

SMALL PROMISE CSPS THAT REDUCE TO LARGE CSPS

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ABSTRACT. For relational structures \mathbf{A}, \mathbf{B} of the same signature, the Promise Constraint Satisfaction Problem $\text{PCSP}(\mathbf{A}, \mathbf{B})$ asks whether a given input structure maps homomorphically to \mathbf{A} or does not even map to \mathbf{B} . We are promised that the input satisfies exactly one of these two cases.

If there exists a structure \mathbf{C} with homomorphisms $\mathbf{A} \rightarrow \mathbf{C} \rightarrow \mathbf{B}$, then $\text{PCSP}(\mathbf{A}, \mathbf{B})$ reduces naturally to $\text{CSP}(\mathbf{C})$. To the best of our knowledge all known tractable PCSPs reduce to tractable CSPs in this way. However Barto [Bar19] showed that some PCSPs over finite structures \mathbf{A}, \mathbf{B} require solving CSPs over infinite \mathbf{C} .

We show that even when such a reduction to some finite \mathbf{C} is possible, this structure may become arbitrarily large. For every integer $n > 1$ and every prime p we give \mathbf{A}, \mathbf{B} of size n with a single relation of arity n^p such that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ reduces via a chain of homomorphisms $\mathbf{A} \rightarrow \mathbf{C} \rightarrow \mathbf{B}$ to a tractable CSP over some \mathbf{C} of size p but not over any smaller structure. In a second family of examples, for every prime $p \geq 7$ we construct \mathbf{A}, \mathbf{B} of size $p - 1$ with a single ternary relation such that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ reduces via $\mathbf{A} \rightarrow \mathbf{C} \rightarrow \mathbf{B}$ to a tractable CSP over some \mathbf{C} of size p but not over any smaller structure. In contrast we show that if \mathbf{A}, \mathbf{B} are graphs and $\text{PCSP}(\mathbf{A}, \mathbf{B})$ reduces to a tractable $\text{CSP}(\mathbf{C})$ for some finite digraph \mathbf{C} , then already \mathbf{A} or \mathbf{B} has a tractable CSP. This extends results and answers a question of [DSM⁺21].

1. INTRODUCTION

The *Constraint Satisfaction Problem* (CSP) for a fixed relational structure \mathbf{A} can be formulated as the following homomorphism problem:

$\text{CSP}(\mathbf{A})$

Input: a relational structure \mathbf{X}

Output: yes, if there exists a homomorphism $\mathbf{X} \rightarrow \mathbf{A}$,
no, otherwise

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In [AHG14, AGH17] Austrin, Guruswami, and Håstad introduced *Promise Satisfaction Problems* (PCSP) as approximations of CSP. For relational structures \mathbf{A}, \mathbf{B} of the same finite signature with a homomorphism $\mathbf{A} \rightarrow \mathbf{B}$, let

PCSP(\mathbf{A}, \mathbf{B})

Input: a relational structure \mathbf{X}

Output: yes, if there exists a homomorphism $\mathbf{X} \rightarrow \mathbf{A}$,
no, if there exists no homomorphism $\mathbf{X} \rightarrow \mathbf{B}$

Since we have a homomorphism $\mathbf{A} \rightarrow \mathbf{B}$ by assumption, the two alternatives (1) \mathbf{X} maps homomorphically to \mathbf{A} or (2) \mathbf{X} does not map homomorphically to \mathbf{B} are mutually exclusive for any input \mathbf{X} . The *promise* is that at least one of the alternatives holds for \mathbf{X} .

PCSPs are motivated by open questions about the (in)approximability of SAT and graph coloring. The classical *approximate graph coloring problem* for $r \leq s$ asks: given an r -colorable graph, find an s -coloring for it [GJ76]. The decision version of this is to distinguish graphs that are r -colorable from those that are not even s -colorable. This is PCSP($\mathbf{K}_r, \mathbf{K}_s$) for $\mathbf{K}_r, \mathbf{K}_s$ the complete graphs on r, s vertices, respectively. It (and consequently the approximate graph coloring problem) has been conjectured to be NP-hard for all $3 \leq r \leq s$. Even after more than 40 years of research this conjecture remains open in its full generality.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be relational structures of the same signature with homomorphisms $\mathbf{A} \rightarrow \mathbf{C} \rightarrow \mathbf{B}$. Then we say \mathbf{C} is *sandwiched* by \mathbf{A} and \mathbf{B} . In this case PCSP(\mathbf{A}, \mathbf{B}) has a trivial reduction to CSP(\mathbf{C}) without changing the instance. In general, the complexity of the PCSP is unknown. Still, to the best of our knowledge, all known tractable (i.e., solvable in polynomial time) PCSPs reduce to tractable CSPs in this way.

As in [AB21] we call PCSP(\mathbf{A}, \mathbf{B}) *finitely tractable* if there exists a finite \mathbf{C} that is sandwiched by \mathbf{A} and \mathbf{B} such that CSP(\mathbf{C}) is tractable. Barto [Bar19] provided the first example of a PCSP over finite structures that is tractable but not finitely tractable. In [DSM⁺21] the second author of this paper and his students gave the first example of a finitely tractable PCSP(\mathbf{A}, \mathbf{B}) for \mathbf{A}, \mathbf{B} of size 2 that does not reduce to a tractable CSP(\mathbf{C}) for any \mathbf{C} of size ≤ 2 .

We extend this result in Theorem 3.1 as follows: For every prime p and every integer $n > 1$ we give \mathbf{A}, \mathbf{B} of size n such that PCSP(\mathbf{A}, \mathbf{B}) reduces to a tractable CSP over some \mathbf{C} of size p but not over any smaller structure. All these structures have a single relation with arity n^p .

In our second family of examples in Theorem 3.3 we observe a similar behaviour even when restricting to structures with a single ternary relation: For every prime $p \geq 7$ we construct \mathbf{A}, \mathbf{B} of size $p - 1$ such that PCSP(\mathbf{A}, \mathbf{B}) reduces to a tractable CSP over some \mathbf{C} of size p but not over any smaller structure.

In contrast we show that if finite undirected graphs \mathbf{A} and \mathbf{B} sandwich some finite directed graph \mathbf{C} with CSP(\mathbf{C}) tractable, then already CSP(\mathbf{A}) or CSP(\mathbf{B}) is tractable (Corollary 4.3). This answers Problem 1 of [DSM⁺21].

More generally, if a finite smooth digraph \mathbf{A} and a digraph \mathbf{B} sandwich some finite digraph \mathbf{C} with CSP(\mathbf{C}) tractable, then already CSP(\mathbf{A}) or CSP(\mathbf{B}) is tractable (Corollary 5.8). It remains open whether every finitely tractable PCSP(\mathbf{A}, \mathbf{B}) for finite digraphs \mathbf{A}, \mathbf{B} reduces to a tractable CSP(\mathbf{C}) for a digraph \mathbf{C} of size $\leq \max(|A|, |B|)$.

2. PRELIMINARIES

We will only define a bare minimum of notions necessary to make sense of this article. For a more detailed introduction to promise constraint satisfaction, see [BBKO21]. For more on digraph homomorphisms, see [HN04].

For $n \in \mathbb{N}$, write $[n] := \{0, \dots, n-1\}$ (note that $n \notin [n]$).

Unless indicated otherwise, arithmetical operations, like $+$ and \cdot , are considered over the integers even if the input numbers come, say, from the set $[p]$. By $a \bmod p$ we mean the operation of taking remainder that returns a number in $[p]$. When writing formulas, \bmod is evaluated after addition; in particular $\sum_i a_i \bmod p$ means $(\sum_i a_i) \bmod p$ where $\sum_i a_i$ is evaluated in \mathbb{Z} .

A *relation* R of arity n on a set A is a subset of A^n . A *signature* is a (finite, in this article) list of relation symbols, each of which is associated with an arity. A *relational structure* of a given signature Γ , denoted by \mathbf{A} , consists of a universe A together with a family of relations, one for each relation symbol from Γ . If R is an n -ary relation symbol from Γ and \mathbf{A} is a structure over Γ with universe A , then $R^{\mathbf{A}}$ is an n -ary relation on A . A relational structure is finite if its universe is finite.

We call a relational structure \mathbf{A} with universe A *affine* if there exists a binary operation $+$, a unary operation $-$ and a constant 0 on A such that $\mathbb{A} := (A, +, -, 0)$ is an abelian group and every relation $R^{\mathbf{A}}$ of \mathbf{A} is closed under $x - y + z$; equivalently, every, say, n -ary relation $R^{\mathbf{A}}$ is a coset of a subgroup of \mathbb{A}^n . Then $\text{CSP}(\mathbf{A})$ can be solved by linear algebra and is in the complexity class P .

Let n be a positive integer and \mathbf{A} be a relational structure. By the *n -th power of \mathbf{A}* , denoted by \mathbf{A}^n , we will understand the relational structure with the universe A^n (the set-theoretic n -th power) and with the same signature as \mathbf{A} . If $R^{\mathbf{A}}$ is an m -ary relation of \mathbf{A} , then $R^{\mathbf{A}^n}$ is the relation on A^n that contains exactly those m -tuples $(u_1, u_2, \dots, u_m) \in (A^n)^m$ such that for all $i = 1, 2, \dots, n$ the i -th projection of the m -tuple, $(\pi_i(u_1), \dots, \pi_i(u_m))$, lies in $R^{\mathbf{A}}$. Here π_i is the mapping $A^n \rightarrow A$ that returns the i -th component of its input.

Let \mathbf{A} and \mathbf{B} be two relational structures of the same signature Γ . A mapping $f: A \rightarrow B$ is a *relational structure homomorphism* (or just homomorphism for short) if for each relation R from Γ (denote its arity by n) and each $(a_1, \dots, a_n) \in R^{\mathbf{A}}$ we have $(f(a_1), \dots, f(a_n)) \in R^{\mathbf{B}}$. An *n -ary polymorphism* from \mathbf{A} to \mathbf{B} is a homomorphism $\mathbf{A}^n \rightarrow \mathbf{B}$.

A finite relational structure \mathbf{A} is a *core* if any homomorphism $\mathbf{A} \rightarrow \mathbf{A}$ is a bijective mapping. It is a well known fact that any finite relational structure has a unique (up to isomorphism) core.

Let Γ be a (finite) signature and let \mathbf{A} and \mathbf{B} be two relational structures with signature Γ such that there exists a homomorphism from \mathbf{A} to \mathbf{B} . The *promise constraint satisfaction problem* with fixed target structures \mathbf{A} and \mathbf{B} , denoted by $\text{PCSP}(\mathbf{A}, \mathbf{B})$, has as its instances finite relational structures \mathbf{X} of signature Γ (the relations of \mathbf{X} are specified by listing all tuples in them). An instance is

- a “yes” instance if there exists a homomorphism $\mathbf{X} \rightarrow \mathbf{A}$, and
- a “no” instance if there does not exist any homomorphism $\mathbf{X} \rightarrow \mathbf{B}$.

It is easy to show that since \mathbf{A} maps homomorphically to \mathbf{B} no instance can be both a “Yes” and a “No” instance. However, there might exist instances that are neither “Yes,” nor “No” instances. On these inputs an algorithm solving $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is allowed to do anything. A *constraint satisfaction problem with target structure \mathbf{A}* , denoted by $\text{CSP}(\mathbf{A})$, is $\text{PCSP}(\mathbf{A}, \mathbf{A})$.

As a reminder, \mathbf{P} is the class of all problems solvable in polynomial time and \mathbf{NP} is the class of all problems solvable in nondeterministic polynomial time. Informally, we will call a (promise) constraint satisfaction problem tractable if there exists a polynomial time algorithm that correctly classifies each instance as a “yes” or “no” instance and we will assume that $\mathbf{P} \neq \mathbf{NP}$ since otherwise all presented complexity results are trivial.

If \mathbf{A} is a relational structure and \mathbf{A}' is the core of \mathbf{A} , then it is easy to verify that the sets of “yes” and “no” instances of $\text{CSP}(\mathbf{A})$ and $\text{CSP}(\mathbf{A}')$ are the same. In particular, the computational complexity of $\text{CSP}(\mathbf{A})$ only depends on the core of \mathbf{A} .

Let \mathbf{A} and \mathbf{A}' be relational structures on the same universe but with possibly distinct signatures. We say that \mathbf{A}' is *pp-definable* from \mathbf{A} if each relation on \mathbf{A}' can be defined using a first order formula which only uses relations of \mathbf{A} , equality, existential quantifiers and conjunction. It is well-known and easy to see that if \mathbf{A}' is pp-definable from \mathbf{A} , then $\text{CSP}(\mathbf{A}')$ is reducible to $\text{CSP}(\mathbf{A})$.

An important special type of relational structures are *directed graphs (digraphs)*; these are relational structures whose signature consists of one binary relation E , the edge relation. The elements of the universe of a digraph are usually called the vertices of the digraph. A symmetric (or undirected) *graph* is a digraph whose edge relation is symmetric. A graph is *bipartite* if its vertices can be partitioned into two sets P, Q so that each edge connects a vertex from P to some vertex from Q . A digraph \mathbf{G} is *smooth* if for each of its vertices v there exist vertices u and w such that (u, v) and (v, w) both lie in $E^{\mathbf{G}}$. A *directed cycle* of length ℓ is a digraph with vertices $[\ell]$ and edge set $\{(i, (i + 1) \pmod{\ell}) : i \in [\ell]\}$.

In the early 1990s, Hell and Nešetřil gave the following characterization of the complexity of CSP for symmetric graphs that we will later use:

Theorem 2.1 [HN90, Theorem 1]. *Let \mathbf{G} be a finite symmetric graph. If \mathbf{G} is bipartite, then $\text{CSP}(\mathbf{G})$ is in \mathbf{P} . If \mathbf{G} is not bipartite, then $\text{CSP}(\mathbf{G})$ is NP-complete.*

Barto, Kozik and Niven later generalized the above theorem to all smooth digraphs:

Theorem 2.2 [BKN09]. *Let \mathbf{G} be a finite smooth digraph that is a core. If \mathbf{G} is a disjoint union of directed cycles then $\text{CSP}(\mathbf{G})$ is in \mathbf{P} . Otherwise, $\text{CSP}(\mathbf{G})$ is NP-complete.*

Our main tool for showing that some PCSP does not reduce to a tractable $\text{CSP}(\mathbf{C})$ for a structure \mathbf{C} of specified size is the following result by Barto and Kozik. Recall that a function $f: A^n \rightarrow B$ is *cyclic* if for any $a_1, \dots, a_n \in A$ we have

$$f(a_1, \dots, a_n) = f(a_2, a_3, \dots, a_n, a_1).$$

Theorem 2.3 [BK12, Theorem 4.1]. *Let \mathbf{C} be finite relational structure, and let p be a prime larger than the size of the universe of \mathbf{C} . If there exists no cyclic p -ary polymorphism $\mathbf{C}^p \rightarrow \mathbf{C}$, then $\text{CSP}(\mathbf{C})$ is NP-complete.*

If a structure \mathbf{C} has a cyclic p -ary polymorphism t and is sandwiched by \mathbf{A} and \mathbf{B} via homomorphisms $\mathbf{A} \xrightarrow{g} \mathbf{C} \xrightarrow{h} \mathbf{B}$, then the composition $f: A^p \rightarrow B$ defined by

$$f(x_1, \dots, x_p) := h(t(g(x_1), \dots, g(x_p)))$$

is a cyclic p -ary polymorphism from \mathbf{A} to \mathbf{B} . We will use the contrapositive of this statement together with Theorem 2.3 in the following form: if for some prime p there is no cyclic p -ary polymorphism from \mathbf{A} to \mathbf{B} , then $\text{PCSP}(\mathbf{A}, \mathbf{B})$ does not reduce to a tractable $\text{CSP}(\mathbf{C})$ for any \mathbf{C} of size less than p .

3. BIG AFFINE SANDWICHES

For every $n > 1$ and every prime p we give structures \mathbf{A}, \mathbf{B} of size n with a single n^p -ary relation such that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ reduces to a tractable $\text{CSP}(\mathbf{C})$ for some sandwiched \mathbf{C} of size p but not to a tractable CSP over any smaller sandwiched structure. For $n = 2$ and $p = 3$ such an example was given in [DSM⁺21].

Theorem 3.1. *For $n, p > 1$, let R be a relation symbol of arity n^p , and let $\mathbf{A} = ([n], R^{\mathbf{A}})$, $\mathbf{B} = ([n], R^{\mathbf{B}})$, $\mathbf{C} = ([p], R^{\mathbf{C}})$ be relational structures with*

$$R^{\mathbf{A}} = \{f: [n]^p \rightarrow [n] : f \text{ is a projection}\},$$

$$R^{\mathbf{B}} = \{f: [n]^p \rightarrow [n] : f \text{ is not cyclic}\},$$

$$R^{\mathbf{C}} = \left\{ f: [n]^p \rightarrow [p], (x_1, \dots, x_p) \mapsto \sum_{i=1}^p a_i x_i \bmod p : a_1, \dots, a_p \in [p], \sum_{i=1}^p a_i \bmod p = 1 \right\}.$$

Then

- (1) \mathbf{C} is affine (hence $\text{CSP}(\mathbf{C})$ is in \mathbf{P}) and is sandwiched by \mathbf{A} and \mathbf{B} via the homomorphisms $\mathbf{A} \xrightarrow{g} \mathbf{C} \xrightarrow{h} \mathbf{B}$ where $g: [n] \rightarrow [p]$, $x \mapsto x \bmod p$, and $h: [p] \rightarrow [n]$, $x \mapsto x \bmod n$.
- (2) If p is prime and \mathbf{D} is a structure with $|D| < p$ that is sandwiched by \mathbf{A} and \mathbf{B} , then $\text{CSP}(\mathbf{D})$ is NP-complete.
- (3) $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is in \mathbf{P} . If $p > 2$, then $\text{CSP}(\mathbf{A})$ is NP-complete; else it is in \mathbf{P} . If $(n, p) \neq (2, 2)$, then $\text{CSP}(\mathbf{B})$ is NP-complete; else it is in \mathbf{P} .

The main technical difficulty in proving Theorem 3.1 is to show that $h: \mathbf{C} \rightarrow \mathbf{B}$ is a homomorphism. For this we first establish several properties of cyclic functions of a form that is slightly more general than of those in $h(R^{\mathbf{C}})$ in the next lemma.

For a rational number q let $\lfloor q \rfloor$ denote the greatest integer less than or equal to q .

Lemma 3.2. *For $n, p > 1$ and $a_1, \dots, a_p \in [p]$, let*

$$f: [n]^p \rightarrow [n], (x_1, \dots, x_p) \mapsto \left(\sum_{i=1}^p a_i x_i \bmod p \right) \bmod n,$$

be cyclic. Then

- (1) a_1, \dots, a_p are all congruent modulo n .
- (2) If n does not divide p , then for any $x_1, \dots, x_p \in [n]$ we have

$$\left\lfloor \frac{\sum_{i=1}^p a_i x_i}{p} \right\rfloor = \left\lfloor \frac{\sum_{i=1}^p a_i \sum_{j=1}^p x_j}{p^2} \right\rfloor.$$

- (3) f is symmetric, i.e., invariant under all permutations of its inputs.
- (4) $\sum_{i=1}^p a_i \bmod p \neq 1$.

Proof. For (1) note that $f(1, 0, \dots, 0) = \dots = f(0, \dots, 0, 1)$ as f is cyclic. Now observe that when the 1 is at the i -th position, then $f(0, \dots, 0, 1, 0, \dots, 0) = a_i \bmod n$. Putting these two facts together gives us that all a_i 's are the same modulo n .

For (2) assume that n does not divide p . We show that the integer part of the quotient obtained by dividing $\sum_{i=1}^p a_i x_i$ by p depends only on the weight $w := \sum_{i=1}^p x_i$ of $(x_1, \dots, x_p) \in [n]^p$ by induction on w .

The base case for $\sum_{i=1}^p x_i = 0$ is immediate. Next take $x_1, \dots, x_p \in [n]$ such that $\sum_{i=1}^p x_i = w > 0$. Without loss of generality assume $x_1 \neq 0$. Let σ be the cyclic permutation $\sigma := (1, \dots, p)$, and let $k := \left\lfloor \frac{\sum_{i=1}^p a_i(w-1)}{p^2} \right\rfloor$. Let $y_1 = x_1 - 1$ and $y_i = x_i$ for $i = 2, 3, \dots, p$. Since $\sum_{i=1}^p y_i = w - 1$, the induction hypothesis yields

$$\left\lfloor \frac{\sum_{i=1}^p a_i y_i}{p} \right\rfloor = \left\lfloor \frac{\sum_{i=1}^p a_i \sum_{j=1}^p y_j}{p^2} \right\rfloor = \left\lfloor \frac{\sum_{i=1}^p a_i (w-1)}{p^2} \right\rfloor = k.$$

The total weight will not change if we permute the y_i 's, so for any $s \in [p]$ we also get $\left\lfloor \frac{\sum_{i=1}^p a_i y_{\sigma^s(i)}}{p} \right\rfloor = k$. In particular, $\sum_{i=1}^p a_i y_{\sigma^s(i)} - kp$ lies in $[p]$ for all s .

Going back from y_i 's to x_i 's, we get

$$\sum_{i=1}^p a_i x_{\sigma^s(i)} - a_{\sigma^{-s}(1)} - kp = \sum_{i=1}^p a_i y_{\sigma^s(i)} - kp \in [p].$$

Given that $a_{\sigma^{-s}(1)} \in [p]$, we get

$$\sum_{i=1}^p a_i x_{\sigma^s(i)} - kp \in [2p - 1] \quad (3.1)$$

for any $s \in [p]$.

Since f is cyclic, $\sum_{i=1}^p a_i x_{\sigma^s(i)} \pmod p$ gives the same remainder modulo n for all $s \in [p]$. Furthermore, from (3.1) we see that for each s we have

$$\sum_{i=1}^p a_i x_{\sigma^s(i)} \pmod p = \begin{cases} \sum_{i=1}^p a_i x_{\sigma^s(i)} - kp, \text{ or} \\ \sum_{i=1}^p a_i x_{\sigma^s(i)} - (k+1)p. \end{cases}$$

We claim that exactly one of the two cases above holds for all $s \in [p]$. Suppose for a contradiction that there are $q, r \in [p]$ such that

$$\begin{aligned} \sum_{i=1}^p a_i x_{\sigma^q(i)} \pmod p &= \sum_{i=1}^p a_i x_{\sigma^q(i)} - kp \\ \sum_{i=1}^p a_i x_{\sigma^r(i)} \pmod p &= \sum_{i=1}^p a_i x_{\sigma^r(i)} - (k+1)p. \end{aligned}$$

Since f is cyclic, we get

$$\left(\sum_{i=1}^p a_i x_{\sigma^q(i)} - kp \right) \pmod n = \left(\sum_{i=1}^p a_i x_{\sigma^r(i)} - (k+1)p \right) \pmod n. \quad (3.2)$$

However, from (1) we get that $\sum_{i=1}^p a_i x_{\sigma^s(i)}$ has the same remainder modulo n for all $s \in [p]$. So (3.2) simplifies to $-kp \equiv -(k+1)p \pmod n$. Thus n divides p , which contradicts our assumption.

We have obtained that one of the following two cases happens:

- Either $\sum_{i=1}^p a_i x_{\sigma^s(i)} - kp$ is in $[p]$ for all $s \in [p]$, or
- $\sum_{i=1}^p a_i x_{\sigma^s(i)} - kp$ is in $\{p, p+1, \dots, 2p-1\}$ for all $s \in [p]$.

In other words we have $k' \in \{k, k+1\}$ and $r_0, \dots, r_{p-1} \in [p]$ such that

$$\sum_{i=1}^p a_i x_{\sigma^s(i)} = k'p + r_s \quad (3.3)$$

for all $s \in [p]$. Dividing both sides of (3.3) by p we get $\left\lfloor \frac{\sum_{i=1}^p a_i x_{\sigma^s(i)}}{p} \right\rfloor = k'$. All that remains is to calculate k' .

Summing (3.3) over all $s \in [p]$, we get

$$\sum_{s=0}^{p-1} \sum_{i=1}^p a_i x_{\sigma^s(i)} = k'p^2 + \sum_{s=0}^{p-1} r_s \leq k'p^2 + p(p-1). \quad (3.4)$$

Counting in two ways, we obtain that the left hand side of (3.4) is

$$\sum_{s=0}^{p-1} \sum_{i=1}^p a_i x_{\sigma^s(i)} = \sum_{i=1}^p a_i \cdot \sum_{j=1}^p x_j.$$

Dividing by p^2 yields $k' = \left\lfloor \frac{\sum_{i=1}^p a_i \sum_{j=1}^p x_j}{p^2} \right\rfloor$. Thus in particular for $s=0$ we get

$$\left\lfloor \frac{\sum_{i=1}^p a_i x_{\sigma^0(i)}}{p} \right\rfloor = \left\lfloor \frac{\sum_{i=1}^p a_i x_i}{p} \right\rfloor = \left\lfloor \frac{\sum_{i=1}^p a_i \sum_{j=1}^p x_j}{p^2} \right\rfloor$$

and the induction step follows. Thus (2) is proved.

Next we show (3). If n divides p , then f simplifies to $f(x_1, \dots, x_p) = \sum_{i=1}^p a_i x_i \pmod n$, which is clearly symmetric by (1). So assume that n does not divide p in the following. Let π be a permutation on $\{1, \dots, p\}$, and let $x_1, \dots, x_p \in [n]$. By (2) we have $k \in \mathbb{N}$ and $r, s \in [p]$ such that $\sum_{i=1}^p a_i x_i = kp + r$ and $\sum_{i=1}^p a_i x_{\pi(i)} = kp + s$. Since these sums give the same remainder modulo n by (1), so do r and s . Hence $f(x_1, \dots, x_p) = f(x_{\pi(1)}, \dots, x_{\pi(p)})$ for all permutations π and (3) is proved.

Finally we prove (4). Since f is symmetric by (3), we may reorder its variables so that

$$a_1 \leq a_2 \leq \dots \leq a_p.$$

Let $q \in \mathbb{N}$ be such that $p = 2q$ or $p = 2q + 1$. For $b := a_{q+1}$, we have

$$r := \sum_{i=1}^q (a_i - b) \leq 0 \quad \text{and} \quad s := \sum_{i=q+1}^p (a_i - b) \geq 0.$$

Further

$$r + s = \sum_{i=1}^p a_i - pb \quad \text{and} \quad r \equiv s \equiv 0 \pmod n, \quad (3.5)$$

with the latter following from (1).

If n divides p , then (3.5) implies that n divides $\sum_{i=1}^p a_i$. In particular $\gcd(\sum_{i=1}^p a_i, p) \geq n > 1$ and (4) is proved in this case. So we assume that n does not divide p for the rest of the proof. We apply the equality in (2) in the situation when either the first q or the last q variables x_i are 1 and the rest of the x_i 's are 0. In both cases $\sum_{j=1}^p x_j = q$, so we get

$$\left\lfloor \frac{\sum_{i=1}^q a_i}{p} \right\rfloor = \left\lfloor \frac{\sum_{i=1}^p a_i q}{p^2} \right\rfloor = \left\lfloor \frac{\sum_{i=p-q+1}^p a_i}{p} \right\rfloor.$$

Replacing $\sum_{i=1}^q a_i$ by $r + qb$ and $\sum_{i=p-q+1}^p a_i$ by $s + qb$ yields

$$\left\lfloor \frac{r + qb}{p} \right\rfloor = \left\lfloor \frac{s + qb}{p} \right\rfloor.$$

Denote the quantity on the line above by k . Multiplying by p , we obtain the inequalities (recall that $r \leq s$)

$$kp \leq r + qb \leq s + qb \leq (k + 1)p - 1. \quad (3.6)$$

Hence $0 \leq s - r \leq p - 1$. Since $r \leq 0$, this yields $0 \leq s \leq p - 1$. Similarly $-p + 1 \leq r \leq 0$. In particular $-p + 1 \leq r + s \leq p - 1$.

Seeking a contradiction we suppose that $\sum_{i=1}^p a_i \equiv 1 \pmod{p}$. Then also $r + s \equiv 1 \pmod{p}$ by (3.5), which leaves $r + s = -p + 1$ or $r + s = 1$. The latter case is impossible since r and s are both multiples of $n > 1$ by (3.5). We are left with the case $r + s = -p + 1$. Due to the inequalities $s \geq 0$ and $r \geq -p + 1$, we must have $r = -p + 1$ and $s = 0$. We will now bring this case to a contradiction.

Since $s - r = p - 1$, the outer inequalities in (3.6) must be equalities, that is,

$$r + qb = kp \quad \text{and} \quad s + qb = (k + 1)p - 1.$$

We will again apply (2); this time we will let either the first $q + 1$ or the last $q + 1$ variables x_i be 1 and the rest 0. From (2) we obtain

$$\left\lfloor \frac{r + qb + b}{p} \right\rfloor = \left\lfloor \frac{s + qb + a_{p-q}}{p} \right\rfloor.$$

Plugging in $r + qb = kp$ and $s + qb = (k + 1)p - 1$, we get

$$\begin{aligned} \left\lfloor \frac{kp + b}{p} \right\rfloor &= \left\lfloor \frac{(k + 1)p - 1 + a_{p-q}}{p} \right\rfloor, \\ \left\lfloor k + \frac{b}{p} \right\rfloor &= \left\lfloor (k + 1) + \frac{a_{p-q} - 1}{p} \right\rfloor. \end{aligned}$$

Since $b \leq p - 1$ the left hand side evaluates to k . In order for the right hand side to also be k , we must have $a_{p-q} = 0$. It follows that $a_1 = \dots = a_q \leq a_{p-q} = 0$ and consequently $r = -qb$. We distinguish two cases depending on the parity of p :

- If $p = 2q + 1$ is odd, then $0 = a_{p-q} = a_{q+1} = b$ yields $r = 0$, which contradicts $r = -p + 1$.
- Else if $p = 2q$ is even, then $r = -qb$ and $r = -2q + 1$ implies $q = b = 1$ and $p = 2$. In this situation we would have $a_1 = 0$ and $a_2 = b = 1$. Item (1) then gives us $0 \equiv 1 \pmod{n}$, a contradiction with $n > 1$.

Either case led to a contradiction. Thus (4) is proved. \square

After this preparation we are now ready to prove our main result.

Proof of Theorem 3.1. For (1) note first that by definition $R^{\mathbf{C}}$ is the closure of $g(R^{\mathbf{A}})$ under $x - y + z \pmod{p}$. Hence \mathbf{C} is affine and $g: \mathbf{A} \rightarrow \mathbf{C}$ is a homomorphism. Next for any $f \in R^{\mathbf{C}}$, the function $h(f)$ is of the form

$$h(f): [n]^p \rightarrow [n], (x_1, \dots, x_p) \mapsto \left(\sum_{i=1}^p a_i x_i \pmod{p} \right) \pmod{n},$$

for some $a_1, \dots, a_p \in [p]$ with $\sum_{i=1}^p a_i \pmod{p} = 1$. Since $h(f)$ cannot be cyclic by part (4) of Lemma 3.2, we have $h(f) \in R^{\mathbf{B}}$ and $h: \mathbf{C} \rightarrow \mathbf{B}$ is a homomorphism.

For (2) assume that p is prime. Let \mathbf{D} be a relational structure whose universe is smaller than p with homomorphisms $\mathbf{A} \xrightarrow{r} \mathbf{D} \xrightarrow{s} \mathbf{B}$. Seeking a contradiction, suppose that \mathbf{D} has some cyclic p -ary polymorphism $t: \mathbf{D}^p \rightarrow \mathbf{D}$. Consequently

$$f: \mathbf{A}^p \rightarrow \mathbf{B}, (x_1, \dots, x_p) \mapsto s(t(r(x_1), \dots, r(x_p))),$$

is a cyclic polymorphism from \mathbf{A} to \mathbf{B} . However, applying f to the graphs of projections $\pi_1, \pi_2, \dots, \pi_p \in R^{\mathbf{A}}$ gives us that $f \in R^{\mathbf{B}}$ in contradiction to the definition of $R^{\mathbf{B}}$.

Hence \mathbf{D} has no cyclic p -ary polymorphisms. Thus $\text{CSP}(\mathbf{D})$ is NP-complete by Theorem 2.3.

For (3) note that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is in P by (1). If $n < p$ and p is prime, then (2) yields that $\text{CSP}(\mathbf{A})$ and $\text{CSP}(\mathbf{B})$ are NP-complete already. We still give brief self-contained arguments for the complexity of $\text{CSP}(\mathbf{A})$ and $\text{CSP}(\mathbf{B})$ in general. Let

$$E_p := \{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\} \subseteq [n]^p.$$

First note that the projection of $R^{\mathbf{A}}$ to E_p ,

$$\{f|_{E_p} : f \in R^{\mathbf{A}}\},$$

is the set of restrictions of the p -ary projection maps to E_p . When $f: [n]^p \rightarrow [n]$ is a projection map, then $f|_{E_p}$ can be identified with a member of E_p in a natural way. Hence $([n], E_p)$ is pp-definable from \mathbf{A} and $\text{CSP}([2], E_p)$ reduces to $\text{CSP}(\mathbf{A})$. The former is known as 1-in- p -SAT and is NP-complete if $p > 2$ by Schaefer's dichotomy for Boolean CSP [Sch78]. Thus $\text{CSP}(\mathbf{A})$ is NP-complete as well if $p > 2$. If $p = 2$, then $R^{\mathbf{A}}$ contains only 2 elements. Hence \mathbf{A} has a majority polymorphism $m: \mathbf{A}^3 \rightarrow \mathbf{A}$ satisfying $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ for all $x, y \in A$. Thus $\text{CSP}(\mathbf{A})$ is tractable by [FV99].

Next consider the projection of $\{f \in R^{\mathbf{B}} : f \text{ is constant on } [n]^p \setminus E_p\}$ to E_p . This is $N_p := [n]^p \setminus \{(a, \dots, a) : a \in [n]\}$, the p -ary not-all-equal relation on $[n]$. Hence $\text{CSP}([n], N_p)$ reduces to $\text{CSP}(\mathbf{B})$. If $n = 2$, the former is Not-All-Equal p -SAT, which is known to be NP-complete for $p > 2$ by Schaefer's dichotomy for Boolean CSP. For $n > 2$, note that the not-equal relation $N_2 = \{(a, b) \in [n]^2 : (a, b, \dots, b) \in N_p\}$ is primitively positively definable from N_p and hence from $R^{\mathbf{B}}$. So $\text{CSP}([n], N_2)$ reduces to $\text{CSP}(\mathbf{B})$. The former is graph n -coloring, which is NP-complete for $n > 2$. It follows that $\text{CSP}(\mathbf{B})$ is NP-complete whenever $(n, p) \neq (2, 2)$.

If $p = n = 2$, then $R^{\mathbf{B}} = \{f: [2]^2 \rightarrow [2] : f(1, 0) = f(0, 1) + 1 \pmod{2}\}$. Hence \mathbf{B} is affine and $\text{CSP}(\mathbf{B})$ in P. \square

In our second class of examples, let $p \geq 7$ be prime. We construct structures \mathbf{A}, \mathbf{B} of size $p - 1$ with a single ternary relation such that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ reduces to a tractable $\text{CSP}(\mathbf{C})$ for some sandwiched \mathbf{C} of size p but not to a tractable CSP over any smaller sandwiched structure.

Theorem 3.3. *For a prime $p \geq 7$, let R be a relation symbol of arity 3, and let $\mathbf{A} = ([p - 1], R^{\mathbf{A}})$, $\mathbf{B} = ([p - 1], R^{\mathbf{B}})$, $\mathbf{C} = ([p], R^{\mathbf{C}})$ be relational structures with*

$$\begin{aligned} R^{\mathbf{C}} &= \{(x, y, z) \in [p]^3 : x - 2y + z \equiv 1 \pmod{p}\}, \\ R^{\mathbf{A}} &= R^{\mathbf{C}} \cap [p - 1]^3, \end{aligned}$$

$$R^{\mathbf{B}} = h(R^{\mathbf{C}}) \text{ for } h: [p] \rightarrow [p - 1], x \mapsto \begin{cases} x & \text{if } x \in [p - 1], \\ 1 & \text{if } x = p - 1. \end{cases}$$

Then

- (1) \mathbf{C} is affine and sandwiched by \mathbf{A} and \mathbf{B} via the homomorphisms $\mathbf{A} \xrightarrow{id} \mathbf{C} \xrightarrow{h} \mathbf{B}$, and
(2) if \mathbf{D} is a structure with $|D| < p$ that is sandwiched by \mathbf{A} and \mathbf{B} , then $\text{CSP}(\mathbf{D})$ is NP-complete. In particular $\text{CSP}(\mathbf{A})$ and $\text{CSP}(\mathbf{B})$ are both NP-complete but $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is in P.

Proof. Item (1) is immediate from the definitions. Assertion (2) will follow from Theorem 2.3 once we have proved that there exists no cyclic p -ary polymorphism $\mathbf{A}^p \rightarrow \mathbf{B}$. We will do that by showing several claims about $R^{\mathbf{A}}$ and $R^{\mathbf{B}}$.

To that end consider the linear system $M \cdot x = b$ over \mathbb{Z}_p for

$$M := \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & 0 \\ 0 & & & & 0 & 1 & -2 & 1 \\ 1 & 0 & & & 0 & 1 & -2 \\ -2 & 1 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathbb{Z}_p^{p \times p}, \quad b := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \end{pmatrix} \in \mathbb{Z}_p^p.$$

To simplify notation we identify the elements of \mathbb{Z}_p and of $[p]$ in the following.

Claim 3.4. The nullspace of M has dimension 2 and is spanned by $a := (0, 1, 2, \dots, p-1)^T$ and b .

Proof. Clearly a, b are linearly independent solutions for $M \cdot x = 0$. So the nullspace of M has dimension at least 2. On the other hand we see that the first $p-2$ rows of M are linearly independent. So the nullspace of M has dimension at most 2. \square

Claim 3.5. $M \cdot x = b$ has a solution $x = (x_1, \dots, x_p)^T \in \mathbb{Z}_p^p$ with $x_{p-1} = x_p = 0$.

Proof. Let r_1, \dots, r_p denote rows 1 to p of the augmented matrix (M, b) of our linear system. Note that $r_{p-1}, r_p, r_1, \dots, r_{p-2}$ are the rows of (M^T, b) . Claim 3.4 and $a^T \cdot b = b^T \cdot b = 0$ yield

$$a^T \cdot (M^T, b) = (0, \dots, 0) \text{ and } b^T \cdot (M^T, b) = (0, \dots, 0).$$

Hence (M^T, b) has rank at most $p-2$. Since (M^T, b) and (M, b) have the same set of rows (only permuted), it follows that (M, b) also has rank at most $p-2$.

We see that simply omitting the last two rows of (M, b) yields a row echelon form for our linear system. Further variables x_{p-1} and x_p are free and can be set to 0 for a solution of $M \cdot x = b$. \square

Claim 3.6. There exist $u_1, \dots, u_p \in [p-1]$ such that each column of

$$U := \begin{pmatrix} u_1 & u_2 & \dots & u_p \\ u_2 & u_3 & \dots & u_1 \\ u_3 & u_4 & \dots & u_2 \end{pmatrix}$$

is in $R^{\mathbf{A}}$.

Proof. By Claim 3.5 $M \cdot x = b$ has a solution $x = (x_1, \dots, x_p)^T \in \mathbb{Z}_p^p$ such that the p entries $\{x_1, \dots, x_p\}$ form a proper subset of \mathbb{Z}_p . Since b is in the nullspace of M by Claim 3.4, by adding an appropriate multiple of b to x we can get a solution $(u_1, \dots, u_p) \in [p]^p$ such that $\{u_1, \dots, u_p\}$ does not contain $p-1$. \square

Claim 3.7. $R^{\mathbf{B}}$ does not contain any constant tuple.

Proof. Seeking a contradiction, suppose we have $x, y, z \in [p]$ such that $x - 2y + z \equiv 1 \pmod{p}$ and $h(x) = h(y) = h(z)$. Since h restricted to $[p - 1]$ is the identity, at least one of x, y, z must be $p - 1$. But then $h(x) = h(y) = h(z) = 1$ and $x, y, z \in \{1, p - 1\}$. We distinguish the following cases:

- If $x \neq z$, then $x - 2y + z \equiv -2y \pmod{p}$. The latter is 1 modulo p only if $y = 1$ and $p = 3$.
- If $x = z = 1$, then $x - 2y + z \equiv 2(1 - y) \pmod{p}$. The latter is 1 modulo p only if $y = p - 1$ and $p = 3$.
- If $x = z = p - 1$, then $x - 2y + z \equiv 2(p - 1 - y) \pmod{p}$. The latter is 1 modulo p only if $y = 1$ and $p = 5$.

Each case yields a contradiction to our assumption that $p \geq 7$. □

Finally let \mathbf{D} be a structure such that $|D| < p$ with homomorphisms $\mathbf{A} \xrightarrow{r} \mathbf{D} \xrightarrow{s} \mathbf{B}$. Seeking a contradiction, suppose that \mathbf{D} has some cyclic p -ary polymorphism $t: \mathbf{D}^p \rightarrow \mathbf{D}$. Consequently

$$f: \mathbf{A}^p \rightarrow \mathbf{B}, (x_1, \dots, x_p) \mapsto s(t(r(x_1), \dots, r(x_p))),$$

is a cyclic polymorphism from \mathbf{A} to \mathbf{B} . However, applying f to the rows of the matrix U (whose columns are elements of $R^{\mathbf{A}}$) given by Claim 3.6 yields a constant tuple in $R^{\mathbf{B}}$. This contradicts Claim 3.7.

Hence \mathbf{D} has no cyclic p -ary polymorphisms and $\text{CSP}(\mathbf{D})$ is NP-complete by Theorem 2.3. □

4. SYMMETRIC STRUCTURES

We show that there are no non-trivial (in the sense of the previous section) examples of finitely tractable PCSPs over (undirected) graphs. For that we start with some observations on symmetric structures in general.

We call a Γ -structure \mathbf{A} *symmetric* if all its relations are invariant under all coordinate permutations, i.e., for every, say k -ary, $R \in \Gamma$ we have

$$\forall \sigma \in S_k \forall (x_1, \dots, x_k) \in R^{\mathbf{A}} : (x_{\sigma(1)}, \dots, x_{\sigma(k)}) \in R^{\mathbf{A}}.$$

Define the *maximal symmetric subset* $R^{\overline{\mathbf{A}}}$ of $R^{\mathbf{A}}$ as

$$R^{\overline{\mathbf{A}}} := \left\{ (x_1, \dots, x_k) \in A^k : \forall \sigma \in S_k (x_{\sigma(1)}, \dots, x_{\sigma(k)}) \in R^{\mathbf{A}}. \right\}$$

Since the condition for membership in $R^{\overline{\mathbf{A}}}$ is a conjunction of $|S_k|$ many conditions, the *symmetric part* $\overline{\mathbf{A}} := (A, \{R^{\overline{\mathbf{A}}} : R \in \Gamma\})$ of \mathbf{A} is pp-definable from \mathbf{A} .

Lemma 4.1. *Let $g: \mathbf{A} \rightarrow \mathbf{C}$ be a homomorphism where \mathbf{A} is symmetric and $\text{CSP}(\mathbf{C})$ is in \mathcal{P} . Then g maps \mathbf{A} into $\overline{\mathbf{C}}$ (the symmetric part of \mathbf{C}) and $\text{CSP}(\overline{\mathbf{C}})$ is in \mathcal{P} .*

Proof. Since \mathbf{A} is symmetric, so is its homomorphic image $g(\mathbf{A})$ in \mathbf{C} . Hence $g(\mathbf{A})$ is a substructure of $\overline{\mathbf{C}}$. Since $\overline{\mathbf{C}}$ is pp-definable from \mathbf{C} , $\text{CSP}(\overline{\mathbf{C}})$ reduces to $\text{CSP}(\mathbf{C})$ and so is in \mathcal{P} . □

Hence when searching for a tractable \mathbf{C} sandwiched by symmetric \mathbf{A} and \mathbf{B} , we may restrict ourselves to searching for symmetric structures \mathbf{C} .

As a consequence we can answer [DSM⁺21, Problem 1] in the negative: Are there some finite graphs \mathbf{A}, \mathbf{B} such that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is finitely tractable but only for a sandwiched digraph \mathbf{C} that is strictly bigger than $\max(|A|, |B|)$?

Corollary 4.2. *If there exists a homomorphism from a finite (undirected) graph \mathbf{A} into a finite digraph \mathbf{C} without loops such that $\text{CSP}(\mathbf{C})$ is in P , then $\text{CSP}(\mathbf{A})$ is in P .*

Proof. Assume $\mathsf{P} \neq \mathsf{NP}$ (clearly the assertion is trivial otherwise). By Lemma 4.1 we may assume that \mathbf{C} is an undirected graph (without loops). Since $\text{CSP}(\mathbf{C})$ is in P , \mathbf{C} is bipartite by Theorem 2.1. From the homomorphism $\mathbf{A} \rightarrow \mathbf{C}$ we see that \mathbf{A} is bipartite as well. Hence $\text{CSP}(\mathbf{A})$ is in P . \square

In particular finite tractability trivializes for PCSP on graphs.

Corollary 4.3. *If $\text{PCSP}(\mathbf{A}, \mathbf{B})$ for finite graphs \mathbf{A}, \mathbf{B} is finitely tractable, then $\text{CSP}(\mathbf{A})$ or $\text{CSP}(\mathbf{B})$ is tractable.*

This also settles a special instance of the following question which remains open in general:

Problem 4.4 (Libor Barto, personal communication). Are there some finite symmetric \mathbf{A}, \mathbf{B} such that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is finitely tractable but only for a sandwiched \mathbf{C} that is strictly bigger than $\max(|A|, |B|)$?

5. DIGRAPHS

In this section we move from symmetric graphs to directed graphs. The situation here is more complicated. In particular finite tractability does not trivialize for PCSP on digraphs like it does on symmetric graphs by Corollary 4.3. The following example was communicated to us by Jakub Bulín: Let \mathbf{A} be a finite tree with NP -complete CSP , let \mathbf{C}, \mathbf{B} be the complete digraphs without loops on 2, 3 vertices, respectively. Clearly there exist homomorphisms $\mathbf{A} \rightarrow \mathbf{C} \rightarrow \mathbf{B}$ and $\text{CSP}(\mathbf{C})$ is tractable. Hence $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is finitely tractable but \mathbf{A}, \mathbf{B} have NP -complete CSP s.

The following generalization of [DSM⁺21, Problem 1] is still open:

Problem 5.1. Are there some finite digraphs \mathbf{A}, \mathbf{B} such that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is finitely tractable but only for a sandwiched \mathbf{C} that is strictly bigger than $\max(|A|, |B|)$?

Brakensiek and Guruswami showed that every PCSP is polynomial time equivalent to $\text{PCSP}(\mathbf{A}, \mathbf{B})$ for some digraphs \mathbf{A}, \mathbf{B} [BG17, Theorem 6.10]. So our examples in Theorems 3.1 and 3.3 can be translated to digraphs $\mathbf{A}' \rightarrow \mathbf{C}' \rightarrow \mathbf{B}'$ but it is unclear whether the obtained \mathbf{C}' is still the smallest sandwiched structure with tractable CSP .

Using the classification of finite smooth digraphs with tractable CSP in Theorem 2.2, we can get a partial answer that also generalizes Corollary 4.2.

Theorem 5.2. *If there exists a homomorphism from a finite smooth digraph \mathbf{A} into a finite digraph \mathbf{C} without loops such that $\text{CSP}(\mathbf{C})$ is in P , then $\text{CSP}(\mathbf{A})$ is in P .*

Proof. Assume $\mathsf{P} \neq \mathsf{NP}$ (clearly the assertion is trivial otherwise). Let \mathbf{A} be a finite smooth digraph, and let \mathbf{C} be a minimal digraph without loops and with tractable $\text{CSP}(\mathbf{C})$ for which there exists a homomorphism $g: \mathbf{A} \rightarrow \mathbf{C}$. We proceed by a series of claims:

Claim 5.3. The digraph \mathbf{C} is smooth.

Proof. Let $n = |C|$ and let $E^{\mathbf{C}}$ denote the edge relation of \mathbf{C} . By the pigeonhole principle the set V of vertices of \mathbf{C} that induces the maximal smooth subgraph of \mathbf{C} is pp-definable as $v \in V$ if and only if

$$\begin{aligned} \exists x_1, \dots, x_n, y_1, \dots, y_n : & (x_1, x_2) \in E^{\mathbf{C}} \wedge (x_2, x_3) \in E^{\mathbf{C}} \wedge \dots \wedge (x_{n-1}, x_n) \in E^{\mathbf{C}} \\ & \wedge (x_n, v) \in E^{\mathbf{C}} \wedge (v, y_1) \in E^{\mathbf{C}} \wedge (y_1, y_2) \in E^{\mathbf{C}} \wedge \dots \wedge (y_{n-1}, y_n) \in E^{\mathbf{C}}. \end{aligned}$$

Then the induced subgraph $\mathbf{C}' := (V, E^{\mathbf{C}}|_{V \times V})$ is smooth and pp-definable from \mathbf{C} . Further $g(\mathbf{A})$ is a subgraph of \mathbf{C}' because $g(\mathbf{A})$ is a smooth subgraph of \mathbf{C} .

Since \mathbf{C}' is pp-definable from \mathbf{C} , $\text{CSP}(\mathbf{C}')$ reduces to $\text{CSP}(\mathbf{C})$. Since the latter is tractable, so is the former. Hence $\mathbf{C} = \mathbf{C}'$ by the minimality of \mathbf{C} and the claim is proved. \square

Claim 5.4. The digraph \mathbf{C} is a core.

Proof. Immediate from the minimality of \mathbf{C} . \square

Claim 5.5. \mathbf{C} is a disjoint union of directed cycles.

Proof. Since $\text{CSP}(\mathbf{C})$ is in \mathbf{P} and \mathbf{C} a core, this follows from Theorem 2.2. \square

Claim 5.6. The graph $g(\mathbf{A})$ is equal to \mathbf{C} .

Proof. Since \mathbf{A} is smooth, $g(\mathbf{A})$ is a smooth subgraph of \mathbf{C} . Now, \mathbf{C} is a disjoint union of directed cycles by Claim 5.5. Since $g(\mathbf{A})$ is a smooth subgraph of a disjoint union of directed cycles, we get that $g(\mathbf{A})$ itself is a disjoint union of some cycles of \mathbf{C} . Claim 5.6 then follows from the minimality of \mathbf{C} . \square

Claim 5.7. The digraph \mathbf{C} is the core of \mathbf{A} .

Proof. Let \mathbf{A}' be a substructure of \mathbf{A} that is isomorphic to the core of \mathbf{A} . Then $g(\mathbf{A}') = \mathbf{C}$ follows as in the proof of Claim 5.6. Hence $\mathbf{A}' \cong \mathbf{C}$. \square

Since $\text{CSP}(\mathbf{A})$ is equivalent to the CSP of its core \mathbf{C} and \mathbf{C} is a disjoint union of cycles (for whose CSP there is a straightforward polynomial time algorithm), we get that $\text{CSP}(\mathbf{A})$ is in \mathbf{P} , concluding the proof. \square

As for undirected graphs we obtain the following consequence:

Corollary 5.8. *If $\text{PCSP}(\mathbf{A}, \mathbf{B})$ for finite digraphs \mathbf{A}, \mathbf{B} is finitely tractable and \mathbf{A} is smooth, then $\text{CSP}(\mathbf{A})$ or $\text{CSP}(\mathbf{B})$ is tractable.*

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