is persisting in $F[I'']$ where $I'' = \lambda z.(z\ \nu)$.

3. Case 1 of Theorem 2.20

We assume in this section that we are in case 1. of Theorem 2.20.

3.1. **A simple argument.**

Let $A' = \lambda x_1 ... \lambda x_l \lambda z.(z\ \nu)$ where l is a bound for the length of the U_k. We show that $\nu \in \beta\Omega(F[A'])$. It is easy to show that ν is never applied in the terms $(A'\ U_k[A'])$ but the fact that ν is never applied in a reduct of $F[A']$ is not so clear. Since, because of the next section, we do not need this point we have not tried to check.

If $F[A'] \triangleright^* H \triangleright^*_\Omega G$. By Lemma 2.5, $F \triangleright^* F' = D[[]_i = w_i\ :\ i \in \mathcal{I}]$, $w_i = (x_{(i)}\ Arg(x_{(i)}, F'))$ and $H = D[[]_i = w'_i\ :\ i \in \mathcal{I}]$ where $w_i[A'] \triangleright^* w'_i$. Let k be such that $F' \triangleright F_k$. Let $i_0 \in \mathcal{I}$ be such that the occurrence of $x_{(k)}$ in F_k chosen by the good branch is a residue of the occurrence of $x_{[i_0]}$ in F'. By Lemma 2.2, the length of $Arg(x_{[i_0]}, F')$ is bounded by l and thus $w'_{i_0} = \lambda x_q ... x_l \lambda z.(z\ \nu)$ for some $q \leq l$.

It remains to show that the sub-term $\lambda z.(z\ \nu)$ of w'_{i_0} cannot be erased in the Ω-reduction from H to G. Assume it is not the case. Then, there is a sub-term D' of D containing the hole $[]_{i_0}$ such that $D'[[]_i = w'_i\ :\ i \in \mathcal{I}]$ is unsolvable. $D'[[]_i = w_i]$ is solvable (since, otherwise, the occurrence $x_{[k]}$ will be in an unsolvable sub-term of F_k and this contradicts the fact that $x_{[k]}$ is good). Since the reduction $D'[[]_i = w_i] \triangleright^* D'[[]_i = w'_i]$ only is inside the w'_i, since the first term is solvable and the second one is not, then, by the Church-Rosser property, the head variable of the head normal form of $D'[[]_i = w_i]$ is an occurrence of x. By Lemma 2.4, $x_{[i_0]}$ is pure in F'. $x_{[i_0]}$ is not the head variable of the head normal form of $D'[[]_i = w_i]$ (since, otherwise, by the Church-Rosser property, $D'[[]_i = w'_i]$ would be solvable). Contradiction.

3.2. **The proof.**

There are actually different situations.

(1) Either, for some $k_1 \geq k_0$, $y_k \in \overrightarrow{z_k}$ for any $k \geq k_1$. Since for $k \geq k_1$, S_k is empty and the head variable of t_k is not substituted, there is $\overrightarrow{z}$ and $z \in \overrightarrow{z}$, such that, for $k \geq k_1$, $t_k = \lambda \overrightarrow{z}.\ (z\ \overrightarrow{w_k})$.

(2) Or $y_k \notin \overrightarrow{z_k}$ for all $k \geq k_0$ and
   (a) Either the situation is unstable (i.e. the set of those k such that $y_k \in dom(\sigma_k)$ is infinite).
   (b) Or, there is $\overrightarrow{z}$, $y \notin \overrightarrow{z}$ and $k_1 \geq k_0$, such that, for $k \geq k_1$, $t_k = \lambda \overrightarrow{z}.\ (y\ \overrightarrow{w_k})$.

We assume that we are in situation 1. or 2.(b) which may be synthesized by: there exists k_0 and some fixed variable y (that may be in $\overrightarrow{z}$ or not), such that, for all $k \geq k_0$, $t_k = \lambda \overrightarrow{z}.\ (y\ \overrightarrow{w_k})$ for some $\overrightarrow{w_k}$.

We fix $p \geq lg(\overrightarrow{z}) + l + 2$ where l is a bound for the length of the U_k. Let $A' = (J\ \nu\ A)$ where J is a new constant with the following reduction rule

$$(J\ \nu\ u) \triangleright \lambda y_1 ... \lambda y_p.(u\ (J\ \nu\ y_1)...(J\ \nu\ y_p))$$

We will prove that ν is persisting in $F[A']$.

The term A' is not a pure λ-term since the constant J occurs in it. We could, of course, replace this constant by a λ-term J' that has the same behavior, e.g. $(Y\ \lambda k \lambda y_1 ... \lambda y_p.(u\ (k\ \nu\ y_1)...(k\ \nu\ y_p)))$ where Y is the Turing fixed point operator. But such a term introduces some problems in Lemma 3.6 because J' and $(J'\ \nu)$ contain redexes and can be reduced. With such a term J', though intuitively true, this lemma (as it is stated) does not remain correct.

Making this lemma correct (with J' instead of a constant) will require the treatment of redexes inside J' and $(J'\ \nu)$. The reader should be convinced that this can be done but, since it would need tedious definitions, we will not do it.

Note that, in situation 2.(a), we cannot do the kind of proof given below. Here is an example. Let G be a λ-term such that $G \triangleright^* \lambda u \lambda v.(v\ (G\ \lambda y.(u\ (K\ y))\ v))$ and $F = (G\ x)$. We can take $\lambda z.(z^k\ (G\ \lambda y.(x\ (K^k\ y))\ z)$ for F_k. We thus have $U_k = (K^k\ y)$, $t_k = \lambda z_1 ... \lambda z_k.\ y$ and $\sigma_k = [y := (K\ y)]$. It is clear that $F[I] \not\simeq F[\Omega]$. But $F[I'] \triangleright^* \lambda z.(z^k\ (G\ I\ z))$ for any $I' \simeq \lambda y \lambda y_1 ... \lambda y_k.\ (y\ w_1 ... w_k)$ where ν is possibly free in the w_i. Thus ν is not persisting in $F[I']$.

3.2.1. *Some preliminary definitions and results.*

When, in a term t, we replace some sub-term u by $(J\ \nu\ u)$ to get t', the reducts u (resp. u'), of t (resp. of t') are very similar. The goal of this section is to make this a bit precise.

Definition 3.1.

(1) We define, for terms u, the sets E_u of terms by the following grammar:
$E_u = u\ |(J\ \nu\ e_u)\ |\lambda \overrightarrow{y}.(e_u\ \overrightarrow{e_y})$

(2) Let t, t' be some terms. We denote by $t \rightsquigarrow t'$ if there is a context C with one hole such that $t = C[u]$ and $t' = C[u']$ where $u' \in E_u$.

Notations and Example 3.2.

(1) Note that, in the previous definition as well as in the sequel, e_u always denotes a term in E_u and, for a sequence $\overrightarrow{y}$ of variables , $\overrightarrow{e_y}$ always denote a sequence of terms $\overrightarrow{v}$ (of the same length as $\overrightarrow{y}$) such that for each variable y in $\overrightarrow{y}$, the corresponding term in $\overrightarrow{v}$ is a member of E_y. Also note that, in the previous definition, for a term $\lambda\overrightarrow{y}.(e_u\ \overrightarrow{e_y})$ to be in E_u, we assume that the variables in $\overrightarrow{y}$ do not occur in e_u.
(2) $t \rightsquigarrow t'$ means that t' is obtained from t by replacing some sub-term u of t by $(J\ \nu\ u)$ or by reducing redexes introduced by J i.e. the one coming from its reduction rule $(J\ \nu\ u) \triangleright \lambda y_1...\lambda y_p.(u\ (J\ \nu\ y_1)...(J\ \nu\ y_p))$ and those whose λ's are among $\lambda y_1...\lambda y_p$.

Lemma 3.3.
(1) *If $t \in E_z$ and $t' \in E_u$ then $t[z := t'] \in E_u$.*
(2) *If $a \rightsquigarrow a'$ and $b \rightsquigarrow b'$, then $a[x := b] \rightsquigarrow^* a'[x := b']$.*
(3) *If $u \rightsquigarrow^* u'$ and if v' is an Ω-redex in u', then $v \rightsquigarrow^* v'$ for some Ω-redex v in u.*
(4) *If $t \in E_u$ then $t \triangleright^* \lambda\overrightarrow{y}(u\ \overrightarrow{e_y})$ for some $\overrightarrow{e_y}$.*

Proof. 1, 2 and 3 are immediate. 4 is proved by induction on the number of rules used to show $t \in E_u$. □

Lemma 3.4.
(1) *Assume $u = C[(\lambda x.a\ b)]$ for some context C and let $u \rightsquigarrow^* u'$. Then $u' = C'[(a'\ b')]$ for some context C' and some terms a', b' such that $C \rightsquigarrow^* C'$, $a' \in E_{\lambda x.a''}$, $a \rightsquigarrow^* a''$ and $b \rightsquigarrow^* b'$.*
(2) *Assume $u \rightsquigarrow^* u'$ and $u \triangleright^* v$. Then, $v \rightsquigarrow^* v'$ for some v' such that $u' \triangleright^* v'$.*

Proof. The first point is immediate because the operations that are done to go from u to u' are either in C (to get C') or in b (to get b') or in a (to get a') or in $\lambda x.a'$. It is enough to check that locals and globals operations on a commute.

For the second point, we do the proof for one step of reduction $u \triangleright v$ and we use the first point and Lemma 3.3. □

Lemma 3.5.
(1) *If $\overrightarrow{\lambda x}.(x\ \overrightarrow{c}) \rightsquigarrow^* u$, then $u \triangleright^* \overrightarrow{\lambda x}\overrightarrow{\lambda y}.(x\ \overrightarrow{c'}\ \overrightarrow{e_y})$ for some $\overrightarrow{c} \rightsquigarrow^* \overrightarrow{c'}$ and some $\overrightarrow{e_y}$.*
(2) *If $u \triangleright^* \overrightarrow{\lambda x}.(x\ \overrightarrow{c})$ and $u \rightsquigarrow^* u'$, then $u' \triangleright^* \overrightarrow{\lambda x}\overrightarrow{\lambda y}.(x\ \overrightarrow{c'}\ \overrightarrow{e_y})$ for some $\overrightarrow{c} \rightsquigarrow^* \overrightarrow{c'}$ and some $\overrightarrow{e_y}$.*

Proof.
(1) We do the proof on an example. It is clear that this is quite general. Assume $\lambda x_1\lambda x_2(x\ c_1\ c_2) \rightsquigarrow^* u$, then $u \in E_{\lambda x_1 u_1}$, $u_1 \in E_{\lambda x_2 u_2}$, $u_2 \in E_{(v_1\ c_2')}$, $v_1 \in E_{(v_2\ c_1')}$, $v_2 \in E_x$, $c_1 \rightsquigarrow^* c_1'$ and $c_2 \rightsquigarrow^* c_2'$. The β-reduction is done starting from inside and the propagation is done by using Lemma 3.3.
(2) It is a consequence of the first point and Lemma 3.4. □

The next lemma means that, when a redex appears in some u' where $u \rightsquigarrow^* u'$, it can either come from the corresponding redex in u, or has been created by the transformation of an application in u that was not already a redex or comes from the replacement of some sub-term u by, essentially, $(J\ \nu\ u)$.

Lemma 3.6.
(1) *Assume $u' = C'[R']$ for some context C' and some redex R' and let $u \rightsquigarrow^* u'$. Then :*

- *either* $R' = (\lambda x.a'\ b')$, $u = C[(\lambda x.a\ b)]$ *for some context* C *and some terms* a, b *such that* $C \rightsquigarrow^* C'$, $a \rightsquigarrow^* a'$ *and* $b \rightsquigarrow^* b'$.
- *or* $R' = (a'\ b')$, $u = C[(a\ b)]$ *for some context* C *and some terms* a, b *such that* $C \rightsquigarrow^* C'$, $a' = \lambda\overrightarrow{y}.(a''\ \overrightarrow{e_y})$, $a \rightsquigarrow^* a''$ *and* $b \rightsquigarrow^* b'$.
- *or* $R' = (J\ \nu\ a')$, $u = C[a]$ *for some context* C *and some term* a *such that* $C \rightsquigarrow^* C'$ *and* $a \rightsquigarrow^* a'$.

(2) *Assume* $u \rightsquigarrow^* u'$ *and* $u' \triangleright^* v'$. *Then* $v \rightsquigarrow^* v'$ *for some* v *such that* $u \triangleright^* v$.

Proof. For the first point, there are two cases. Either the redex R' is the residue of a redex in u or it has been created by the operations from the grammar E.

For the second point, it is enough to prove the result for one step of reduction $u' \triangleright v'$. Use the first point. □

Definition 3.7. Let t be a solvable term. We say that:

(1) ν occurs nicely in t if the only occurrences of ν are in a sub-term of the form $(J\ \nu)$.
(2) ν occurs correctly in t if it occurs nicely in t and the head normal form of t looks like $\overrightarrow{\lambda x}.(x\ \overrightarrow{c}\ \overrightarrow{e_y})$ for some final subsequence $\overrightarrow{y}$ of $\overrightarrow{x}$ of length at least 1 such that ν does occur in $\overrightarrow{e_y}$.

Lemma 3.8. *Let* t *be a solvable term. Assume that* ν *occurs nicely (resp. correctly) in* t. *Then* ν *occurs nicely (resp. correctly) in every reduct of* t.

Proof. By the properties of J. □

Lemma 3.9. *The variable* ν *is never applied in a reduct of* $F[A']$.

Proof. The variable ν occurs nicely in $F[A']$, then, by Lemma 3.8, it occurs nicely in every reduct of $F[A']$, thus ν is never applied in a reduct of $F[A']$. □

3.2.2. *End of the proof.*

Proposition 3.10. *For* $k \geq k_0$, ν *occurs correctly in* $(A'\ V_k[A])$.

Proof. $(A\ V_k[A]) \rightsquigarrow^* (A\ (J\ \nu\ V_k[A]))$ (note that this last term may be misunderstood: it actually means $(A\ (J\ \nu\ a_1)\ ...\ (J\ \nu\ a_q))$ where $V_k[A]$ is the sequence $a_1 ... a_q$) and $(A\ V_k[A]) \triangleright_h^* \lambda\overrightarrow{z}.(y\ \overrightarrow{w_k})$. Thus, by Lemma 3.5, $(A\ (J\ \nu\ V_k[A])) \triangleright^* \overrightarrow{\lambda z}\overrightarrow{\lambda x}.(y\ \overrightarrow{w_k'}\ \overrightarrow{e_x})$ for some $\overrightarrow{w_k} \rightsquigarrow^* \overrightarrow{w_k'}$ and $\overrightarrow{e_x}$.

But $(A'\ V_k[A]) \triangleright^* \lambda\overrightarrow{y}(A\ (J\ \nu\ V_k[A])\ \overrightarrow{(J\ \nu\ y)})$ and thus $(A'\ V_k[A]) \triangleright^* \lambda\overrightarrow{y}.(\lambda\overrightarrow{z}\lambda\overrightarrow{x}.(y\ \overrightarrow{w_k'}\ \overrightarrow{w_x})\ \overrightarrow{(J\ \nu\ y)})$. Using then $p - l \geq lg(\overrightarrow{z}) + 2$ and distinguishing $y \notin \overrightarrow{z}$ or $y \in \overrightarrow{z}$ it follows easily that $(A'\ V_k[A]) \triangleright^* \lambda\overrightarrow{y_1}(y_2\ \overrightarrow{w_k''}\ \overrightarrow{w_{y_3}})$ where $\overrightarrow{y_3}$ is a final subsequence of $\overrightarrow{y_1}$, $\overrightarrow{e_{y_3}} \in \overrightarrow{E_{y_3}}$ and $lg(\overrightarrow{y_3}) \geq 1$.

The fact that ν occurs nicely is clear. □

Proposition 3.11. *Assume that there is a sequence $(j_k)_{k\in\mathbb{N}}$ of integers such that, for each k, ν occurs correctly in $(A'\ U_{j_k}[A])$. Then ν is persisting in $F[A']$.*

Proof. Let t_1 be a reduct of $F[A']$. By Lemma 3.9, it is enough to show that ν does occur in t_1. By Lemma 1.3, let t_2 be such that $F[A'] \rhd^* t_2 \rhd_\Omega^* t_1$. By Lemma 2.5, $F \rhd^* G = D[[]_i = w_i \ : \ i \in \mathcal{I}]$ where $w_i = (x_{[i]}\ Arg(x_{[i]}, G))$ and $t_2 = D[[]_i = w'_i \ : \ i \in \mathcal{I}]$ where w'_i is a β-reduct of $w_i[A']$.

By Lemma 2.6, $G \rhd^* F_{j_k}$ for some k. Let $x_{[j]}$ be the occurence of x in G which has $x_{(j_k)}$ as residue in F_{j_k}. By Lemma 2.2, there is a substitution σ and a sequence V of terms such that $U_{j_k} = W :: V$ and $\sigma(M_j)$ reduces to W where $M_j = Arg(x_{[j]}, G)$. By Lemma 3.6, let w''_j be such that $(A'\ \ Arg(x_{[j]}, G)[A])$ reduces to w''_j and $w''_j \rightsquigarrow^* w'_j$.

Since $(A'\ \sigma(M_j)[A])$ reduces both to $\sigma(w''_j)$ and to $(A'\ W[A])$, let s be a common reduct of $(\sigma(w''_j)\ V[A])$ and $(A'\ W[A]\ V[A])$. Since ν occurs correctly in $(A'\ W[A]\ V[A])$, it occurs correctly (by Lemma 3.8) in s. But $(\sigma(w''_j)\ V[A])$ reduces to s and ν does not occur in σ neither in $V[A]$, then ν occurs in w''_j. Thus it occurs in w'_j and thus in t_2.

Since an Ω-reduction of w''_j cannot erase ν (otherwise, by Church-Rosser, ν will not occur in a reduct of s) and $w''_j \rightsquigarrow^* w'_j$, then, by Lemma 3.3(item 3), an Ω-reduction of w'_j cannot erase all its ν. The same proof as the one in section 3.1 shows that ν cannot be totally erased by the Ω-reduction from t_2 to t_1 and thus it occurs in t_1. ☐

Corollary 3.12. *ν is persisting in $F[A']$.*

Proof. By Proposition 3.10, for $k \geq k_0$, ν occurs correctly in $(A'\ V_k[A])$. By Lemma 3.8, ν occurs correctly in $(A'\ U_k[A])$ and we conclude by Proposition 3.11. ☐

4. Case 2(a) of Theorem 2.20

Here is an example of this situation. Let $F = (G\ x)$ where G is a λ-term such that $G \rhd^* \lambda u \lambda v.(v\ (G\ (u\ K\ K)\ v))$. We can take $\lambda z.(z^k\ (G\ (x\ K^{\sim 2k})\ z)$ for F_k. Thus $U_k = K^{\sim 2k}$, $S_k = K^{\sim 2}$, $t_k = K$ and $\sigma_k = id$. $F[K] \not\simeq F[\Omega]$. We are in situation 2.(a) of Theorem 2.20 because $(\sigma_k(t_k)\ S_k[K]) = (K\ K^{\sim 2}) \rhd^* K$.

The idea of the construction

The desired A' looks like the one for the previous case. It will be $(\widehat{J}\ c_0\ \nu\ A)$ for *some other term* $\widehat{J}$ that behaves mainly as J but has to be a bit more clever. The difference with the previous one is the following. For the J of section 3.2, $(J\ \nu\ u)$ reduces to a term where the J that occurs in the arguments of u is *the same*. Here $\widehat{J}$ has to be parameterized by some integers. This is the role of its first argument. We will define a term $\widehat{J}$ and will denote $(\widehat{J}\ c_n)$ by J_n i.e. intuitively, the $\widehat{J}$ at "step" n. Here $(J_n\ \nu\ u)$ will reduce to a term where J_{n+1} and not J_n occurs in the arguments of u. See Definition 4.3 and Lemma 4.4 below.

The reason is the following. In section 3.2, after some steps, the head variable of $(A\ V_k[A])$ does not change anymore and the variable ν can no more disappear. Here, to be able to ensure that ν does not disappear, $\widehat{J}$ must occur in head position infinitely often and, for that, it has to be able to introduce more and more λ's.

The λ's that are introduced by J_n (i.e. $\lambda y_1 ... \lambda y_{h_n}$ of Lemma 4.4) are used for two things. First they ensure that a $\widehat{J}$ will be added to the next S_k, say S_{k_n}, coming in head position. Secondly, they ensure that ν occurs correctly in $(A'\ V_{j_n}[A])$ where j_n is large enough to let S_{k_n} come in head position.

This is the idea but there is one difficulty. If S_{k_n} comes in head position during the head reduction of $(A\ V_{j_n}[A])$, it will also come in head position during the reduction of $(A'\ V_{j_n}[A])$ but (see Lemma 3.5) some λ's are put in front (the $\overrightarrow{\lambda y}$ of Lemma 4.7). If we could compute their number it will not be problematic but, actually, this is not possible. For the following reason: these λ's come from the J_k that came in head positions in previous steps but, when we are trying to put a $\widehat{J}$ in front of S_{k_n} it is possible that J_p has already appeared in head position for some $p > n$.

We thus proceed as follows. J_0 introduces enough λ's to put J_1 in front of S_{k_0} (the first S_k that comes in head position during ρ) and to ensure that ν will occur correctly in $(A'\ V_{j_0}[A])$. Then we look at the first term of V_{j_0} that comes in head position during ρ. This term, say a_0, has J_1 in front of it in the head reduction of $(A'\ V_{j_0}[A])$. This is the term (a_0 may be a term in S_{k_0} but it may be some other term) that will allow to define the number of $\lambda's$ that J_1 introduces. It introduces enough $\lambda's$ to put J_2 in front of S_{k_1}, the next S_k coming in head position, and to ensure that ν occurs correctly in the reduction of $(A'\ V_{j_1}[A])$. Note that k_1 has to be large enough to avoid the λ's that occur at the beginning of the term where a_0 occurs in head position. We keep going in this way to define the J_n. It is clear (this could be formally proved by using standard fixed point theorems) that the function h (see definitions 4.1 and 4.3 below) that computes the parameter c_n at step n is recursive. A formalization of this is given in Definition 4.1 below.

There is a final, though not essential, difficulty to make this precise. The definition of $\widehat{J}$ needs to know the function h. But to compute h we need some approximation of $\widehat{J}$. More precisely to compute $h(n)$ we need to know the behavior of $\widehat{J}$ where only the values of $h(p)$ for $p < n$ will be used. To do that, we introduce fake J_n (they are denoted as $\tilde{J}_n$ below). They are as J_n but the term J_{n+1} (which is not yet known since $h(n+1)$ is not yet known) is replaced by some fresh constant γ. Note that, using the standard fixed point theorem, we could avoid these fake $\tilde{J}_n$ but then, Definition 4.1 below should be more complicated because the constant γ would be replaced by a term computed from a code of $\widehat{J}$ and we should explain how this is computed ...

Finally note that it is at this point that we use the fact that the branch in $\mathcal{T}$ is recursive.

Back to the example

With the example given before, we can take for $\widehat{J}$ a λ-term such that $\widehat{J} \rhd^* \lambda u \lambda v \lambda y_1 \lambda y_2.(v\ (\widehat{J}\ u\ y_1)\ \ (\widehat{J}\ u\ y_2))$ and for K' the term $(\widehat{J}\ \nu\ K)$. Then $K' \rhd^* K'' = \lambda y_1 \lambda y_2 \lambda z_1 \lambda z_2.(y_1\ (\widehat{J}\ u\ z_1)\ (\widehat{J}\ u\ z_2))$. Since, if $F[K'] \rhd^* t$, then $t \rhd^* \lambda z.(z^k\ (G\ K''\ z))$. It follows that ν is persisting in $F[K']$. Note that here the function h is constant. Given any recursive function h' it will not be difficult to build terms F, A such that the corresponding function h is precisely h'.

Definition 4.1. For each i, let l_i be the number of λ's at the head of t_i.

The definition of $\widehat{J}$ needs some new objects. Let γ be a new constant. We define, by induction, the integers j_n, h_n, the terms $\tilde{J}_n, a_{j_n}$ and the sequence of terms B_{j_n}.

In this definition a term (or a sequence of terms) marked with $'$ is a term in relation $\rightsquigarrow$ with the corresponding unmarked term.

- (*Step 0*) Let j_0 be the least integer j such that some S_k comes in head position during ρ_j (the head reduction of $(A\ V_j[A])$), $k_0 = lg(V_{j_0})$, $h_0 = max(l_0, k_0)$. Let $V_{j_0} = d_1 ... d_{k_0}$

and $\tilde{J}_0 = \lambda n\lambda x\lambda x_1...\lambda x_{h_0}(x\ (\gamma\ n\ x_1)\ ...\ (\gamma\ n\ x_{h_0}))$. Then

$$((\tilde{J}_0\ \nu\ A)\ V_{j_0}[A]) \rhd^* \lambda y_1...y_{r_0}(A\ (\gamma\ \nu\ d_1[A])\ ...\ (\gamma\ \nu\ d_{k_0}[A])\ (\gamma\ \nu\ y_1)\ ...\ (\gamma\ \nu\ y_{r_0}).$$

Since, for some i, $d_i[A]$ comes in head position during the head reduction of $(A\ V_{j_0}[A])$, then, for some i', $(\gamma\ \nu\ d_{i'}[A])$ comes in head position during the head reduction of $((\tilde{J}_0\ \nu\ A)\ V_{j_0}[A])$ and thus $((\tilde{J}_0\ \nu\ A)\ V_{j_0}[A]) \rhd^* \lambda X_{j_0}((\gamma\ \nu\ a_{j_0})\ B_{j_0})$ for some a_{j_0}, some sequence X_{j_0} of variables and some sequence B_{j_0} of terms.

- (*Step 1*) Let j_1 be the least integer $j > j_0$ such that some S_k comes in head position during ρ_j for k large enough ($lg(X_{j_0}) < lg(S_{j_0} :: ... :: S_{k-1})$ is needed). Then
$$((\tilde{J}_0\ \nu\ A)\ V_{j_1}[A]) = ((\tilde{J}_0\ \nu\ A)\ V'_{j_0}[A]\ S'_{j_0}...S'_k...S'_{j_1-1}) \rhd^* ((\gamma\ \nu\ a'_{j_0})\ B'_{j_0})\ T_0.$$
Let $h_1 = l_{j_0} + lg(B'_{j_0} :: T_0) + 1$ and $\tilde{J}_0 = \tilde{J}_1[\gamma := \lambda n\lambda x\lambda x_1...\lambda x_{h_1}(x\ (\gamma\ n\ x_1)\ ...\ (\gamma\ n\ x_{h_1}))]$. Then $((\tilde{J}_1\ \nu\ A)\ V_{j_1}[A]) \rhd^* \lambda X_{j_1}((\gamma\ \nu\ a_{j_1})\ B_{j_1})$ for some term a_{j_1} and some sequence B_{j_1} of terms.
- (*Step n+1*) Assume the integers j_n, h_n and the term $\tilde{J}_n$ are already defined. Let j_{n+1} be the least integer $j > j_n$ such that some S_k comes in head position during ρ_j for k large enough ($lg(X_{j_n}) < lg(S_{j_1} :: ... :: S_{k-1})$ is needed). Then
$$((\tilde{J}_n\ \nu\ A)\ V_{j_{n+1}}[A]) = ((\tilde{J}_n\ \nu\ A)\ V'_{j_1}[A]\ S'_{j_n}...S'_k...S'_{j_{n+1}-1}) \rhd^* ((\gamma\ \nu\ a'_{j_n})\ B'_{j_n})\ T_n.$$
Let $h_{n+1} = l_{j_n} + lg(B'_{j_n} :: T_n) + 1$ and $\tilde{J}_{n+1} = \tilde{J}_n[\gamma := \lambda p\lambda x\lambda x_1...\lambda x_{h_{n+1}}(x\ (\gamma\ p\ x_1)\ ...\ (\gamma\ p\ x_{h_{n+1}}))]$. Then $((\tilde{J}_{n+1}\ \nu\ A)\ V_{j_{n+1}}[A]) \rhd^* \lambda X_{j_{n+1}}((\gamma\ \nu\ a_{j_{n+1}})\ B_{j_{n+1}})$ for some term $a_{j_{n+1}}$ and some sequence $B_{j_{n+1}}$ of terms.

Comment 4.2. Note that, since the branch in $\mathcal{T}$ is recursive, it follows, by standard arguments, that the function h defined by $h(i) = h_i$ is computable.

Definition 4.3.
- Let H be a λ-term that represents the function h.
- Let $T = \lambda a\lambda b\lambda c.(a\ (b\ (z\ (suc\ k)\ n\ c)))$, $D = \lambda k\lambda n\lambda x.(H\ k\ T\ I\ x)$ and $\widehat{J} = (Y\ \lambda z.D)$ where Y is the Turing fixed point operator.
- For each $n \in \mathbb{N}$, we denote $(\widehat{J}\ c_n)$ by J_n.

Lemma 4.4. *For each $n \in \mathbb{N}$, $(J_n\ \nu\ u) \rhd^* \lambda y_1...\lambda y_{h_n}.(u\ (J_{n+1}\ \nu\ y_1)...(J_{n+1}\ \nu\ y_{h_n}))$.*

Proof. Easy. □

As in the previous section, we consider J_n as constants with the reduction rules of the previous Lemma. Let $A' = (J_0\ \nu\ A)$. We prove that ν is persisting in $F[A']$.

Definition 4.5.
(1) We define, for terms u, the sets E_u of terms by the following grammar:
$E_u = u\ |(J_n\ \nu\ e_u)\ |\lambda\overrightarrow{y}.(e_u\ \overrightarrow{e_y})$
(2) Let t, t' be some terms. We denote by $t \rightsquigarrow t'$ if there is a context C with one hole such that $t = C[u]$ and $t' = C[u']$ where $u' \in E_u$.

Lemma 4.6. (1) *Assume $u \rightsquigarrow^* u'$ and $u \rhd^* v$. Then, $v \rightsquigarrow^* v'$ for some v' such that $u' \rhd^* v'$.*
(2) *Assume $u \rightsquigarrow^* u'$ and $u' \rhd^* v'$. Then, $v \rightsquigarrow^* v'$ for some v such that $u \rhd^* v$.*

Proof. Same proof as Lemmas 3.4 and 3.6. □

Lemma 4.7. *If* $u \triangleright^* \overrightarrow{\lambda x}.(x\ \overrightarrow{c})$ *and* $u \rightsquigarrow^* u'$, *then* $u' \triangleright^* \overrightarrow{\lambda x}\overrightarrow{\lambda y}.(x\ \overrightarrow{c'}\ \overrightarrow{e_y})$ *for some* $\overrightarrow{c} \rightsquigarrow^* \overrightarrow{c'}$ *and* $\overrightarrow{e_y}$.

Proof. This follows from Lemma 4.6 □

Definition 4.8. Let t be a solvable term. We say that:

(1) ν occurs nicely in t if the only occurrences of ν are in a sub-term of the form $(J_n\ \nu)$.
(2) ν occurs correctly in t if it occurs nicely in t and the head normal form of t looks like $\overrightarrow{\lambda x}.(x\ \overrightarrow{c}\ \overrightarrow{e_y})$ for some final subsequence $\overrightarrow{y}$ of $\overrightarrow{x}$ of length at least 1 such that ν does occur in $\overrightarrow{e_y}$.

The intuitive meaning of Lemma 4.9 below is "for each $n \in \mathbb{N}$, $t_{j_n} \rightsquigarrow^* (a_{j_n}\ B_{j_n})$ and $(A'\ V_{j_n}[A]) \triangleright^* \lambda X_{j_n}.(J_{n+1}\ \nu\ a_{j_n}\ B_{j_n})$".

Strictly speaking, this is not true, because the a_{j_n} and B_{j_n} are not the "real" ones i.e. the ones that occur in the reduction with the real $\widehat{J}$. For two reasons:

- The first one is easily corrected: in a_{j_n} and B_{j_n} the constant γ must be replaced by J_{n+1}
- The second one is more subtle. In the correct lemma, $\rightsquigarrow$ and the J_n should be the "real" ones. But the a_{j_n} are defined using $\tilde{J}_p$ which are only fake J_p. Stating the correct lemma will need complicated, and useless, definitions. We will not do it and thus we state the lemma in the way it should be, intuitively, understood.

Lemma 4.9. *For each* $n \in \mathbb{N}$, $t_{j_n} \rightsquigarrow^* (a_{j_n}[\gamma := J_{n+1}]\ B_{j_n}[\gamma := J_{n+1}])$ *and* $(A'\ V_{j_n}[A]) \triangleright^* \lambda X_{j_n}.(J_{n+1}\ \nu\ a_{j_n}[\gamma := J_{n+1}]\ B_{j_n}[\gamma := J_{n+1}])$.

Proof. By induction on n. □

Proposition 4.10. *For each* $n \in \mathbb{N}$, ν *occurs correctly in* $(A'\ V_{j_n}[A])$.

Proof. We have $(A'\ V_{j_n}[A]) \triangleright^* \lambda X_{j_n}.(J_{n+1}\ \nu\ a_{j_n}[\gamma := J_{n+1}]\ B_{j_n}[\gamma := J_{n+1}]) = \lambda X_{j_n}.((\lambda z_1 ... \lambda z_{h_{n+1}}.(\ a_{j_n}[\gamma := J_{n+1}]\ (J_{n+2}\ \nu\ z_1)\ ...\ (J_{n+2}\ \nu\ z_{h_{n+1}}))\ B_{j_n}[\gamma := J_{n+1}])$. Since $h_{n+1} = l_{j_n} + lg(B'_{j_n} T_n) + 1$, then, by Lemma 4.9, $(A'\ V_{j_n}[A]) \triangleright^* \lambda X_{j_n} \lambda Z.(t'_{j_n}\ e_Z)$ where $t'_{j_n} \rightsquigarrow^* t_{j_n}$, $e_Z \rightsquigarrow^* Z$ and $lg(Z) > l_{j_n}$. Therefore, by Lemma 4.7, ν occurs correctly in $(A'\ V_{j_n}[A])$. □

Lemma 4.11. *Let* t *be a solvable term. Assume that* ν *occurs correctly in* t. *Then* ν *occurs (and it occurs correctly) in every reduct of* t.

Proof. This follows immediately from the fact that if $(J_n\ \nu\ y) \rightsquigarrow^* u$, then $\nu \in \beta\Omega(u)$. □

Lemma 4.12. *Let* k *be an integer such that* ν *occurs correctly in* $(A'\ V_k[A])$. *Then* ν *occurs correctly in* $(A'\ U_k[A'])$.

Proof. This follows immediately from Lemmas 4.7 and 4.11. □

Proposition 4.13. ν *is persisting in* $F[A']$.

Proof. Same proof as Proposition 3.11. □

5. The other assumption

As we already said the fact that ν is persisting in $F[A']$ does not imply that, letting $A_n = A'[\nu = c_n]$, $F[A_n] \not\simeq F[A_m]$ for $n \neq m$. To ensure that the range of $\lambda x.F$ is infinite, we need another assumption on F.

Let $\lambda x.F$ be a closed term and A be such that $F[A] \not\simeq F[\Omega]$. *In propositions 5.1 and 5.3 below we assume that F has the Barendregt's persistence property. Let A' be the corresponding term. We also assume that A' has been obtained by the way developed in section 3.2 or in section 4.* Note that, in these cases, ν is never applied in a reduct of $F[A']$.

Proposition 5.1. *Assume there is a sequence $(t_n)_{n\in\mathbb{N}}$ of distinct closed and normal terms such that, for every n, t_n never occurs as a sub-term of some t' such that $F[A'] \triangleright^*_{\beta\Omega} t'$. Then, the range of $\lambda x.F$ is infinite.*

Proof. Let $A_n = A'[\nu := t_n]$. It is enough to show that $F[A_n] \not\simeq F[A_m]$ for $n \neq m$. Assume $F[A_n] \simeq F[A_m]$ for some $n \neq m$ and let u be a common reduct. Since ν is never applied in a reduct of $F[A']$, there are reducts a_n and a_m of $F[A']$ such that $u = a_m[\nu := t_m] = a_n[\nu := t_n]$. This implies that t_n occurs in a_m. Contradiction. $\square$

Remark

Say that u is a universal generator if, for every closed λ-term t, there is a reduct of u where t occurs as a sub-term. Before Plotkin gave his counterexample, Barendregt had proved that the omega-rule is valid when t, t' are not universal generators. Our hypothesis on the existence of the sequence $(t_n)_{n\in\mathbb{N}}$ may look similar. Assuming that $F[A']$ is not a universal generator, there is a term t that never occurs as a sub-term of a reduct of $F[A']$. Letting $t_0 = t$ and $t_{n+1} = \lambda x.t_n$, the reducts of $F[A']$ never contain one of these terms. Also, they are not equal, because otherwise t is Ω. This is however not enough to show that the $F[\nu := t_n]$ are distinct because we need that no reduct of $F[A']$ contains *a reduct* of one of the t_n (this is why, in our hypothesis, we have assumed that the t_n are normal). This raises two questions :

(1) Say that a term u is a weak generator if for every closed λ-term t one of its reducts occurs as a sub-term of a reduct of u. A universal generator is, trivially, a weak generator. Is the converse true ?
(2) Is it true that, if $F[A']$ is a universal generator then so is $F[A]$. Note that, somehow, the A' we have constructed is a kind of η-infinite expansion of A.

If these two propositions were true, we could replace the assumption of Proposition 5.1 by the (more elegant) fact that $F[A]$ is not a universal generator.

Definition 5.2. Let $A_n = A'[\nu = c_n]$. Say that F, A satisfy the Scope lemma if, for every n, m, the fact that $F[A_n] \simeq F[A_m]$ implies that, for some k, $(x\ U_k)[x := A_n] \simeq (x\ U_k)[x := A_m]$.

The terminology "Scope lemma" is borrowed to A. Polonsky. In his paper he stated an hypothesis (denoted as the scope Lemma) which corresponds to the previous property.

Proposition 5.3. *Assume F, A satisfy the scope lemma. Then the range of $\lambda x.F$ is infinite.*

Proof. Immediate. $\square$

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