
WEIGHTED LOGICS FOR NESTED WORDS AND ALGEBRAIC FORMAL POWER SERIES

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ABSTRACT. Nested words, a model for recursive programs proposed by Alur and Madhusudan, have recently gained much interest. In this paper we introduce quantitative extensions and study nested word series which assign to nested words elements of a semi-ring. We show that regular nested word series coincide with series definable in weighted logics as introduced by Droste and Gastin. For this we establish a connection between nested words and the free bisemigroup. Applying our result, we obtain characterizations of algebraic formal power series in terms of weighted logics. This generalizes results of Lautemann, Schwentick and Thérien on context-free languages.

1. INTRODUCTION

Model checking of finite state systems has become an established method for automatic hardware and software verification and led to numerous verification programs used in industrial application. In order to verify recursive programs it is necessary to model them as pushdown systems rather than finite automata. This has motivated Alur and Madhusudan [3, 4] to define regular nested word languages and visibly pushdown languages. The latter is a proper subclass of the context-free languages and exceeds the regular languages. Both classes are closely related. Nested words on the one hand have a linear sequential structure and on the other hand have a hierarchical structure. This way they may also be used to model linguistic data as well as semistructured data such as XML documents. Nested words and visibly pushdown languages gained much interest and set a starting point for a new research field (see e.g. [1, 2, 7] among many others).

The goal of this paper is: 1. to introduce a quantitative automaton model and a quantitative logic for nested words that are equally expressive, 2. to establish a connection between nested words and alternating texts, a graph representation of the free bisemigroup which is an object studied by Ésik and Németh [17] and Hashiguchi et al. [19–21], 3. to give a characterization of the important class of algebraic formal power series by means of weighted logics.

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In order to model quantitative aspects, extensions of existing models such as weighted automata were investigated. There, transitions of automata additionally carry a weight which can be of very different nature (e.g. counting, probabilities, etc.). In fact, weighted automata have found many different applications e.g. in image processing [10], in speech recognition [34] or as a model for probabilistic systems [5, 6]. In this paper we introduce and investigate weighted nested word automata which may serve as a quantitative model for sequential programs with recursive procedure calls. Due to the fact that we define them over arbitrary semirings, they are very flexible and can model, for example, probabilistic or stochastic programs of recursive nature as well as quantitative database queries.

Since weighted nested word automata and weighted pushdown automata are closely related, one should also mention that weighted pushdown systems have been applied to data flow analysis (see e.g. [23, 24]). There, however, the emphasis lies on the (weighted) configuration graph of the system which is used to model the state space of a program. Weights are incorporated in order to model, for example, the data of the program. In [23, 24] weighted versions of reachability problems in such graphs were considered.

In this paper we are interested in the semantics of a weighted automaton given as a mapping which assigns a value to each nested word. As the first main result of this paper we characterize the expressiveness of weighted nested word automata using weighted logics, generalizing a result of Alur and Madhusudan. Weighted logics were introduced by Droste and Gastin [11]. They enriched the classical language of monadic second-order logic with values from a semiring in order to add quantitative expressiveness. This way one may now e.g. express *how often* a certain property holds, *how much* execution time a process needs or *how reliable* it is. The result of Droste and Gastin has been extended to infinite words, (infinite) trees, texts, pictures and traces [14, 15, 18, 28, 33, 36]. We note, moreover, that a restriction of Lukasiewicz multi-valued logic coincides with this weighted logics [38].

In order to prove our result mentioned above we establish a new connection between alternating texts and nested words and reduce the result to an analogous one for alternating texts. The class of alternating texts, introduced by Ehrenfeucht and Rozenberg [16], forms the free bisemigroup which was also investigated by Hashiguchi et al. [19–21]. Moreover, a language theory for series-parallel-biposets, a different representation of the free bisemigroup, was developed by Ésik and Németh [17]. Besides the author’s opinion that a reduction to a previously known result is mathematically more elegant than e.g. a structural induction, the approach admits the advantage that it gives insight into relationships and similarities between different structures considered in the literature and therefore offers benefits. For example, decidability results for the emptiness and equivalence problem come almost for free as a corollary. Note that this extends the classical satisfiability problem for monadic second order logic, which is one motivation of transforming formulas in automata.

Furthermore, we can use the connection again in this paper to obtain a new characterization of algebraic formal power series. The latter form an important generalization of context-free languages. Algebraic formal power series were considered initially already by Chomsky and Schützenberger [8] and have since been intensively studied by Kuich and others. For a survey see [25] or [26]. Using projections of nested word series and applying the logical characterization of weighted nested word automata, we are able to give a characterization of algebraic formal power series in terms of weighted logics, generalizing a result of Lautemann, Schwentick and Thérien [27] on context-free languages. The connection between alternating texts and nested words is then used to also generalize a second

characterization of [27], thereby giving a different proof also for the result of Lautemann, Schwentick and Thérien.

The paper is organized as follows. In Section 2 we introduce nested words, weighted automata for nested words and give an example for them. In Section 3 we introduce weighted logics for nested words, introduce different fragments of the latter and state the first main result, the characterization of regular nested word series in terms of weighted logics. In Section 4 we introduce alternating texts, a graph representation of the free bisemigroup and define a weighted version of Ésik and Németh’s parenthesizing automata operating over elements of the free bisemigroup. Next, in Section 5, we define an embedding of nested words into alternating texts and show that we can translate weighted formulae as well as automata back and forth with respect to this embedding. This gives the proof of the first main result. After that, in Section 6, we apply the result and obtain characterizations of algebraic formal power series in terms of weighted logics.

An extended abstract of this paper appeared as [29]. This paper differs from it in the following way. First, full proofs are included. Second, the first main result, the logical characterization of regular nested word series, has been extended and it is shown that an existential fragment of weighted logics suffices to characterize weighted automata over nested words. Third, rather than translating nested words to sp-biposets, the graph representation of the free bisemigroup used by Ésik and Németh [17], we translate it to alternating texts, a different representation. This admits the advantage that we can more easily obtain a second characterization of algebraic formal power series in terms of weighted logics. This second characterization, which we include here in full length, was only sketched in the concluding remarks of [29] and gives the fourth main difference.

2. WEIGHTED AUTOMATA ON NESTED WORDS

In this section we recall the notion of nested words which was introduced by Alur and Madhusudan [4] and we define weighted automata for them. Let Δ be a finite alphabet and let Δ^+ be the free semigroup of finite but non-empty words. Let $w = a_1 \dots a_n \in \Delta^+$. The length of w is $|w| = n$. A *nesting relation* ν of width n ($n \in \mathbb{N}$) is a binary relation on $[n] = \{1, \dots, n\}$ such that for all $1 \leq i, j \leq n$:

- (1) if $\nu(i, j)$, then $i < j$,
- (2) if $\nu(i, j)$ and $\nu(i, j')$, then $j = j'$ and if $\nu(i, j)$ and $\nu(i', j)$, then $i = i'$,
- (3) if $\nu(i, j)$ and $\nu(i', j')$ and $i < i'$ then either $j < j'$ or $j' < j$.

If $\nu(i, j)$, we say i is a *call position* and j is a *return position*. Any $1 \leq i \leq n$ which is neither a call nor a return position is called an *internal position*. We collect all nesting relations of width n in Nest_n .

Definition 2.1 (Alur & Madhusudan [4]). A *nested word* (over Δ) is a pair (w, ν) such that $w \in \Delta^+$ and ν is a *nesting relation of width* $|w|$.

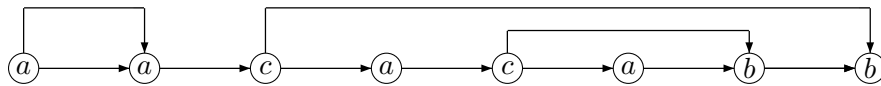


Figure 1: A visualization of the nested word $(aacacabb, \{(1, 2), (3, 8), (5, 7)\})$

We collect all nested words over Δ in $\text{NW}(\Delta)$. Let $nw = (w, \nu) \in \text{NW}(\Delta)$ where $w = a_1 \dots a_n$. The *factor* $nw[i, j]$ for $1 \leq i \leq j \leq n$ is the restriction of nw to the positions from i to j ; more formally $nw[i, j] = (a_i \dots a_j, \nu[i, j])$ where $\nu[i, j] = \{(k, \ell) \mid 1 \leq k, \ell \leq j - i + 1, (k + i - 1, \ell + i - 1) \in \nu\}$. Furthermore, we say a pair $(k, \ell) \in \nu$ is a *surface arch* of nw if there does not exist $(k', \ell') \in \nu$ with $k' < k < \ell < \ell'$.

Nested words have been introduced in order to model executions of recursive programs as well as nested data structures such as XML documents. Here, we model quantitative behavior of systems or documents such as the runtime or the probability of an execution of a randomized program, or the number of occurrences of a certain type of entry in an XML document. We do this by assigning to a nested word a quantity expressing, for example, the runtime or the probability or the number of entries.

Example 2.2.

- (1) As Alur and Madhusudan point out, XML documents or bibtex databases can naturally be modeled as nested words, where the nesting relation captures open and close tags [4]. Suppose we model bibtex databases as nested words. Then we may assign to a nested word e.g. the number of technical reports it stores.
- (2) Probabilistic automata have been used to model systems with uncertainty, such as communication systems over lossy channels, to model fault-tolerant systems or to model randomized programs. Consider the randomized recursive pseudo-procedure `bar` where `flip(Y)` means flipping a fair coin Y . Consider furthermore the alphabet $\Delta = \{r, w, b, \text{call}, \text{ret}\}$ of atomic events which stand for read, write, beep, call and return. Now, an execution of `bar` could be as follows: `read(x)`, flip a coin and see tail, call recursively `bar`, `read(x)`, flip a coin and see head, `beep`, flip a coin and see tail, return from the recursive call, flip a coin and see head, `write(x)`, flip a coin and see head, `write(x)`, flip a coin and see tail, exit the program. Then the nested word $nw = (w, \nu)$ defined by $w = r.\text{call}.r.b.\text{ret}.w.w.\text{ret}$ and $\nu = \{(2, 5)\}$ models this execution of `bar` where ν encodes the recursive call of `bar`. We calculate the probability of the execution by multiplying the probability of each atomic action (probability $1/2$ for those actions that depend on a coin flip), i.e. $1 \cdot 1/2 \cdot 1 \cdot 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 = 1/64$. We will model `bar` using a weighted nested word automaton in Example 2.4, below.

```

proc bar() {
  read(x);
  flip(Y); if(Y==head)
    beep;
  else
    bar();
  flip(Y); while(Y==head)
    write(x);
    flip(Y);
  exit;}

```

To be as flexible as possible, we take the quantities we assign to a nested word from a commutative semiring. A *commutative semiring* \mathbb{K} is an algebraic structure $(\mathbb{K}, +, \cdot, 0, 1)$ such that $(\mathbb{K}, +, 0)$ and $(\mathbb{K}, \cdot, 1)$ are commutative monoids, multiplication distributes over addition and 0 is absorbing, i.e. $0 \cdot k = k \cdot 0 = 0$ for all $k \in \mathbb{K}$. For example the natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ form a commutative semiring. Other important examples are also the tropical semiring $(\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$ and the arctic or max-plus semiring $(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$ which have been used to model real-time systems or discrete event systems. These semirings possess the property that any finitely generated submonoid of $(\mathbb{K}, +, 0)$ is finite. Such semirings are called *additively locally finite*. Another important example of an additively locally finite semiring is the probabilistic semiring $([0, 1], \max, \cdot, 0, 1)$. We call a semiring *locally finite* if any finitely generated subsemiring is finite. Examples

include any Boolean algebra such as the trivial Boolean algebra $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ as well as $(\mathbb{R}_+ \cup \{\infty\}, \max, \min, 0, \infty)$ and the fuzzy semiring $([0, 1], \max, \min, 0, 1)$.

In the following let \mathbb{K} be a commutative semiring such that $0 \neq 1$.

Definition 2.3. A *weighted nested word automaton* (WNWA for short) is a quadruple $\mathcal{A} = (Q, \iota, \delta, \kappa)$ where $\delta = (\delta_{\text{call}}, \delta_{\text{int}}, \delta_{\text{ret}})$ such that

- (1) Q is a finite set of states,
- (2) $\delta_{\text{call}}, \delta_{\text{int}} : Q \times \Delta \times Q \rightarrow \mathbb{K}$ are the *call* and *internal transition functions*,
- (3) $\delta_{\text{ret}} : Q \times Q \times \Delta \times Q \rightarrow \mathbb{K}$ is the *return transition function*,
- (4) $\iota, \kappa : Q \rightarrow \mathbb{K}$ are the *initial* and *final distribution*.

A *run* of \mathcal{A} on $nw = (a_1 \dots a_n, \nu)$ is a *sequence of states* $r = (q_0, \dots, q_n)$; we also write $r : q_0 \xrightarrow{nw} q_n$. The *weight* of r at position $1 \leq j \leq n$ is given by

$$\text{wgt}_{\mathcal{A}}(r, j) = \begin{cases} \delta_{\text{call}}(q_{j-1}, a_j, q_j) & \text{if } \nu(j, i) \text{ for some } j < i \leq n \\ \delta_{\text{int}}(q_{j-1}, a_j, q_j) & \text{if } j \text{ is an internal position} \\ \delta_{\text{ret}}(q_{j-1}, q_{i-1}, a_j, q_j) & \text{if } \nu(i, j) \text{ for some } 1 \leq i < j. \end{cases}$$

Now, the *weight* of r is $\text{wgt}_{\mathcal{A}}(r) = \prod_{1 \leq j \leq n} \text{wgt}_{\mathcal{A}}(r, j)$ and the *behavior* $\|\mathcal{A}\| : \text{NW}(\Delta) \rightarrow \mathbb{K}$ of \mathcal{A} is defined by

$$\|\mathcal{A}\|(nw) = \sum_{q_0, q_n \in Q} \iota(q_0) \cdot \sum_{r: q_0 \xrightarrow{nw} q_n} \text{wgt}_{\mathcal{A}}(r) \cdot \kappa(q_n).$$

A function $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ is called a *nested word series*. As for formal power series we write (S, nw) for $S(nw)$. We define the *scalar multiplication* \cdot and the *sum* $+$ pointwise, i.e. for $k \in \mathbb{K}$ and any two nested word series S_1, S_2 we let $(k.S_1, nw) = k \cdot (S_1, nw)$ and $(S_1 + S_2, nw) = (S_1, nw) + (S_2, nw)$ for all $nw \in \text{NW}(\Delta)$. For $L \subseteq \text{NW}(\Delta)$ let $\mathbb{1}_L$ be the *characteristic series* of L , i.e. the series that assumes 1 for all $nw \in L$ and 0 otherwise. A nested word series S is *regular* if there is a WNWA \mathcal{A} such that $\|\mathcal{A}\| = S$. For $\mathbb{K} = \mathbb{B}$, i.e. when $\delta_{\text{call}}, \delta_{\text{int}}$ and δ_{ret} are subsets of $Q \times \Delta \times Q$ and $Q \times Q \times \Delta \times Q$, or in other words when the transitions do not carry a weight, Definition 2.3 is equivalent to the definition of a (unweighted) nested word automaton [4]. A language of nested words $L \subseteq \text{NW}(\Delta)$ is then called regular if it is accepted by a nested word automaton. It is easy to see that this is the case iff the characteristic series $\mathbb{1}_L : \text{NW}(\Delta) \rightarrow \mathbb{B}$ is regular.

Example 2.4. The procedure `bar` of Example 2.2 can be modeled by a WNWA over $\mathbb{K} = ([0, 1], \max, \cdot, 0, 1)$ with four states $\{q_1, \dots, q_4\}$. The transitions (only those with non-zero weight) are given as follows. We let $\iota(q_1) = 1$ and $\kappa(q_4) = 1$. Moreover,

$$\begin{aligned} \delta_{\text{int}}(q_1, r, q_2) &= 1, & \delta_{\text{int}}(q_2, b, q_3) &= \delta_{\text{int}}(q_3, w, q_3) = \delta_{\text{int}}(q_3, \text{ret}, q_4) = 1/2 \\ \delta_{\text{call}}(q_2, \text{call}, q_1) &= 1/2, & \delta_{\text{ret}}(q_3, q_2, \text{ret}, q_3) &= 1/2. \end{aligned}$$

Intuitively, each of the states corresponds to a line in the procedure `bar` which is the next to be executed. q_1 corresponds to line 2, q_2 corresponds to line 3, q_3 corresponds to line 7 and q_4 is only reached at the end of an execution. Consider the nested word nw of Example 2.2(2). There is exactly one run $r : q_1 \xrightarrow{nw} q_4$ with $\text{wgt}(r) \neq 0$. We start in state q_1 execute r and change to q_2 . We then call and change back to q_1 . After that we execute r again and change to state q_2 . We then execute b and change to q_3 . We return and stay in q_3 . Now we execute w twice while staying in q_3 and finally end at state q_4 . Observe that the automaton assigns $1/64$ to the nested word nw .

3. WEIGHTED LOGICS

In this section we introduce another formalism for specifying nested word series. For this we interpret a nested word $nw = (a_1 \dots a_n, \nu)$ as a relational structure consisting of the domain $\text{dom}(nw) = [n]$ together with the unary relations $\text{Lab}_a = \{i \in \text{dom}(nw) \mid a_i = a\}$ for all $a \in \Delta$, the binary relation ν and the usual \leq relation on $\text{dom}(nw)$.

First, we recall classical monadic second-order logic. The set $\text{MSO}(\Delta, \leq, \nu)$ (we also write MSO for short) is given by the following grammar.

$$\varphi ::= x = y \mid \text{Lab}_a(x) \mid x \leq y \mid \nu(x, y) \mid x \in X \mid \varphi \vee \psi \mid \neg \varphi \mid \exists x. \varphi \mid \exists X. \varphi$$

where a ranges over Δ , where x, y are first-order variables and where X is a second-order variable. As usual we abbreviate $x < y = \neg(y \leq x)$, $\varphi \rightarrow \psi = \neg \varphi \vee \psi$ and $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ for any $\varphi, \psi \in \text{MSO}$.

Let $\varphi \in \text{MSO}$ and let $\text{Free}(\varphi)$ denote the set of variables that occur free in φ . Let \mathcal{V} be a finite set of first-order and second-order variables such that $\text{Free}(\varphi) \subseteq \mathcal{V}$. A (\mathcal{V}, nw) -assignment γ is a mapping from \mathcal{V} to the powerset $\mathcal{P}(\text{dom}(nw))$ such that first-order variables are mapped to singletons. For $i \in \text{dom}(nw)$ and $T \subseteq \text{dom}(nw)$ we denote by $\gamma[x \rightarrow i]$ (resp. $\gamma[X \rightarrow T]$) the $(\mathcal{V} \cup \{x\}, nw)$ -assignment (resp. $(\mathcal{V} \cup \{X\}, nw)$ -assignment) which equals γ on $\mathcal{V} \setminus \{x\}$ (resp. $\mathcal{V} \setminus \{X\}$) and assumes $\{i\}$ for x (resp. T for X). We write $(nw, \gamma) \models \varphi$ if φ holds in nw under the assignment γ . We write $\varphi(x_1, \dots, x_n, X_1, \dots, X_m)$ if $\text{Free}(\varphi) \subseteq \{x_1, \dots, x_n, X_1, \dots, X_m\}$. In this case write $nw \models \varphi[i_1, \dots, i_n, T_1, \dots, T_m]$ whenever we have $(nw, \gamma) \models \varphi$ if $\gamma(x_j) = \{i_j\}$ and $\gamma(X_j) = T_j$. This is justified by the fact that $(nw, \gamma) \models \varphi$ only depends on the restriction $\gamma|_{\text{Free}(\varphi)}$ of γ to $\text{Free}(\varphi)$. Let $\mathcal{L}_{\mathcal{V}}(\varphi) = \{(nw, \gamma) \mid nw \in \text{NW}(\Delta), \gamma \text{ is a } (\mathcal{V}, nw)\text{-assignment, } (nw, \gamma) \models \varphi\}$. Abbreviate $\mathcal{L}(\varphi) = \mathcal{L}_{\text{Free}(\varphi)}(\varphi)$. Note that in case that φ is a sentence, i.e. $\text{Free}(\varphi) = \emptyset$, we consider $\mathcal{L}(\varphi)$ as a subset of $\text{NW}(\Delta)$.

Let $Z \subseteq \text{MSO}$. A language $L \subseteq \text{NW}(\Delta)$ is Z -definable if $L = \mathcal{L}(\varphi)$ for a sentence $\varphi \in Z$. Formulae containing no quantification at all are called *propositional*. First-order formulae, i.e. formulae containing only quantification over first-order variables are collected in FO. The class EMSO consists of all formulae φ of the form $\exists X_1. \dots \exists X_m. \psi$ where $\psi \in \text{FO}$. Alur and Madhusudan showed that monadic second-order logic and nested word automata are equally expressive.

Theorem 3.1 (Alur & Madhusudan [3, 4]). *A nested word language $L \subseteq \text{NW}(\Delta)$ is regular iff L is MSO-definable iff L is EMSO-definable.*

We now turn to weighted monadic second-order logic as introduced in [11]. The set $\text{MSO}(\mathbb{K}, \Delta, \leq, \nu)$ (once again we shortly write $\text{MSO}(\mathbb{K})$) of *weighted MSO formulae* over \mathbb{K} is given by the following grammar:

$$\begin{aligned} \varphi ::= & k \mid x = y \mid \text{Lab}_a(x) \mid x \leq y \mid \nu(x, y) \mid x \in X \\ & \mid \neg(x = y) \mid \neg \text{Lab}_a(x) \mid \neg x \leq y \mid \neg \nu(x, y) \mid \neg(x \in X) \\ & \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x. \varphi \mid \exists X. \varphi \mid \forall x. \varphi \mid \forall X. \varphi \end{aligned}$$

where $k \in \mathbb{K}$, where a ranges over Δ , where x, y are first-order variables and where X is a second-order variable. Note that we allow negation only for *atomic formulae*, i.e. for the formulae $x = y$, $\text{Lab}_a(x)$, $x \leq y$, $\nu(x, y)$ and $x \in X$. This is because in general semirings we do not have a natural complement and hence it is not clear how to define the semantics of negation for values other than 0 and 1 (cf. [11]).

Let $\varphi \in \text{MSO}(\mathbb{K})$ and $\text{Free}(\varphi) \subseteq \mathcal{V}$. The weighted semantics $\llbracket \varphi \rrbracket_{\mathcal{V}}$ of φ is a function assigning a value in \mathbb{K} to a nested word nw and a (\mathcal{V}, nw) -assignment γ . To each such pair (nw, γ) we assign an element of \mathbb{K} inductively as follows. For $k \in \mathbb{K}$ we put $\llbracket k \rrbracket_{\mathcal{V}}(nw, \gamma) = k$. For every other atomic formula or negated atomic formula φ the semantics $\llbracket \varphi \rrbracket_{\mathcal{V}}$ is given by the characteristic function $\mathbb{1}_{\mathcal{L}_{\mathcal{V}}(\varphi)}$. Moreover, we define

$$\begin{aligned} \llbracket \varphi \vee \psi \rrbracket_{\mathcal{V}}(nw, \gamma) &= \llbracket \varphi \rrbracket_{\mathcal{V}}(nw, \gamma) + \llbracket \psi \rrbracket_{\mathcal{V}}(nw, \gamma), \\ \llbracket \varphi \wedge \psi \rrbracket_{\mathcal{V}}(nw, \gamma) &= \llbracket \varphi \rrbracket_{\mathcal{V}}(nw, \gamma) \cdot \llbracket \psi \rrbracket_{\mathcal{V}}(nw, \gamma), \\ \llbracket \exists x. \varphi \rrbracket_{\mathcal{V}}(nw, \gamma) &= \sum_{i \in \text{dom}(nw)} \llbracket \varphi \rrbracket_{\mathcal{V} \cup \{x\}}(nw, \gamma[x \rightarrow i]), \\ \llbracket \exists X. \varphi \rrbracket_{\mathcal{V}}(nw, \gamma) &= \sum_{T \subseteq \text{dom}(nw)} \llbracket \varphi \rrbracket_{\mathcal{V} \cup \{X\}}(nw, \gamma[X \rightarrow T]), \\ \llbracket \forall x. \varphi \rrbracket_{\mathcal{V}}(nw, \gamma) &= \prod_{i \in \text{dom}(nw)} \llbracket \varphi \rrbracket_{\mathcal{V} \cup \{x\}}(nw, \gamma[x \rightarrow i]), \\ \llbracket \forall X. \varphi \rrbracket_{\mathcal{V}}(nw, \gamma) &= \prod_{T \subseteq \text{dom}(nw)} \llbracket \varphi \rrbracket_{\mathcal{V} \cup \{X\}}(nw, \gamma[X \rightarrow T]). \end{aligned}$$

We put $\llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket_{\text{Free}(\varphi)}$. Observe that in the case where φ is a sentence, $\llbracket \varphi \rrbracket$ can be considered as a series from $\text{NW}(\Delta)$ to \mathbb{K} .

Remark 3.2. A formula $\varphi \in \text{MSO}(\mathbb{K})$ which does not contain a subformula $k \in \mathbb{K}$ can be interpreted as an unweighted formula. We will use this implicitly in the sequel. Moreover, note that if \mathbb{K} is the Boolean semiring \mathbb{B} , then weighted logics and classical MSO logic coincide. In this case k is either 0 (false) or 1 (true).

Example 3.3.

- (1) As in Example 2.2 suppose we model bibtex databases as nested words. Moreover, assume that $\text{tecrep} \in \Delta$ marks the beginning of an entry containing a technical report. Now, let $\mathbb{K} = \mathbb{N}$ be the semiring of the natural numbers. Then $(\llbracket \exists x. \text{Lab}_{\text{tecrep}}(x) \rrbracket, nw)$ counts the number of technical reports of the bibtex database modeled by nw .
- (2) Again let $\mathbb{K} = \mathbb{N}$. Consider the formula $\varphi = \forall x. \exists y. 1$. Then $(\llbracket \exists x. 1 \rrbracket, (a_1 \dots a_n, \nu)) = n$ and $(\llbracket \forall y. \exists x. 1 \rrbracket, (a_1 \dots a_n, \nu)) = n^n$. It can be shown as for words that $\llbracket \varphi \rrbracket$ is not regular as it grows too fast (cf. Example 3.4 in [11]).

Let $Z \subseteq \text{MSO}(\mathbb{K})$. A series $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ is *Z-definable* if $S = \llbracket \varphi \rrbracket$ for a sentence $\varphi \in Z$. Example 3.3(2) shows that unrestricted application of universal quantification does not preserve regularity. Therefore we now define different fragments of $\text{MSO}(\mathbb{K})$.

Note that the fragment $\text{RMSO}(\mathbb{K})$, the collection of *restricted* formulae, which was considered in [11] and which characterizes regular formal power series is a semantic restriction, and it is not clear whether membership in $\text{RMSO}(\mathbb{K})$ can be decided. In order to have a decidable fragment, we now syntactically define the fragment $\text{sRMSO}(\mathbb{K})$. For this we follow the approach of [12].

The idea is to restrict universal first-order quantification to formulae having a semantics that takes on only finitely many values. To this aim we start by identifying a class of formulae φ that take on values 0 and 1 only, more precisely we will have $\mathbb{1}_{\mathcal{L}_{\mathcal{V}}(\varphi)} = \llbracket \varphi \rrbracket_{\mathcal{V}}$. The problem that arises is that by definition of the semantics, \forall gets translated by means of $+$. Hence, for a formula $\varphi = \varphi_1 \vee \varphi_2$ we only want to evaluate φ_2 if φ_1 evaluates to 0, otherwise we might end up with a sum greater than one. A similar problem occurs for $\exists x$. and $\exists X$.

Given a classical (unweighted) MSO-formula φ we assign to it formulae φ^+ and φ^- such that $\llbracket \varphi^+ \rrbracket = \mathbb{1}_{\mathcal{L}(\varphi)}$ and $\llbracket \varphi^- \rrbracket = \mathbb{1}_{\mathcal{L}(\neg\varphi)}$. The crucial point is that we have a linear order at disposal.

- (1) If φ is of the form $x = y$, $\text{Lab}_a(x)$, $x \leq y$, $\nu(x, y)$, $x \in X$ then $\varphi^+ = \varphi$ and $\varphi^- = \neg\varphi$.
- (2) If $\varphi = \neg\psi$, then $\varphi^+ = \psi^-$ and $\varphi^- = \psi^+$.
- (3) If $\varphi = \psi \vee \psi'$, then $\varphi^+ = \psi^+ \vee (\psi^- \wedge \psi'^+)$ and $\varphi^- = \psi^- \wedge \psi'^-$.
- (4) If $\varphi = \exists x.\psi(x)$, then $\varphi^+ = \exists x.\psi(x)^+ \wedge \forall y.(y < x \wedge \psi(y))^-$ and $\varphi^- = \forall x.\psi(x)^-$.

In order to disambiguate set quantification, we have to define a linear order on the subsets of the domain of a nested word or equivalently on nested words (of fixed length) over the alphabet $\{0, 1\}$. We take the lexicographic order $<$ which is given by the following formula.

$$X < Y = \exists y.y \in Y \wedge \neg y \in X \wedge \forall z.[z < y \rightarrow (z \in X \leftrightarrow z \in Y)]^+$$

Now we proceed:

- (5) If $\varphi = \exists X.\psi(X)$, then $\varphi^+ = \exists X.\psi(X)^+ \wedge \forall Y.(Y < X \wedge \psi(Y))^-$ and $\varphi^- = \forall X.\psi(X)^-$.

Formulae of the form φ^+ or φ^- for some $\varphi \in \text{MSO}$ are called *syntactically unambiguous*. Observe, if φ is syntactically unambiguous, then $\llbracket \varphi \rrbracket_{\mathcal{V}} = \mathbb{1}_{\mathcal{L}_{\mathcal{V}}(\varphi)}$ for any finite set of variables $\mathcal{V} \supseteq \text{Free}(\varphi)$. In the following, we shortly write $\varphi \xrightarrow{+} \psi$ for $\varphi^- \vee (\varphi^+ \wedge \psi)$ for any two weighted formulae φ, ψ where φ does not contain subformulae of the form k ($k \in \mathbb{K}$) and hence is also a classical formula.

We define $\text{aUMSO}(\mathbb{K})$, the collection of *almost unambiguous* formulae, to be the smallest subset of $\text{MSO}(\mathbb{K})$ containing all constants k ($k \in \mathbb{K}$) and all syntactically unambiguous formulae which is closed under conjunction and disjunction. Using the distributivity, observe that for any $\psi \in \text{aUMSO}(\mathbb{K})$ there is a formula ψ' of the form $\psi' = \bigvee_{i=1}^n (k_i \wedge \psi_i)$ for some $k_i \in \mathbb{K}$ and syntactically unambiguous ψ_i such that $\llbracket \psi \rrbracket = \llbracket \psi' \rrbracket$ (cf. [12]). We are now ready to define the fragment $\text{sRMSO}(\mathbb{K})$.

Definition 3.4. A weighted formula φ is in $\text{sRMSO}(\mathbb{K})$ (syntactically restricted MSO) if for every subformula ϑ of φ the following two conditions hold:

- (1) If $\vartheta = \forall X.\psi$ for some $\psi \in \text{MSO}(\mathbb{K})$, then ψ is syntactically unambiguous.
- (2) If $\vartheta = \forall x.\psi$ for some $\psi \in \text{MSO}(\mathbb{K})$, then $\psi \in \text{aUMSO}(\mathbb{K})$.

We collect in $\text{sRFO}(\mathbb{K})$ all $\varphi \in \text{sRMSO}(\mathbb{K})$ which do not contain any set quantification and we collect in $\text{sREMSO}(\mathbb{K})$ all $\varphi \in \text{sRMSO}(\mathbb{K})$ of the form $\exists X_1 \dots \exists X_m.\psi$ with $\psi \in \text{sRFO}(\mathbb{K})$.

Let now $\text{wUMSO}(\mathbb{K})$, the collection of *weakly unambiguous* formulae, be the smallest subset of $\text{MSO}(\mathbb{K})$ containing all constants k ($k \in \mathbb{K}$) and all syntactically unambiguous formulae which is closed under conjunction, disjunction and existential quantification (both first- and second-order). We define the fragment $\text{swRMSO}(\mathbb{K})$.

Definition 3.5. A weighted formula φ is in $\text{swRMSO}(\mathbb{K})$ (syntactically weakly restricted MSO) if for every subformula ϑ of φ the following two conditions hold:

- (1) If $\vartheta = \forall X.\psi$ for some $\psi \in \text{MSO}(\mathbb{K})$, then ψ is syntactically unambiguous.
- (2) If $\vartheta = \forall x.\psi$ for some $\psi \in \text{MSO}(\mathbb{K})$, then $\psi \in \text{wUMSO}(\mathbb{K})$.

Clearly, $\text{aUMSO}(\mathbb{K}) \subset \text{wUMSO}(\mathbb{K}) \subset \text{sRMSO}(\mathbb{K}) \subset \text{swRMSO}(\mathbb{K}) \subset \text{MSO}(\mathbb{K})$. The first main result of this paper is the characterization of regular nested word series using weighted logics. It reads as follows.

Theorem 3.6. *Let \mathbb{K} be a commutative semiring and let $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ be a nested word series. Then the following holds.*

- (a) *S is regular iff it is sRMSO(\mathbb{K})-definable iff it is sREMSO(\mathbb{K})-definable.*
- (b) *If \mathbb{K} is additively locally finite, then S is regular iff it is swRMSO(\mathbb{K})-definable.*
- (c) *If \mathbb{K} is locally finite, then S is regular iff it is MSO(\mathbb{K})-definable.*

We prove the result at the end of Section 5 by interpreting nested words in alternating texts. In the next section we introduce alternating texts and weighted automata for them.

Example 3.7. The *nesting depth* of a position i of a nested word nw is the number of open call positions (i.e. where the corresponding return position has not occurred yet including the position itself). The nesting depth of a nested word is the maximum nesting depth of its positions. Let $\mathbb{K} = (\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$.

$$\begin{aligned} \text{open}(x) &= \forall y. (y \leq x \wedge \text{call}(y)) \xrightarrow{+} 1 \wedge (y \leq x \wedge \text{return}(y)) \xrightarrow{-} -1 \text{ where} \\ \text{call}(x) &= \exists y. \nu(x, y) \text{ and } \text{return}(x) = \exists y. \nu(y, x) \end{aligned}$$

Then $\llbracket \exists x. \text{open}(x) \rrbracket$ assigns to a nested word its nesting depth. Hence, since $\exists x. \text{open}(x) \in \text{sRMSO}(\mathbb{K})$, the series is regular by Theorem 3.6.

4. ALTERNATING TEXTS

A bisemigroup is a set together with two associative operations. Several authors investigated the free bisemigroup as a fundamental, two-dimensional extension of classical automaton theory, see e.g. Ésik and Németh [17] and Hashiguchi et al. (e.g. [19–21]). Ésik and Németh considered as a representation for the free bisemigroup the so-called *sp-biposets*, a certain class of biposets. A different representation of the free bisemigroup over some finite set Δ are the so-called alternating texts [16, 22]. A *text* over Δ is a tuple $(V, \lambda, \leq_1, \leq_2)$ where \leq_1 and \leq_2 are linear orders over a finite but non-empty domain V and $\lambda : V \rightarrow \Delta$ is a labeling function. Of course we consider texts only up to isomorphism. Therefore, unless otherwise specified, the domain of a text will be $[n] = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ and \leq_1 will correspond to the canonical order on $[n]$.

We define the binary operations \circ and \bullet , called the horizontal and vertical product, on texts as follows: Let $\tau = (V, \lambda, \leq_1, \leq_2)$ and $\tau' = (V', \lambda', \leq'_1, \leq'_2)$ be two texts where we assume that V and V' are disjoint. Then

$$\begin{aligned} \tau_1 \circ \tau_2 &= (V \uplus V', \lambda \cup \lambda', \leq_1 \cup \leq'_1 \cup V \times V', \leq_2 \cup \leq'_2 \cup V \times V'), \\ \tau_1 \bullet \tau_2 &= (V \uplus V', \lambda \cup \lambda', \leq_1 \cup \leq'_1 \cup V \times V', \leq_2 \cup \leq'_2 \cup V' \times V). \end{aligned}$$

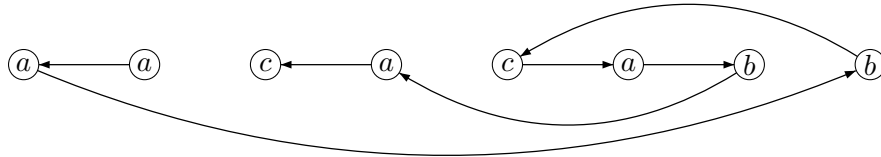


Figure 2: A visualization of the alternating text given by $(a \bullet a) \circ (c \bullet a \bullet (c \circ a \circ b) \bullet b)$. Here we only give the successor relation of the second order. The first order is given simply from the left to the right.

Let $\text{TXT}(\Delta)$ be the class of texts which can be obtained from the singleton texts by finite applications of \circ, \bullet . This class was named the class of *alternating texts* in [16]. The class $\text{TXT}(\Delta)$ together with the operations \circ, \bullet is the free bisemigroup over Δ [22]. Let monadic second-order logic $\text{MSO}(\Delta, \leq_1, \leq_2)$ and weighted logics for texts, denoted $\text{MSO}(\mathbb{K}, \Delta, \leq_1, \leq_2)$ be defined along the same lines as for nested words. Moreover, define $\text{sRMSO}(\mathbb{K}, \Delta, \leq_1, \leq_2)$ and $\text{swRMSO}(\mathbb{K}, \Delta, \leq_1, \leq_2)$ using the linear order \leq_1 .

Now we introduce weighted parenthesizing automata (cf. [28]) operating on the free bisemigroup generalizing parenthesizing automata as introduced by Ésik and Németh [17].

Definition 4.1. A tuple $\mathcal{A} = (\mathcal{H}, \mathcal{V}, \Omega, \mu, \mu_{\text{op}}, \mu_{\text{cl}}, \lambda, \gamma)$ is a *weighted parenthesizing automaton* (WPA) provided that

- \mathcal{H} and \mathcal{V} are finite, disjoint sets of *horizontal* and *vertical states*, respectively,
- Ω is a finite set of *parentheses*,¹
- $\mu : (\mathcal{H} \times \Delta \times \mathcal{H}) \cup (\mathcal{V} \times \Delta \times \mathcal{V}) \rightarrow \mathbb{K}$ is the *transition function*,
- $\mu_{\text{op}}, \mu_{\text{cl}} : (\mathcal{H} \times \Omega \times \mathcal{V}) \cup (\mathcal{V} \times \Omega \times \mathcal{H}) \rightarrow \mathbb{K}$ are the *opening* and *closing parenthesizing functions*, respectively,
- $\lambda, \gamma : \mathcal{H} \cup \mathcal{V} \rightarrow \mathbb{K}$ are the *initial* and *final weight functions*, respectively.

We now come to the notion of a run r of \mathcal{A} . We give an inductive definition where we also define its *label* $\text{lab}(r) \in \text{TXT}(\Delta)$, its *weight* $\text{wgt}_{\mathcal{A}}(r) \in \mathbb{K}$, its *initial state* $\text{init}(r) \in \mathcal{H} \cup \mathcal{V}$ and its *final state* $\text{fin}(r) \in \mathcal{H} \cup \mathcal{V}$. Formally the set of *runs* of \mathcal{A} is the smallest set of words over the alphabet $\Delta \cup \Omega \cup \mathcal{H} \cup \mathcal{V} \cup \{(\cdot, \cdot)\} \cup \{ \cdot, \cdot \}$ such that:

- (1) The word (q_1, a, q_2) is a run for all $(q_1, q_2) \in (\mathcal{H} \times \mathcal{H}) \cup (\mathcal{V} \times \mathcal{V})$ and $a \in \Delta$. We set

$$\begin{aligned} \text{lab}((q_1, a, q_2)) &= a \in \text{TXT}(\Delta), & \text{wgt}_{\mathcal{A}}((q_1, a, q_2)) &= \mu(q_1, a, q_2), \\ \text{init}((q_1, a, q_2)) &= q_1 \text{ and } \text{fin}((q_1, a, q_2)) &= q_2. \end{aligned}$$
- (2) If r_1 and r_2 are runs such that $\text{fin}(r_1) = \text{init}(r_2) \in \mathcal{H}$ (respectively such that $\text{fin}(r_1) = \text{init}(r_2) \in \mathcal{V}$), then $r = r_1 r_2$ is a run having

$$\begin{aligned} \text{lab}(r) &= \text{lab}(r_1) \circ \text{lab}(r_2), & \text{(resp. } \text{lab}(r) &= \text{lab}(r_1) \bullet \text{lab}(r_2)), \\ \text{wgt}_{\mathcal{A}}(r) &= \text{wgt}_{\mathcal{A}}(r_1) \cdot \text{wgt}_{\mathcal{A}}(r_2), & \text{init}(r) &= \text{init}(r_1) \text{ and } \text{fin}(r) = \text{fin}(r_2). \end{aligned}$$
- (3) If a run r resulting from 2 has $\text{init}(r) \in \mathcal{H}$ (resp. $\text{init}(r) \in \mathcal{V}$) and if $q_1, q_2 \in \mathcal{V}$ (resp. $q_1, q_2 \in \mathcal{H}$) and $s \in \Omega$, then $r' = (q_1, (s, \text{init}(r)) r (\text{fin}(r),)_s, q_2)$ is a run. We set

$$\begin{aligned} \text{lab}(r') &= \text{lab}(r), & \text{init}(r') &= q_1 \text{ and } \text{fin}(r') = q_2, \\ \text{wgt}_{\mathcal{A}}(r') &= \mu_{\text{op}}((q_1, (s, \text{init}(r))) \cdot \text{wgt}_{\mathcal{A}}(r) \cdot \mu_{\text{cl}}((\text{fin}(r),)_s, q_2)). \end{aligned}$$

Let $\tau \in \text{TXT}(\Delta)$. Since in (3) above we require that the run r we start with results from (2), we do not allow repeated application of (3) and therefore there are only finitely many runs r of \mathcal{A} with label τ . Intuitively, we do not allow for doubled parentheses. If r is a run of \mathcal{A} with $\text{lab}(r) = \tau$, $\text{init}(r) = q_1$, $\text{fin}(r) = q_2$, we write $r : q_1 \xrightarrow{\tau} q_2$. The behavior of \mathcal{A} is a text series $\|\mathcal{A}\| : \text{TXT}(\Delta) \rightarrow \mathbb{K}$. It is given by

$$\|\mathcal{A}\|(\tau) = \sum_{q_1, q_2 \in \mathcal{H} \cup \mathcal{V}} \lambda(q_1) \cdot \sum_{r: q_1 \xrightarrow{\tau} q_2} \text{wgt}_{\mathcal{A}}(r) \cdot \gamma(q_2).$$

A text series S is *regular* if there is a WPA \mathcal{A} such that $\|\mathcal{A}\| = S$.

¹ We let $s \in \Omega$ represent both an opening and a closing parentheses. To help the intuition we also write $(s$ or $)_s$ for s .

Theorem 4.2 (see [30]). *Let \mathbb{K} be a commutative semiring and let $S : \text{TXT}(\Delta) \rightarrow \mathbb{K}$ be an alternating text series. Then the following holds.*

- (a) *S is regular iff it is $\text{sRMSO}(\mathbb{K})$ -definable iff it is $\text{sREMSO}(\mathbb{K})$ -definable.*
- (b) *If \mathbb{K} is additively locally finite, then S is regular iff it is $\text{swRMSO}(\mathbb{K})$ -definable.*
- (c) *If \mathbb{K} is locally finite, then S is regular iff it is $\text{MSO}(\mathbb{K})$ -definable.*

We note that the proof in [30] is effective, i.e. given an $\text{sRMSO}(\mathbb{K})$ (resp. $\text{swRMSO}(\mathbb{K})$, resp. $\text{MSO}(\mathbb{K})$) formula φ we can effectively construct a WPA \mathcal{A} such that $\llbracket \varphi \rrbracket = \|\mathcal{A}\|$, and conversely, given a WPA \mathcal{A} we can effectively construct $\varphi \in \text{sREMSO}(\mathbb{K})$ such that $\llbracket \varphi \rrbracket = \|\mathcal{A}\|$.

5. INTERPRETING NESTED WORDS IN ALTERNATING TEXTS

We will now derive similar results for nested words as for alternating texts by interpreting the different structures within each other. For this we utilize definable transductions as introduced by Courcelle [9]. We only have to ensure that they preserve definability, now with respect to weighted logics. First, we introduce the notion of definable transductions. For this let σ_1 and $\sigma_2 = ((R_i)_{i \in I}, \rho)$ be two relational signatures where $\rho : I \rightarrow \mathbb{N}_+$ assigns to each relation symbol R_i a positive arity. Moreover, let \mathcal{C}_1 and \mathcal{C}_2 be classes of finite σ_1 - and σ_2 -structures, respectively. Let monadic second-order logic $\text{MSO}(\sigma_1)$ and $\text{MSO}(\sigma_2)$ be defined along the lines as for nested words.

By a (σ_1, σ_2) -1-copying definition scheme with parameters X_1, \dots, X_n we mean a tuple $\mathcal{D} = (\vartheta, \delta, (\varphi_i)_{i \in I})$ of formulae in $\text{MSO}(\sigma_1)$ such that $\text{Free}(\vartheta) \subseteq \{X_1, \dots, X_n\}$, $\text{Free}(\delta) \subseteq \{x_1, X_1, \dots, X_n\}$ and $\text{Free}(\varphi_i) \subseteq \{x_1, \dots, x_{\rho(i)}, X_1, \dots, X_n\}$ for all $i \in I$.

Let \mathcal{D} be a (σ_1, σ_2) -1-copying definition scheme, let $s_1 \in \mathcal{C}_1$ and let T_1, \dots, T_n subsets of the domain $\text{dom}(s_1)$ of s_1 such that $s_1 \models \vartheta[T_1, \dots, T_n]$. Then define the σ -structure $\mathbf{def}_{\mathcal{D}}(s_1, T_1, \dots, T_n) = s_2$ with domain $\text{dom}(s_2) \subseteq \text{dom}(s_1)$ and interpretations of relation symbols $R_i^{s_2}$ given as follows:

$$\begin{aligned} v \in \text{dom}(s_2) &\Leftrightarrow s_1 \models \delta[v, T_1, \dots, T_n] \text{ for all } v \in \text{dom}(s_1). \\ (v_1, \dots, v_{\rho(i)}) \in R_i^{s_2} &\Leftrightarrow s_1 \models \varphi_i[v_1, \dots, v_{\rho(i)}, T_1, \dots, T_n] \text{ for all } i \in I \text{ and} \\ &\quad \text{all } v_1, \dots, v_{\rho(i)} \in \text{dom}(s_2). \end{aligned}$$

By abusing notation, we define the transduction $\mathbf{def}_{\mathcal{D}} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ by letting $(s_1, s_2) \in \mathbf{def}_{\mathcal{D}}$ iff $s_1 \in \mathcal{C}_1$ and there are sets $T_1, \dots, T_n \subseteq \text{dom}(s_1)$ with $s_1 \models \vartheta[T_1, \dots, T_n]$ such that $s_2 = \mathbf{def}_{\mathcal{D}}(s_1)$. Let us call a definition scheme \mathcal{D} with parameters X_1, \dots, X_n *unambiguous* if for any pair $(s_1, s_2) \in \mathbf{def}_{\mathcal{D}}$ there is at most one assignment of parameters $\gamma : \{X_1, \dots, X_n\} \rightarrow \mathcal{P}(\text{dom}(s_1))$ such that $\mathbf{def}_{\mathcal{D}}(s_1, \gamma(X_1), \dots, \gamma(X_n)) = s_2$.

Definition 5.1. A transduction $\Phi \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ is *unambiguously definable* if there is a unambiguous definition scheme \mathcal{D} such that $\Phi = \mathbf{def}_{\mathcal{D}}$. It is unambiguously FO-definable if there is an unambiguous definition scheme $\mathcal{D} = (\vartheta, \delta, (\varphi_i)_{i \in I})$ defining Φ with $\vartheta, \delta, (\varphi_i)_{i \in I} \in \text{FO}$.

A transduction which is given by a less restricted definition scheme, where one allows for more than one copy of s_1 and which is not necessarily unambiguous, is called definable. Courcelle [9] showed that the preimage of a definable set under a definable transduction is again definable. We will show a similar result for series. Let $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a partial function with domain $\text{dom}(\Phi)$ and let $S : \mathcal{C}_2 \rightarrow \mathbb{K}$. Define $\Phi^{-1}(S)$ by letting $(\Phi^{-1}(S), s_1) =$

$(S, \Phi(s_1))$ for all $s_1 \in \text{dom}(\Phi)$ and $(\Phi^{-1}(S), s_1) = 0$ otherwise. If Φ is injective, we let $\Phi(S) = (\Phi^{-1})^{-1}(S)$.

Clearly, $\text{MSO}(\mathbb{K})$ can be defined for \mathcal{C}_1 and \mathcal{C}_2 along the same lines as for nested words. In order to disambiguate a formula, we need a linear order on each $s \in \mathcal{C}_1$ (resp. \mathcal{C}_2). For the next proposition we therefore assume that there are binary relation symbols $\leq_1 \in \sigma_1$ and $\leq_2 \in \sigma_2$ such that the interpretation of \leq_i in s is a linear order for any $s \in \mathcal{C}_i$ ($i = 1, 2$). Using these linear orders we can define syntactically unambiguous formulae and then $\text{sRMSO}(\mathbb{K})$ and $\text{swRMSO}(\mathbb{K})$ over σ_1 and σ_2 .

Proposition 5.2. *Let $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be an unambiguously definable partial function. Then the following holds:*

- (1) *If $S : \mathcal{C}_2 \rightarrow \mathbb{K}$ is $\text{MSO}(\mathbb{K})$ -definable, then so is $\Phi^{-1}(S)$.*
- (2) *If $S : \mathcal{C}_2 \rightarrow \mathbb{K}$ is $\text{sRMSO}(\mathbb{K})$ -definable, then so is $\Phi^{-1}(S)$.*
- (3) *If $S : \mathcal{C}_2 \rightarrow \mathbb{K}$ is $\text{swRMSO}(\mathbb{K})$ -definable, then so is $\Phi^{-1}(S)$.*
- (4) *If Φ is unambiguously FO-definable and $S : \mathcal{C}_2 \rightarrow \mathbb{K}$ is $\text{sREMSO}(\mathbb{K})$ -definable, then $\Phi^{-1}(S)$ is $\text{sREMSO}(\mathbb{K})$ -definable.*

Proof sketch. Full proof and more general results can be found in [30, 31].

Let $\mathcal{D} = (\vartheta, \delta, (\varphi_i)_{i \in I})$ be an unambiguous definition scheme defining Φ . Let $\varphi \in \text{MSO}(\mathbb{K})$. By induction on the structure of φ we now define the formula $\widehat{\varphi} \in \text{MSO}(\mathbb{K}, \sigma_1)$.

$$\widehat{k} = k, \quad \widehat{x = y} = (x = y) \quad \widehat{x \in X} = x \in X \quad R_i(x_1, \dots, x_{\rho(i)}) = \varphi_i(x_1 \dots x_{\rho(i)}, X_1, \dots, X_n)^+$$

If φ is $x = y$, $x \in X$ or $R_i(x_1, \dots, x_{\rho(i)})$ let $\widehat{\neg\psi} = (\widehat{\psi})^-$. Moreover, let

$$\begin{aligned} \widehat{\psi_1 \wedge \psi_2} &= \widehat{\psi_1} \wedge \widehat{\psi_2} \\ \widehat{\psi_1 \vee \psi_2} &= \begin{cases} (\widehat{\psi_1} \vee \widehat{\psi_2})^+ & \text{if } \psi_1 \vee \psi_2 \text{ is syntactically unambiguous} \\ \widehat{\psi_1} \vee \widehat{\psi_2} & \text{otherwise} \end{cases} \\ \widehat{\exists x.\psi} &= \begin{cases} [\exists x.(\delta(x, X_1, \dots, X_n) \wedge \widehat{\psi})]^+ & \text{if } \exists x.\psi \text{ is syntactically unambiguous} \\ \exists x.(\delta(x, X_1, \dots, X_n)^+ \wedge \widehat{\psi}) & \text{otherwise} \end{cases} \\ \widehat{\exists X.\psi} &= \begin{cases} [\exists X.\forall x.(x \in X \rightarrow \delta(x, X_1, \dots, X_n)) \wedge \widehat{\psi}]^+ & \text{if } \exists X.\psi \text{ is synt. unambiguous} \\ \exists X.\forall x.(x \in X \rightarrow \delta(x, X_1, \dots, X_n))^+ \wedge \widehat{\psi} & \text{otherwise} \end{cases} \\ \widehat{\forall x.\psi} &= \forall x.\delta(x, X_1, \dots, X_n) \xrightarrow{+} \widehat{\psi} \\ \widehat{\forall X.\psi} &= \forall X.(\forall x.x \in X \rightarrow \delta(x, X_1, \dots, X_n)) \xrightarrow{+} \widehat{\psi}. \end{aligned}$$

Now let φ be as required such that $\llbracket \varphi \rrbracket = S$. One can show by induction on the structure of φ that $\llbracket \exists X_1, \dots, X_n.\vartheta(X_1, \dots, X_n)^+ \wedge \widehat{\varphi} \rrbracket = \Phi^{-1}(S)$. By construction we get that if φ is syntactically unambiguous, then so is its translation $\widehat{\varphi}$. Again by induction it is therefore not hard to see that $\widehat{\varphi}$ is in $\text{aUMSO}(\mathbb{K})$ (resp. $\text{wUMSO}(\mathbb{K})$) if φ is in $\text{aUMSO}(\mathbb{K})$ (resp. $\text{wUMSO}(\mathbb{K})$). From this we conclude that the translation is as required. \square

We are now going to show that regular series coincide with $\text{sRMSO}(\mathbb{K})$ -definable ones. For this we define two embeddings of nested words into alternating texts and use the characterizations of text series. The connection we establish turns out to be useful again in Section 6. Define $\Phi_\bullet, \Phi_\circ : \text{NW}(\Delta) \rightarrow \text{TXT}(\Delta)$ as follows. Let $nw = (w, \nu) \in \text{NW}(\Delta)$ where $w = a_1 \dots a_n$. If $\nu = \emptyset$, then let $\Phi_\circ(nw) = a_1 \circ \dots \circ a_n$ and $\Phi_\bullet(nw) = a_1 \bullet \dots \bullet a_n$.

If $\nu \neq \emptyset$, let i be the minimal call position and j the corresponding return position. Let $nw' = nw[i+1, j-1]$ and $nw'' = nw[j+1, n]$. Suppose for the moment that $i+1 \leq j-1$ and $j+1 \leq n$. We define

$$\begin{aligned}\Phi_{\circ}(nw) &= a_1 \circ \dots \circ a_{i-1} \circ (a_i \bullet \Phi_{\bullet}(nw') \bullet a_j) \circ \Phi_{\circ}(nw''), \\ \Phi_{\bullet}(nw) &= a_1 \bullet \dots \bullet a_{i-1} \bullet (a_i \circ \Phi_{\circ}(nw') \circ a_j) \bullet \Phi_{\bullet}(nw'').\end{aligned}$$

If $i+1 = j$ or $j = n$, then we just ignore the terms $\Phi_{\circ}(nw')$, $\Phi_{\circ}(nw'')$, $\Phi_{\bullet}(nw')$ and $\Phi_{\bullet}(nw'')$, respectively, in the definition above. Intuitively, we transform the nesting relation into well-matched brackets. As an example consider the nested word nw given in Figure 1. Its coding $\Phi_{\circ}(nw)$ is the alternating text in Figure 2.

Let $\Phi_{\circ}(nw) = (V^{\circ}, \lambda^{\circ}, \leq_1^{\circ}, \leq_2^{\circ})$ and $\Phi_{\bullet}(nw) = (V^{\bullet}, \lambda^{\bullet}, \leq_1^{\bullet}, \leq_2^{\bullet})$. The following observations can easily be made by induction either on n or on $|\nu|$:

- (a) Both V° and V^{\bullet} have cardinality n . We therefore assume from now on that $V^{\circ} = V^{\bullet} = [n]$ such that \leq_1° as well as \leq_1^{\bullet} is the usual order on $[n]$. It is easy to see that $\lambda^{\circ}(i) = \lambda^{\bullet}(i) = a_i$.
- (b) Both Φ_{\circ} and Φ_{\bullet} are injective.

Recall that a position of nw has odd *nesting depth* if the number of open call positions is odd (see Example 3.7).

Lemma 5.3. *Let $nw = (a_1 \dots a_n, \nu) \in \text{NW}(\Delta)$, let $\Phi_{\circ}(nw) = ([n], \lambda, \leq_1^{\circ}, \leq_2^{\circ})$ and let $\Phi_{\bullet}(nw) = ([n], \lambda, \leq_1^{\bullet}, \leq_2^{\bullet})$. Moreover, let $1 \leq i < j \leq n$. Then we have, $i \geq_2^{\circ} j$ iff $i \leq_2^{\bullet} j$ iff there is some $(k, \ell) \in \nu$ with $1 \leq k \leq i < j \leq \ell \leq n$ such that there is no $(k', \ell') \in \nu$ with $k < k' \leq i < j \leq \ell' < \ell$ and k has odd nesting depth.*

Proof. The proof is by induction on $|\nu|$. For $|\nu| = 0$ this is trivial. Now let $|\nu| \geq 1$. We only prove that $i \geq_2^{\circ} j$ iff there is some $(k, \ell) \in \nu$ with $1 \leq k \leq i < j \leq \ell \leq n$ such that there is no $(k', \ell') \in \nu$ with $k < k' \leq i < j \leq \ell' < \ell$ and k has odd nesting depth. That this holds iff $i \leq_2^{\bullet} j$ can be shown analogously. Let i' be the minimal call position and j' the corresponding return position. Let $nw' = nw[i'+1, j'-1]$ and $nw'' = nw[j'+1, n]$ provided they exist. Moreover, let $\Phi_{\bullet}(nw') = ([j' - i' - 1], \lambda', \leq_1', \leq_2')$ and $\Phi_{\circ}(nw'') = ([n - j'], \lambda'', \leq_1'', \leq_2'')$. We consider three cases:

- (1) Assume $i < i'$ or $i \leq j' < j$. Then $i \leq_2^{\circ} j$ and there is no $(k, \ell) \in \nu$ with $1 \leq k \leq i < j \leq \ell \leq n$.
- (2) Assume $i' \leq i < j \leq j'$. If $i = i'$ or $j = j'$, then $i \geq_2^{\circ} j$ and choosing $(k, \ell) = (i, j)$ gives (k, ℓ) as required since i has nesting depth 1. If $i' < i < j < j'$, then we get:

$$\begin{aligned}i \geq_2^{\circ} j &\iff i - i' \geq_2' j - i' \\ &\iff \text{not } i - i' \leq_2' j - i'\end{aligned}$$

$$\iff \begin{aligned} &\text{either there is some } (k, \ell) \in \nu[i'+1, j'-1] \text{ with } 1 \leq \\ &k \leq i - i' < j - i' \leq \ell \leq j' - i' - 1 \text{ such that there is no} \\ &(k', \ell') \in \nu[i'+1, j'-1] \text{ with } k < k' \leq i - i' < j - i' \leq \ell' < \ell \\ &\text{and } k \text{ has even nesting depth in } nw', \text{ or there is no } (k, \ell) \in \\ &\nu[i'+1, j'-1] \text{ with } 1 \leq k \leq i - i' < j - i' \leq \ell \leq j' - i' - 1 \end{aligned}$$

$$\iff \begin{aligned} &\text{there is some } (k, \ell) \in \nu \text{ with } 1 \leq k \leq i < j \leq \ell \leq n \text{ such} \\ &\text{that there is no } (k', \ell') \in \nu \text{ with } k < k' \leq i < j \leq \ell' < \ell \\ &\text{and } k \text{ has odd nesting depth.} \end{aligned}$$

(3) Assume $j' < i$. Then we get

$$\begin{aligned}
i \geq_2^{\circ} j &\iff i - j' \geq_2'' j - j' \\
&\iff \text{there is some } (k, \ell) \in \nu[j' + 1, n] \text{ with } 1 \leq k \leq i - j' < \\
&\quad j - j' \leq \ell \leq n - j' \text{ such that there is no } (k', \ell') \in \nu[j' + 1, n] \\
&\quad \text{with } k < k' \leq i - j' < j - j' \leq \ell' < \ell \text{ and } k \text{ has odd} \\
&\quad \text{nesting depth} \\
&\iff \text{there is some } (k, \ell) \in \nu \text{ with } 1 \leq k \leq i < j \leq \ell \leq n \text{ such} \\
&\quad \text{that there is no } (k', \ell') \in \nu \text{ with } k < k' \leq i < j \leq \ell' < \ell \\
&\quad \text{and } k \text{ has odd nesting depth.}
\end{aligned}$$

□

Corollary 5.4. *The functions Φ_{\circ} and Φ_{\bullet} are unambiguously FO-definable.*

Proof. We only show that Φ_{\circ} is FO-definable. For Φ_{\bullet} the claim can be shown analogously. We give a 1-copying definition scheme $(\vartheta, \delta, (\varphi_{\text{Lab}_a})_{a \in \Delta}, \varphi_{\leq 1}, \varphi_{\leq 2})$ with four parameters X_1, X_2, Y_1, Y_2 .

Let the macros $\text{call}(x)$ and $\text{return}(x)$ be as in Example 3.7. Moreover, let

$$\text{Frst}_{\nu}(x) = \text{call}(x) \wedge \forall y. \text{call}(y) \rightarrow x \leq y$$

The next macro defines y , the next call or return position following position x .

$$\text{next}_{\nu}(x, y) = x < y \wedge (\text{call}(y) \vee \text{return}(y)) \wedge \forall z. (x < z < y) \rightarrow (\neg \text{call}(z) \wedge \neg \text{return}(z))$$

We now define the formula $\vartheta(X_1, X_2, Y_1, Y_2)$ which for all $nw = (a_1 \dots a_n, \nu) \in \text{NW}(\Delta)$ and $C_1, C_2, R_1, R_2 \subseteq [n]$ has the property that $nw \models \vartheta[C_1, C_2, R_1, R_2]$ iff C_1 is the set of all call positions of odd nesting depth, C_2 is the set of all call positions of even nesting depth, R_1 is the set of all return positions of even nesting depth and R_2 is the set of all return positions of odd nesting depth.

$$\begin{aligned}
\vartheta(X_1, X_2, Y_1, Y_2) = & (X_1 \cap X_2 = \emptyset) \wedge \forall z. (z \in X_1 \vee z \in X_2) \rightarrow \text{call}(z) \\
& \wedge (Y_1 \cap Y_2 = \emptyset) \wedge \forall z. (z \in Y_1 \vee z \in Y_2) \rightarrow \text{return}(z) \\
& \wedge \forall z. \text{Frst}_{\nu}(z) \rightarrow z \in X_1 \\
& \wedge \forall z_1, z_2. ((z_1 \in X_1 \wedge \text{next}_{\nu}(z_1, z_2) \wedge \text{return}(z_2)) \rightarrow z_2 \in Y_1) \\
& \wedge \forall z_1, z_2. ((z_1 \in X_1 \wedge \text{next}_{\nu}(z_1, z_2) \wedge \text{call}(z_2)) \rightarrow z_2 \in X_2) \\
& \wedge \forall z_1, z_2. ((z_1 \in X_2 \wedge \text{next}_{\nu}(z_1, z_2) \wedge \text{return}(z_2)) \rightarrow z_2 \in Y_2) \\
& \wedge \forall z_1, z_2. ((z_1 \in X_2 \wedge \text{next}_{\nu}(z_1, z_2) \wedge \text{call}(z_2)) \rightarrow z_2 \in X_1) \\
& \wedge \forall z_1, z_2. ((z_1 \in Y_1 \wedge \text{next}_{\nu}(z_1, z_2) \wedge \text{return}(z_2)) \rightarrow z_2 \in Y_2) \\
& \wedge \forall z_1, z_2. ((z_1 \in Y_1 \wedge \text{next}_{\nu}(z_1, z_2) \wedge \text{call}(z_2)) \rightarrow z_2 \in X_1) \\
& \wedge \forall z_1, z_2. ((z_1 \in Y_2 \wedge \text{next}_{\nu}(z_1, z_2) \wedge \text{return}(z_2)) \rightarrow z_2 \in Y_1) \\
& \wedge \forall z_1, z_2. ((z_1 \in Y_2 \wedge \text{next}_{\nu}(z_1, z_2) \wedge \text{call}(z_2)) \rightarrow z_2 \in X_2)
\end{aligned}$$

where $X \cap Y = \emptyset$ abbreviates $\neg(\exists z. z \in X \wedge z \in Y)$. We let $\delta(x, X_1, X_2, Y_1, Y_2)$ be some tautology. Now we define the interpreting formulae. We set $\varphi_{\text{Lab}_a}(x, X_1, X_2, Y_1, Y_2) = \text{Lab}_a(x)$ and let $\varphi_{\leq 1}(x, y, X_1, X_2, Y_1, Y_2) = x \leq y$. Furthermore, we define $\varphi_{\circ}(x, y, X_1)$ to

be the following formula which expresses the condition of Lemma 5.3.

$$\begin{aligned} \varphi_o(x, y, X_1) = & \left[x < y \wedge \left(\exists z_1, z_2. (z_1 \leq x \leq y \leq z_2) \wedge \nu(z_1, z_2) \wedge z_1 \in X_1 \wedge \right. \right. \\ & \left. \left. \wedge \forall z'_1, z'_2. (z_1 < z'_1 \leq x \leq y \leq z'_2 < z_2) \rightarrow \neg \nu(z'_1, z'_2) \right) \right] \end{aligned}$$

and let

$$\varphi_{\leq_2}(x, y, X_1) = x = y \vee (y < x \wedge \varphi_o(y, x, X_1)) \vee (x < y \wedge \neg \varphi_o(x, y, X_1)).$$

This completes the definition scheme for Φ_o which is unambiguous. \square

Let $\tau = ([n], \lambda, \leq_1, \leq_2)$ be a text. An interval $[i, j] = \{k \in [n] \mid i \leq_1 k \leq_1 j\}$ of the first order is a *clan* if it is an interval also of the second order. A *prime clan* is a clan that does not overlap with any other, i.e. there is no clan $[k, \ell]$ such that $k <_1 i <_1 \ell <_1 j$ or $i <_1 k <_1 j <_1 \ell$.

Lemma 5.5. *Let $nw = (a_1 \dots a_n, \nu) \in \text{NW}(\Delta)$, let $\Phi_o(nw) = ([n], \lambda, \leq_1^\circ, \leq_2^\circ)$ and let $\Phi_\bullet(nw) = ([n], \lambda, \leq_1^\bullet, \leq_2^\bullet)$. Moreover, let $1 \leq i < j \leq n$.*

Then $(i, j) \in \nu$ iff $[i, j]$ is a prime clan of $\Phi_o(nw)$ and we have either $i \neq 1, j \neq n$ or $1 \geq_2^\circ n$ iff $[i, j]$ is a prime clan of $\Phi_\bullet(nw)$ and we have either $i \neq 1, j \neq n$ or $1 \leq_2^\bullet n$.

Proof. The proof is again by induction on $|\nu|$. If $\nu = \emptyset$, then $[1, n]$ is the only prime clan of both $\Phi_o(nw)$ and $\Phi_\bullet(nw)$ (since any other clan can be overlapped) and we have $1 \leq_2^\circ n$ and $1 \geq_2^\bullet n$. Now let $|\nu| \geq 1$ and let $(i_1, j_1), (i_2, j_2), \dots, (i_t, j_t)$ with $i_1 < i_2 < \dots < i_t$ be the sequence of surface arches (see definition after Def. 2.1). By definition we have

$$\begin{aligned} \Phi_o(nw) = & \Phi_o(nw[1, i_1 - 1]) \circ \Phi_o(nw[i_1, j_1]) \circ \dots \circ \\ & \circ \Phi_o(nw[j_{t-1} + 1, i_t - 1]) \circ \Phi_o(nw[i_t, j_t]) \circ \Phi_o(nw[j_t + 1, n]), \end{aligned}$$

where we ignore a factor if the corresponding interval is empty. We show that $(i, j) \in \nu$ iff $[i, j]$ is a prime clan of $\Phi_o(nw)$ and we have either $i \neq 1, j \neq n$ or $1 \geq_2^\circ n$. That this holds iff $[i, j]$ is a prime clan of $\Phi_\bullet(nw)$ and we have either $i \neq 1, j \neq n$ or $1 \leq_2^\bullet n$ can again be shown analogously.

(Only if). Let $(i, j) \in \nu$. Then there is some r such that $i_r \leq i < j \leq j_r$.

If $i = i_r$ or $j = j_r$, then $i = i_r$ and $j = j_r$. Clearly, $[i_r, j_r]$ is a clan. Suppose for contradiction that there is a clan $[\ell, k]$ overlapping $[i_r, j_r]$. Assume $\ell < i_r < k < j_r$ (the case $i_r < \ell < j_r < k$ is similar). By definition of Φ_o we get $\ell \leq_2^\circ j_r \leq_2^\circ i_r$. Contradiction. Thus $[i_r, j_r]$ is a prime clan. In particular if $i_r = 1$ and $j_r = n$, we get $1 \geq_2^\circ n$.

Otherwise, in case of $i_r < i < j < j_r$, the interval $[i - i_r, j - i_r]$ is a prime clan of $\Phi_\bullet(nw[i_r + 1, j_r - 1])$ by induction hypothesis. Thus, $[i, j]$ must be a clan, since $[i_r, j_r]$ is a clan, too. Suppose for contradiction that there is a clan $[\ell, k]$ overlapping $[i, j]$. As $[i - i_r, j - i_r]$ is a prime clan of $\Phi_\bullet(nw[i_r + 1, j_r - 1])$ we get either $\ell \leq i_r$ or $k \geq j_r$. Assume $\ell \leq i_r$ (the other case is similar). Now, if $\ell < i_r$, we can argue as above and separate ℓ and i_r . Contradiction. If $i_r = \ell$ and $i_r + 1 < i$, then $[1, k - i_r]$ is a clan in $\Phi_\bullet(nw[i_r + 1, j_r - 1])$ which overlaps $[i - i_r, j - i_r]$. Contradiction. And if $\ell = i_r$ and $i_r + 1 = i$, we get by definition $i \leq_2^\circ j \leq_2^\circ i_r$. Again contradiction. Thus $[i, j]$ must be a prime clan.

(If). Let $[i, j]$ be a prime clan such that not $i = 1, j = n$ and $1 \leq_2^\circ n$. If $i = 1$ and $j = n$, then $1 \geq_2^\circ n$ and $(i, j) \in \nu$ by definition of Φ_o . Now suppose $1 < i$ or $j < n$. The following intervals (provided they exist) can easily be seen to be clans: $[1, i_1 - 1]$, $[i_1, j_1]$, $[j_1 + 1, n]$, $[1, j_1]$ and $[\ell, n]$ for any $\ell \leq i_1$. From this we conclude that either $i_1 \leq i < j \leq j_1$ or $j_1 < i$ since otherwise one of the clans above would overlap $[i, j]$. If $i = i_1$ or $j = j_1$ then $i = i_1$ and

$j = j_1$, since $[i_1, j_1 - 1]$ and $[i_1 + 1, j_1]$ are clans, and hence $(i, j) \in \nu$. In the case where $i_1 < i < j < j_1$, we get that $[i - i_1, j - i_1]$ must be a prime clan of $\Phi_\bullet(nw[i_1 + 1, j_1 - 1])$ and if $j_1 < i$, we get that $[i - j_1, j - j_1]$ must be a prime clan of $\Phi_\circ(nw[j_1 + 1, n])$. Hence, in both cases $(i, j) \in \nu$ by induction hypothesis. \square

It is not hard to see that the domains of the partial functions Φ_\circ^{-1} and Φ_\bullet^{-1} are FO-definable. Hence, by the last lemma there is a definition scheme without parameters consisting of FO-formulae which defines Φ_\circ^{-1} (or alternatively Φ_\bullet^{-1}).

Corollary 5.6. *The partial functions Φ_\circ^{-1} and Φ_\bullet^{-1} are unambiguously FO-definable.*

So far we have seen that we can translate a formula over nested words into a formula over texts (and vice versa) such that the formulae correspond to each other with respect to Φ_\circ resp. Φ_\bullet . We will now show that also WPA can simulate WNWA (and vice versa) with respect to Φ_\circ resp. Φ_\bullet .

Proposition 5.7. *Let $S : \text{TXT}(\Delta) \rightarrow \mathbb{K}$ be regular. Then $\Phi_\circ^{-1}(S), \Phi_\bullet^{-1}(S) : \text{NW}(\Delta) \rightarrow \mathbb{K}$ are regular.*

Proof. We show that $\Phi_\circ^{-1}(S)$ is regular. Analogously one can show that $\Phi_\bullet^{-1}(S)$ is regular. Let $\mathcal{P} = (\mathcal{H}, \mathcal{V}, \Omega, \mu, \mu_{\text{op}}, \mu_{\text{cl}}, \lambda, \gamma)$ be a WPA such that $\|\mathcal{P}\| = S$. We construct a WNWA $\mathcal{A} = (Q, \iota, \delta, \kappa)$ with state space $Q = (\mathcal{H} \uplus \mathcal{V}) \times (\Omega \uplus \{i\})$ such that for all $h_0, h_n \in \mathcal{H}$, $v_0, v_n \in \mathcal{V}$ and $\omega \in \Omega \uplus \{i\}$ we have

$$\sum_{r:(h_0, \omega) \xrightarrow{nw} (h_n, \omega)} \text{wgt}_{\mathcal{A}}(r) = \sum_{r:h_0 \xrightarrow{\Phi_\circ(nw)} h_n} \text{wgt}_{\mathcal{P}}(r) \quad \text{and} \quad \sum_{r:(v_0, \omega) \xrightarrow{nw} (v_n, \omega)} \text{wgt}_{\mathcal{A}}(r) = \sum_{r:v_0 \xrightarrow{\Phi_\bullet(nw)} v_n} \text{wgt}_{\mathcal{P}}(r). \quad (5.1)$$

Intuitively, in the first component one simulates the states of the WPA and in the second component one stores the most recent open bracket. This has to be updated when reading a return position using the look-back ability of the WNWA. We give now the formal definition of the transition functions. We give it only on certain subsets of their domains. In all other cases we set the values to 0. Let $a \in \Delta$, $h_1, h_2 \in \mathcal{H}$, $v_1, v_2 \in \mathcal{V}$, $\omega_1 \in \Omega \uplus \{i\}$ and $\omega_2 \in \Omega$. Define

$$\begin{aligned} \delta_{\text{int}}((h_1, \omega_1), a, (h_2, \omega_1)) &= \mu(h_1, a, h_2) \\ \delta_{\text{int}}((v_1, \omega_1), a, (v_2, \omega_1)) &= \mu(v_1, a, v_2) \\ \delta_{\text{call}}((h_1, \omega_1), a, (v_1, \omega_2)) &= \sum_{v \in \mathcal{V}} \mu_{\text{op}}(h_1, (\omega_2, v)) \cdot \mu(v, a, v_1) \\ \delta_{\text{call}}((v_1, \omega_1), a, (h_1, \omega_2)) &= \sum_{h \in \mathcal{H}} \mu_{\text{op}}(v_1, (\omega_2, h)) \cdot \mu(h, a, h_1) \\ \delta_{\text{ret}}((h_1, \omega_2), (v_1, \omega_1), a, (v_2, \omega_1)) &= \sum_{h \in \mathcal{H}} \mu(h_1, a, h) \cdot \mu_{\text{cl}}(h,)_{\omega_2, v_2} \\ \delta_{\text{ret}}((v_1, \omega_2), (h_1, \omega_1), a, (h_2, \omega_1)) &= \sum_{v \in \mathcal{V}} \mu(v_1, a, v) \cdot \mu_{\text{cl}}(v,)_{\omega_2, h_2}. \end{aligned}$$

Observe that for any $nw \in \text{NW}(\Delta)$ and any run $r : q_0 \xrightarrow{nw} q_n$ of \mathcal{A} such that $\text{wgt}_{\mathcal{A}}(r) \neq 0$ the second components of q_0 and q_n coincide and the first components are either both in \mathcal{H} or both in \mathcal{V} .

Let $nw = (a_1 \dots a_n, \nu)$. We show Equation 5.1 by induction on $|\nu|$. First let $\nu = \emptyset$. Then for all $h_0, h_n \in \mathcal{H}$ and $\omega \in \Omega \uplus \{i\}$ we have

$$\begin{aligned} \sum_{r:(h_0, \omega) \xrightarrow{nw} (h_n, \omega)} \text{wgt}_{\mathcal{A}}(r) &= \sum_{h_1, \dots, h_{n-1} \in \mathcal{H}} \prod_{j=1}^n \delta_{\text{int}}((h_{j-1}, \omega), a_j, (h_j, \omega)) = \\ &= \sum_{h_1, \dots, h_{n-1} \in \mathcal{H}} \prod_{j=1}^n \mu(h_{j-1}, a_j, h_j) = \sum_{r:h_0 \xrightarrow{\Phi_{\circ}(nw)} h_n} \text{wgt}_{\mathcal{P}}(r). \end{aligned}$$

Similarly we get the claim for Φ_{\bullet} . Now, let $\nu \neq \emptyset$, let k be the minimal call position and let ℓ be the corresponding return position. Let $nw_1 = nw[1, k-1]$, $nw_2 = nw[k+1, \ell-1]$ and $nw_3 = nw[\ell+1, n]$ (we assume that all nested words exist, the cases where they do not exist are similar). Then for all $h_0, h_n \in \mathcal{H}$ and $\omega \in \Omega \uplus \{i\}$ we have

$$\begin{aligned} \sum_{r:(h_0, \omega) \xrightarrow{nw} (h_n, \omega)} \text{wgt}_{\mathcal{A}}(r) &= \\ &= \sum_{\substack{h_{k-1}, h_{\ell} \in \mathcal{H} \\ v_k, v_{\ell-1} \in \mathcal{V} \\ \omega_1 \in \Omega}} \sum_{r_1:(h_0, \omega) \xrightarrow{nw_1} (h_{k-1}, \omega)} \text{wgt}_{\mathcal{A}}(r_1) \cdot \delta_{\text{call}}((h_{k-1}, \omega), a_k, (v_k, \omega_1)) \cdot \\ &\quad \cdot \sum_{r_2:(v_k, \omega_1) \xrightarrow{nw_2} (v_{\ell-1}, \omega_1)} \text{wgt}_{\mathcal{A}}(r_2) \cdot \delta_{\text{ret}}((v_{\ell-1}, \omega_1), (h_{k-1}, \omega), a_{\ell}, (h_{\ell}, \omega)) \cdot \sum_{r_3:(h_{\ell}, \omega) \xrightarrow{nw_3} (h_n, \omega)} \text{wgt}_{\mathcal{A}}(r_3) \\ &= \sum_{\substack{h_{k-1}, h_{\ell} \in \mathcal{H} \\ v_k, v_{\ell-1} \in \mathcal{V} \\ \omega_1 \in \Omega}} \sum_{r_1:h_0 \xrightarrow{a_1 \circ \dots \circ a_{k-1}} h_{k-1}} \text{wgt}_{\mathcal{P}}(r_1) \cdot \sum_{v \in \mathcal{V}} \mu_{\text{op}}(h_{k-1}, (\omega_1, v)) \cdot \mu(v, a_k, v_k) \cdot \\ &\quad \cdot \sum_{r_2:v_k \xrightarrow{\Phi_{\bullet}(nw_2)} v_{\ell-1}} \text{wgt}_{\mathcal{P}}(r_2) \cdot \sum_{v' \in \mathcal{V}} \mu(v_{\ell-1}, a_{\ell}, v') \cdot \mu_{\text{cl}}(v', \omega_1, h_{\ell}) \cdot \sum_{r_3:h_{\ell} \xrightarrow{\Phi_{\circ}(nw_3)} h_n} \text{wgt}_{\mathcal{P}}(r_3) \\ &= \sum_{r:h_0 \xrightarrow{\Phi_{\circ}(nw)} h_n} \text{wgt}_{\mathcal{P}}(r). \end{aligned}$$

Again, the claim is shown similarly for Φ_{\bullet} . This concludes the proof of Equation (5.1).

Now consider the WNWA with states $Q' = \{\perp, ?, s, \circ, \bullet\}$ and transition functions $\delta'_{\text{call}}, \delta'_{\text{int}}, \delta'_{\text{ret}}$ given for all $a \in \Delta$ and $p \in Q' \setminus \{\perp\}$ by

$$\begin{aligned} \delta'_{\text{call}}(\perp, a, ?) &= \delta'_{\text{int}}(\perp, a, s) = \delta'_{\text{call}}(s, a, \circ) = \delta'_{\text{int}}(s, a, \circ) = \delta'_{\text{call}}(?, a, ?) = \delta'_{\text{int}}(?, a, ?) = \\ &= \delta'_{\text{ret}}(?, p, a, ?) = \delta'_{\text{ret}}(?, \perp, a, \bullet) = \delta'_{\text{call}}(\bullet, a, \circ) = \delta'_{\text{int}}(\bullet, a, \circ) = \delta'_{\text{call}}(\circ, a, \circ) = \\ &= \delta'_{\text{int}}(\circ, a, \circ) = \delta'_{\text{ret}}(\circ, p, a, \circ) = 1. \end{aligned}$$

Set any other values of $\delta'_{\text{call}}, \delta'_{\text{int}}, \delta'_{\text{ret}}$ to 0 and let the initial distribution ι' be given by $\iota'(q') = 1$ if $q' = \perp$ and 0 otherwise. Observe that in the case where the final distribution κ' is given by $\kappa'(q') = 1$ if $q' = \circ$ and 0 otherwise, the behavior of the automaton is the characteristic series of the set of nested words nw such that $\Phi_{\circ}(nw)$ is a \circ -product. We collect such nested words in NW° . In the case where the final distribution κ' is given by $\kappa'(q') = 1$ if $q' = \bullet$ and 0 otherwise, the behavior of the automaton is the characteristic series of the set of nested words nw such that $\Phi_{\bullet}(nw)$ is a \bullet -product. We collect such nested

words in NW^\bullet . Finally, in the case where the final distribution κ' is given by $\kappa'(q') = 1$ if $q' = s$ and 0 otherwise, the behavior of the automaton is the characteristic series of the set of all singleton nested words, i.e. Δ .

Now consider the product of this automaton with \mathcal{A} which has states $Q \times Q'$ and whose transition functions $\delta_{\text{call}}^\times, \delta_{\text{int}}^\times, \delta_{\text{ret}}^\times$ is given by letting $\delta_{\text{call}}^\times((q, q'), a, (p, p')) = \delta_{\text{call}}(q, a, p) \cdot \delta(q', a, p')$ for all $q, p \in Q$ and $q', p' \in Q'$. If we define the initial and final distribution ι^\times and κ^\times by letting for all $h \in \mathcal{H}$ and $\omega \in \Omega$

$$\begin{aligned} \iota^\times((h, i), \perp) &= \lambda(h) & \iota((h, \omega), \perp) &= \sum_{v \in \mathcal{V}} \lambda(v) \cdot \mu_{\text{op}}(v, (\omega, h)) \\ \kappa((h, i), \circ) &= \gamma(h) & \kappa((h, i), \circ) &= \sum_{v \in \mathcal{V}} \mu_{\text{cl}}(h,)_{\omega}, v) \cdot \gamma(v), \end{aligned}$$

and in any other case by setting the value to 0, then the behavior of the resulting automaton is $\mathbb{1}_{\text{NW}^\circ} \odot \Phi_\circ^{-1}(S)$. Changing the definitions of $\iota^\times, \kappa^\times$ appropriately gives automata with behavior $\mathbb{1}_{\text{NW}^\bullet} \odot \Phi_\circ^{-1}(S)$ and $\mathbb{1}_\Delta \odot \Phi_\circ^{-1}(S)$. The automaton obtained from disjoint copies of these three automata has hence the behavior $\Phi_\circ^{-1}(S)$. \square

Proposition 5.8. *Let $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ be a regular series. Then $\Phi_\circ(S), \Phi_\bullet(S) : \text{TXT}(\Delta) \rightarrow \mathbb{K}$ are regular.*

Proof. Let $\mathcal{A} = (Q, \iota, \delta, \kappa)$ be a WNWA. We define a WPA $\mathcal{P} = (\mathcal{H}, \mathcal{V}, \Omega, \mu, \mu_{\text{op}}, \mu_{\text{cl}}, \lambda, \gamma)$ with

$$\mathcal{H} = \{q^{\mathcal{H}} \mid q \in Q\} \times (\{c, i\} \uplus \Delta) \quad \text{and} \quad \mathcal{V} = \{q^{\mathcal{V}} \mid q \in Q\} \times (\{c, i\} \uplus \Delta)$$

as well as $\Omega = Q$ such that $(\|\mathcal{P}\|, \Phi_\circ(nw)) = (\|\mathcal{A}\|, nw)$ for all $nw \in \text{NW}(\Delta)$. To prove the result for Φ_\bullet only λ and γ have to be changed.

Intuitively, in the first component one simulates the states of the WNWA, in the second component one either selects whether the next transition is a call or an internal transition, or one stores the letter to simulate a return position with the next bracket. Look-back behavior is simulated by storing a state in the opening bracket and closing it at the appropriate return position.

We formally define $\mu, \mu_{\text{op}}, \mu_{\text{cl}}$ as follows. We give the definition only on certain subsets of their domains. In all other cases we set their values to 0.

$$\begin{aligned} \mu((q_1^{\mathcal{H}}, i), a, (q_2^{\mathcal{H}}, i)) &= \delta_{\text{int}}(q_1, a, q_2) & \mu((q_1^{\mathcal{V}}, i), a, (q_2^{\mathcal{V}}, i)) &= \delta_{\text{int}}(q_1, a, q_2) \\ \mu((q_1^{\mathcal{H}}, c), a, (q_2^{\mathcal{H}}, i)) &= \delta_{\text{call}}(q_1, a, q_2) & \mu((q_1^{\mathcal{V}}, c), a, (q_2^{\mathcal{V}}, i)) &= \delta_{\text{call}}(q_1, a, q_2) \\ \mu((q_1^{\mathcal{H}}, i), a, (q_1^{\mathcal{H}}, a)) &= 1 & \mu((q_1^{\mathcal{V}}, i), a, (q_1^{\mathcal{V}}, a)) &= 1 \\ \mu_{\text{op}}((q_1^{\mathcal{H}}, i), (q_1, (q_1^{\mathcal{V}}, c))) &= 1 & \mu_{\text{op}}((q_1^{\mathcal{V}}, i), (q_1, (q_1^{\mathcal{H}}, c))) &= 1 \\ \mu_{\text{cl}}((q_1^{\mathcal{H}}, a),)_{q_2}, (q_3^{\mathcal{V}}, i)) &= \delta_{\text{ret}}((q_1, q_2, a, q_3)) & \mu_{\text{cl}}((q_1^{\mathcal{V}}, a),)_{q_2}, (q_3^{\mathcal{H}}, i)) &= \delta_{\text{ret}}((q_1, q_2, a, q_3)) \\ \lambda(q_1^{\mathcal{H}}, i) &= \iota(q_1) & \gamma(q_1^{\mathcal{H}}, i) &= \gamma(q_1) \end{aligned}$$

We use induction on $nw = (a_1 \dots a_n, \nu) \in \text{NW}(\Delta)$ to show that the defined WPA behaves as required. More precisely we show that for all $q_1, q_2 \in Q$

$$\sum_{r: (q_1^{\mathcal{H}}, i) \xrightarrow{\Phi_\circ(nw)} (q_2^{\mathcal{H}}, i)} \text{wgt}_{\mathcal{P}}(r) = \sum_{r: q_1 \xrightarrow{nw} q_2} \text{wgt}_{\mathcal{A}}(r) = \sum_{r: (q_1^{\mathcal{V}}, i) \xrightarrow{\Phi_\bullet(nw)} (q_2^{\mathcal{V}}, i)} \text{wgt}_{\mathcal{P}}(r).$$

This is easy to see if $\nu = \emptyset$. Let $\nu \neq \emptyset$ and let k be the minimal call position and let ℓ be the corresponding return position. Let $nw_1 = nw[1, k - 1]$, $nw_2 = nw[k + 1, \ell - 1]$ and $nw_3 = nw[\ell + 1, n]$ (we assume that all nested words exist, the cases where they do not exist are similar). Then

$$\begin{aligned}
 & \sum_{r: (q_1^{\mathcal{H}}, i) \xrightarrow{\Phi_\circ(nw)} (q_2^{\mathcal{H}}, i)} \text{wgt}_{\mathcal{P}}(r) = \\
 &= \sum_{q_3, q_4, q_5, q_6 \in Q} \sum_{r_1: (q_1^{\mathcal{H}}, i) \xrightarrow{a_1 \circ \dots \circ a_{k-1}} (q_3^{\mathcal{H}}, i)} \text{wgt}_{\mathcal{P}}(r_1) \cdot \mu_{\text{op}}((q_3^{\mathcal{H}}, i), (q_3, (q_3^{\mathcal{V}}, c))) \cdot \mu((q_3^{\mathcal{V}}, c), a_k, (q_4^{\mathcal{V}}, i)) \cdot \\
 & \quad \cdot \sum_{r_2: (q_4^{\mathcal{V}}, i) \xrightarrow{\Phi_\bullet(nw_2)} (q_5^{\mathcal{V}}, i)} \text{wgt}_{\mathcal{P}}(r_2) \cdot \mu((q_5^{\mathcal{V}}, i), a_\ell, (q_5, a_\ell)) \cdot \mu_{\text{cl}}((q_5^{\mathcal{V}}, a_\ell),)_{q_3, (q_6^{\mathcal{H}}, i)} \cdot \\
 & \quad \cdot \sum_{r_3: (q_6^{\mathcal{H}}, i) \xrightarrow{\Phi_\circ(nw_3)} (q_2^{\mathcal{H}}, i)} \text{wgt}_{\mathcal{P}}(r_3) \\
 &= \sum_{q_3, q_4, q_5, q_6 \in Q} \sum_{r_1: q_1 \xrightarrow{nw_1} q_3} \text{wgt}_{\mathcal{A}}(r_1) \cdot \delta_{\text{call}}(q_3, a_k, q_4) \cdot \sum_{r_2: q_4 \xrightarrow{nw_2} q_5} \text{wgt}_{\mathcal{A}}(r_2) \cdot \delta_{\text{ret}}(q_5, q_3, a_\ell, q_6) \cdot \\
 & \quad \cdot \sum_{r_3: q_6 \xrightarrow{nw_3} q_2} \text{wgt}_{\mathcal{A}}(r_3) \\
 &= \sum_{r: q_1 \xrightarrow{nw} q_2} \text{wgt}_{\mathcal{A}}(r).
 \end{aligned}$$

We can proceed analogously for Φ_\bullet . Now the result follows from the definition of λ and γ . \square

We can now prove Theorem 3.6

Proof of Theorem 3.6. We prove Theorem 3.6(a). Let $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ be regular. By Proposition 5.8, $\Phi_\circ(S) : \text{TXT}(\Delta) \rightarrow \mathbb{K}$ is regular and hence sREMSO(\mathbb{K})-definable by Theorem 4.2. Now we get that $\Phi_\circ^{-1}(\Phi_\circ(S)) = S$ is sREMSO(\mathbb{K})-definable by Proposition 5.2 and Corollary 5.4.

Conversely, let $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ be sRMSO(\mathbb{K})-definable. By Corollary 5.6 and Proposition 5.2, $\Phi_\circ(S) : \text{TXT}(\Delta) \rightarrow \mathbb{K}$ is sRMSO(\mathbb{K})-definable and thus by Theorem 4.2 regular. From Proposition 5.7 we conclude that $\Phi_\circ^{-1}(\Phi_\circ(S)) = S$ is regular, too.

Similarly we get Theorem 3.6(b) from Theorem 4.2(b). Theorem 3.6(c) follows from Theorem 4.2(c). \square

Again note that all proofs are constructive. Hence, given a sentence φ in sRMSO(\mathbb{K}) (resp. swRMSO(\mathbb{K}), MSO(\mathbb{K})) we can effectively construct a WNWA \mathcal{A} such that $\|\mathcal{A}\| = \llbracket \varphi \rrbracket$. Conversely, given a WNWA \mathcal{A} we can construct an sREMSO(\mathbb{K}) sentence φ such that $\|\mathcal{A}\| = \llbracket \varphi \rrbracket$. The following results follow now easily from the corresponding results for series over alternating texts [30].

Corollary 5.9. *Let \mathbb{K} be a locally finite semiring or let \mathbb{K} be a ring and let $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ be regular such that $S(\text{NW}(\Delta)) \subseteq \mathbb{K}$ is finite. Moreover, let $A \subseteq \mathbb{K}$. Then $S^{-1}(A)$ is regular.*

Corollary 5.10. *Let \mathbb{K} be a computable field or a computable locally finite semiring and let $S_1, S_2 : \text{NW}(\Delta) \rightarrow \mathbb{K}$ be regular. It is decidable whether $S_1 = S_2$.*

Corollary 5.11. *Let \mathbb{K} be a computable zero-sum free semiring and let $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ be regular. It is decidable whether $(S, nw) = 0$ for all $nw \in \text{NW}(\Delta)$.*

Note that one motivation of transforming formulae in automata is solving their satisfiability problem. The last two corollaries can be seen as an extension of this: We have shown that given a formula in $\varphi \in \text{sRMSO}(\mathbb{K})$ (resp. $\varphi \in \text{swRMSO}(\mathbb{K})$, resp. $\varphi \in \text{MSO}(\mathbb{K})$) we can effectively translate it into a weighted nested word automaton \mathcal{A} . Now, provided the semiring is either zero-sum free or locally finite or a field, using the last two corollaries we can test whether there is a nested word nw which gets assigned a non-zero value, i.e. $(\|\mathcal{A}\|, nw) = (\|\varphi\|, nw) \neq 0$.

6. AN APPLICATION TO ALGEBRAIC FORMAL POWER SERIES

In this section we consider algebraic formal power series and show that they arise as the projections of regular nested word series and regular alternating text series. Applying then our logical characterizations of the latter we obtain characterizations of algebraic formal power series in terms of weighted logics generalizing results of Lautemann, Schwentick and Thérien [27] on context-free languages. Algebraic formal power series have been considered initially already by Chomsky and Schützenberger [8] and have since been intensively studied by Kuich and others. Textbooks containing several aspects of algebraic formal power series are [37] and [26]. The reader is also referred to the survey articles [25] and [35].

Let Δ^* be the free monoid over Δ and let ε denote the empty word. A formal power series is a function $S : \Delta^* \rightarrow \mathbb{K}$. We denote the empty word by ε . Given two formal power series S_1, S_2 , their *Cauchy product*, denoted $S_1 \cdot S_2$ or $S_1 S_2$, is given by $(S_1 \cdot S_2, w) = \sum_{w_1 w_2 = w} (S_1, w_1)(S_2, w_2)$ for all $w \in \Delta^*$. By $S_1 \odot S_2$ we denote the pointwise product also called the *Hadamard product* and by $S_1 + S_2$ their pointwise sum. Moreover, if $k \in \mathbb{K}$, then the formal power series $k.S$ is given by $(k.S, w) = k \cdot (S, w)$ for all $w \in \Delta^*$. Let $\mathbb{1}_L$ denote the characteristic series of a language $L \subseteq \Delta^*$. We identify w and $\mathbb{1}_{\{w\}}$. Let \mathcal{X} be an alphabet of variables such that $\Delta \cap \mathcal{X} = \emptyset$. A polynomial P over $(\Delta \cup \mathcal{X})$ is a mapping $P : (\Delta \cup \mathcal{X})^* \rightarrow \mathbb{K}$ such that its support is finite, i.e. the set $\text{supp}(P) = \{w \in (\Delta \cup \mathcal{X})^* \mid (P, w) \neq 0\}$ is finite.

Definition 6.1. A collection of polynomials $(P_X)_{X \in \mathcal{X}}$ over $(\Delta \cup \mathcal{X})$ is called an *algebraic system* with variables in \mathcal{X} .

The supports of the polynomials P_X in the last definition are thus finite sets consisting of words of the form $u_1 X_1 \dots u_k X_k u_{k+1}$ where $u_j \in \Delta^*$ and $X_j \in \mathcal{X}$. We say that a collection $(S_X)_{X \in \mathcal{X}}$ of formal power series $S_X : \Delta^* \rightarrow \mathbb{K}$ is a *solution* of the algebraic system $(P_X)_{X \in \mathcal{X}}$ if for all $X \in \mathcal{X}$,

$$S_X = \sum_{u_1 X_1 \dots u_k X_k u_{k+1} \in \text{supp}(P_X)} (P_X, u_1 X_1 \dots u_k X_k u_{k+1}) \cdot u_1 S_{X_1} \cdots u_k S_{X_k} u_{k+1}.$$

An algebraic system $(P_X)_{X \in \mathcal{X}}$ is *proper* if $(P_X, Y) = (P_X, \varepsilon) = 0$ for all $X, Y \in \mathcal{X}$. A formal power series S having the property that $(S, \varepsilon) = 0$ is called *quasiregular*. A proper algebraic system has a unique quasiregular solution [37], more precisely a proper algebraic system has exactly one solution $(S_X)_{X \in \mathcal{X}}$ such that $(S_X, \varepsilon) = 0$ for all $X \in \mathcal{X}$.

Definition 6.2. A formal power series $S : \Delta^* \rightarrow \mathbb{K}$ is an *algebraic formal power series* if it is a component of the quasiregular solution of a proper algebraic system.²

² This definition is given in [37]. In [25, 26] a series S is called algebraic if its quasiregular part $\mathbb{1}_{\Delta^+} \odot S$ is the component of the quasiregular solution of a proper algebraic system.

We note that over the 2-valued Boolean algebra \mathbb{B} these series correspond exactly to the ε -free context-free languages. The bijection is given by supp .

To warm up let us discuss some easy manipulations of algebraic systems. For this, let us consider some algebraic system $(P_X)_{X \in \mathcal{X}}$. Let $X, Y \in \mathcal{X}$. Clearly, it follows directly from the definition of a solution that we can substitute an occurrence of Y in some word of the support of P_X by P_Y without altering the solutions of the system. More formally: Let $uYv \in \text{supp}(P_X)$. Let $(P'_X)_{X \in \mathcal{X}}$ be given from $(P_X)_{X \in \mathcal{X}}$ by replacing P_X with the polynomial

$$\left(\mathbb{1}_{\text{supp}(P_X) \setminus \{uYv\}} \odot P_X \right) + (P_X, uYv).uP_Yv.$$

Then $(P_X)_{X \in \mathcal{X}}$ and $(P'_X)_{X \in \mathcal{X}}$ are *equivalent*, i.e. any solution of $(P_X)_{X \in \mathcal{X}}$ is a solution of $(P'_X)_{X \in \mathcal{X}}$ and vice versa. An algebraic system $(P_X)_{X \in \mathcal{X}}$ is called *weakly strict*, if $\text{supp}(P_X) \subseteq \{\varepsilon\} \cup \Delta(\Delta \cup \mathcal{X})^*$ for all $X \in \mathcal{X}$. Let us now assume that $(P_X)_{X \in \mathcal{X}}$ is weakly strict. Then for any fixed $k \in \mathbb{N}$ by repeated substitution we can obtain an equivalent algebraic system $(P_X^k)_{X \in \mathcal{X}}$ such that for all $X \in \mathcal{X}$ any $w \in \text{supp}(P_X^k) \setminus \Delta^*$ contains at least k letters from Δ . We conclude that any weakly strict algebraic system $(P_X)_{X \in \mathcal{X}}$ has a unique solution $(S_X)_{X \in \mathcal{X}}$ which is given by $(S_X, w) = (P_X^k, w)$ for all $w \in \Delta^*$ such that $|w| < k$.

Now, we continue by manipulating $(P_X^k)_{X \in \mathcal{X}}$. Let again $X \in \mathcal{X}$ and let $w \in \text{supp}(P_X^k)$ with $|w| < k$. Let $Y \in \mathcal{X} \setminus \{X\}$. For any possible choice of occurrences of X in the support of P_Y we substitute these occurrences by w . More precisely, for all $Y \in \mathcal{X} \setminus \{X\}$ replace P_Y^k by the polynomial

$$\sum_{\substack{i \in \mathbb{N} \\ u_1, u_2, \dots, u_i, u_{i+1} \in (\Delta \cup \mathcal{X})^*}} (P_Y^k, u_1 X u_2 \dots u_i X u_{i+1}).u_1 \cdot (P_X^k, w).w \cdot u_2 \cdots u_i \cdot (P_X^k, w).w \cdot u_{i+1}.$$

Furthermore, replace P_X^k by the polynomial

$$\mathbb{1}_{(\Delta \cup \mathcal{X})^* \setminus \{w\}} \odot \sum_{\substack{i \in \mathbb{N} \\ u_1, u_2, \dots, u_i, u_{i+1} \in (\Delta \cup \mathcal{X})^*}} (P_X^k, u_1 X u_2 \dots u_i X u_{i+1}).u_1 \cdot (P_X^k, w).w \cdot u_2 \cdots u_i \cdot (P_X^k, w).w \cdot u_{i+1}.$$

Observe that these sums are in fact finite and note that in these definitions the factors $u_1, u_2, \dots, u_i, u_{i+1} \in (\Delta \cup \mathcal{X})^*$ may contain occurrences of X . The resulting system is again weakly strict and has thus a unique solution $(S'_X)_{X \in \mathcal{X}}$. A straightforward but cumbersome calculation, which we omit here, shows, using the distributivity of the semiring of formal power series, that $S'_Y = S_Y$ for all $Y \in \mathcal{X} \setminus \{X\}$ and $S'_X = \mathbb{1}_{(\Delta \cup \mathcal{X})^* \setminus \{w\}} \odot S_X$. For fixed $0 \leq k' < k$ by repeated application we can thus obtain a proper and weakly strict algebraic system $(R_X)_{X \in \mathcal{X}}$ such that the quasiregular and unique solution $(T_X)_{X \in \mathcal{X}}$ is given by $(\mathbb{1}_{\{w \mid k' < |w|\}} \odot S_X)_{X \in \mathcal{X}}$. In particular, it follows that the quasiregular part $\mathbb{1}_{\Delta^+} \odot S_X$ of S_X is algebraic for any $X \in \mathcal{X}$.

6.1. Nested Word Series and Their Projections. Next, we consider the projections of regular nested word series and show that they give rise exactly to the algebraic series. The projection $\pi(nw)$ of a nested word $nw = (w, \nu) \in \text{NW}(\Delta)$ is simply the word w ,

i.e. we forget the nesting relation. This projection is canonically generalized to languages $L \subseteq \text{NW}(\Delta)$ by setting $\pi(L) = \{\pi(nw) \mid nw \in L\}$ and to series $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ by letting

$$\begin{aligned} \pi(S) : \Delta^* &\rightarrow \mathbb{K} \\ w &\mapsto \sum_{\substack{nw \in \text{NW}(\Delta) \\ w = \pi(nw)}} (S, nw). \end{aligned}$$

Proposition 6.3. *Let $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ be regular. Then $\pi(S) : \Delta^* \rightarrow \mathbb{K}$ is an algebraic formal power series.*

Proof. Let $\mathcal{A} = (Q, \iota, \delta, \kappa)$ be a WNWA such that $\|\mathcal{A}\| = S$. We define a weakly strict algebraic system $(P_{(q_1, q_2)})_{q_1, q_2 \in Q}$ with variables in Q^2 such that for its solution $(S_{(q_1, q_2)})_{q_1, q_2 \in Q}$ we have for all $w \in \Delta^*$ with $|w| \geq 1$:

$$(S_{(q_1, q_2)}, w) = \sum_{\substack{nw \in \text{NW}(\Delta) \\ \pi(nw) = w}} \sum_{r: q_1 \xrightarrow{nw} q_2} \text{wgt}_{\mathcal{A}}(r). \quad (6.1)$$

The idea is to simulate the transitions of a weighted nested word automaton. For this we will partition the set of nested words of length at least two in three different classes. First the class of nested words where the first and the last position are either corresponding call and return positions or both internal positions. The second class consists of nested words where either the first position is a call position and the last position is an internal position or the last position is a return position and the first position is an internal position. And the last class consists of any other, i.e. where the first position is a call position and the last position is a return position which do not correspond to each other. Using this partition we define for all $q_1, q_2 \in Q$ the polynomial $P_{(q_1, q_2)} : (\Delta \cup Q^2)^* \rightarrow \mathbb{K}$ as follows:

$$(P_{(q_1, q_2)}, w) = \begin{cases} 1 & \text{if } q_1 = q_2 \text{ and } w = \varepsilon \\ \delta_{\text{int}}(q_1, a, q_2) & \text{if } w = a \text{ for some } a \in \Delta \\ \delta_{\text{int}}(q_1, a, q_3) \cdot \delta_{\text{int}}(q_4, b, q_2) + & \text{if } w = a(q_3, q_4)b \\ \delta_{\text{call}}(q_1, a, q_3) \cdot \delta_{\text{ret}}(q_4, q_1, b, q_2) & \text{for some } a, b \in \Delta, q_3, q_4 \in Q \\ \delta_{\text{call}}(q_1, a, q_3) \cdot \delta_{\text{ret}}(q_4, q_1, b, q_5) \cdot \delta_{\text{int}}(q_6, c, q_2) + & \text{if } w = a(q_3, q_4)b(q_5, q_6)c \\ \delta_{\text{int}}(q_1, a, q_3) \cdot \delta_{\text{call}}(q_4, b, q_5) \cdot \delta_{\text{ret}}(q_6, q_4, c, q_2) & \text{for some } a, b, c \in \Delta \\ & \text{and } q_3, q_4, q_5, q_6 \in Q \\ \delta_{\text{call}}(q_1, a, q_3) \cdot \delta_{\text{ret}}(q_4, q_1, b, q_5) \cdot & \text{if } w = a(q_3, q_4)b(q_5, q_6)c(q_7, q_8)d \\ \delta_{\text{call}}(q_6, c, q_7) \cdot \delta_{\text{ret}}(q_8, q_6, d, q_2) & \text{for some } a, b, c, d \in \Delta \\ & \text{and } q_3, q_4, q_5, q_6, q_7, q_8 \in Q \\ 0 & \text{otherwise.} \end{cases}$$

This is a weakly strict algebraic system having a necessarily unique solution $(S_{(q_1, q_2)})_{q_1, q_2 \in Q}$. We show by induction on the length of w that (6.1) holds. For $|w| = 1$ this is easy to see. Now let $|w| > 1$. Then

$$\begin{aligned}
& (S_{(q_1, q_2)}, w) = \\
& = \sum_{q_3, q_4} \in Q \left[\delta_{\text{int}}(q_1, a_1, q_3) \cdot \delta_{\text{int}}(q_4, a_n, q_2) + \delta_{\text{call}}(q_1, a_1, q_3) \cdot \delta_{\text{ret}}(q_4, q_1, a_n, q_2) \right] \\
& \quad \cdot (S_{(q_3, q_4)}, a_2 \dots a_{n-1}) + \\
& + \sum_{2 \leq i \leq n-1} \sum_{q_3, q_4, q_5, q_6 \in Q} \left[\delta_{\text{call}}(q_1, a_1, q_3) \cdot \delta_{\text{ret}}(q_4, q_1, a_i, q_5) \cdot \delta_{\text{int}}(q_6, a_n, q_2) + \right. \\
& \quad \left. + \delta_{\text{int}}(q_1, a_1, q_3) \cdot \delta_{\text{call}}(q_4, a_i, q_5) \cdot \delta_{\text{ret}}(q_6, q_4, a_n, q_2) \right] \\
& \quad \cdot (S_{(q_3, q_4)}, a_2 \dots a_{i-1}) \cdot (S_{(q_5, q_6)}, a_{i+1} \dots a_{n-1}) \\
& + \sum_{2 \leq i < j \leq n-1} \sum_{\substack{q_3, q_4, \\ q_5, q_6, q_7, q_8 \in Q}} \delta_{\text{call}}(q_1, a_1, q_3) \cdot \delta_{\text{ret}}(q_4, q_1, a_i, q_5) \cdot \delta_{\text{call}}(q_6, a_j, q_7) \cdot \delta_{\text{ret}}(q_8, q_6, a_n, q_2) \\
& \quad \cdot (S_{(q_3, q_4)}, a_2 \dots a_{i-1}) \cdot (S_{(q_5, q_6)}, a_{i+1} \dots a_{j-1}) \cdot (S_{(q_7, q_8)}, a_{j+1} \dots a_{n-1}) \\
& = \sum_{q_3, q_4 \in Q} \left[\delta_{\text{int}}(q_1, a_1, q_3) \cdot \sum_{\substack{nw \in \text{NW}(\Delta) \\ \pi(nw) = a_2 \dots a_{n-1}}} \sum_{r: q_3 \xrightarrow{nw} q_4} \text{wgt}_{\mathcal{A}}(r) \cdot \delta_{\text{int}}(q_4, a_n, q_2) + \right. \\
& \quad \left. + \delta_{\text{call}}(q_1, a_1, q_3) \cdot \sum_{\substack{nw \in \text{NW}(\Delta) \\ \pi(nw) = a_2 \dots a_{n-1}}} \sum_{r: q_3 \xrightarrow{nw} q_4} \text{wgt}_{\mathcal{A}}(r) \cdot \delta_{\text{ret}}(q_4, q_1, a_n, q_2) \right] + \\
& + \sum_{2 \leq i \leq n-1} \sum_{\substack{q_3, q_4, \\ q_5, q_6 \in Q}} \left[\delta_{\text{call}}(q_1, a_1, q_3) \cdot \sum_{\substack{nw_1 \in \text{NW}(\Delta) \\ \pi(nw_1) = a_2 \dots a_{i-1}}} \sum_{r_1: q_3 \xrightarrow{nw_1} q_4} \text{wgt}_{\mathcal{A}}(r_1) \cdot \delta_{\text{ret}}(q_4, q_1, a_i, q_5) \cdot \right. \\
& \quad \cdot \sum_{\substack{nw_2 \in \text{NW}(\Delta) \\ \pi(nw_2) = a_{i+1} \dots a_{n-1}}} \sum_{r_2: q_5 \xrightarrow{nw_2} q_6} \text{wgt}_{\mathcal{A}}(r_2) \cdot \delta_{\text{int}}(q_6, a_n, q_2) + \\
& \quad \left. + \delta_{\text{int}}(q_1, a_1, q_3) \cdot \sum_{\substack{nw_1 \in \text{NW}(\Delta) \\ \pi(nw_1) = a_2 \dots a_{i-1}}} \sum_{r_1: q_3 \xrightarrow{nw_1} q_4} \text{wgt}_{\mathcal{A}}(r_1) \cdot \right. \\
& \quad \left. \cdot \delta_{\text{call}}(q_4, a_i, q_5) \cdot \sum_{\substack{nw_2 \in \text{NW}(\Delta) \\ \pi(nw_2) = a_{i+1} \dots a_{n-1}}} \sum_{r_2: q_5 \xrightarrow{nw_2} q_6} \text{wgt}_{\mathcal{A}}(r_2) \cdot \delta_{\text{ret}}(q_6, q_4, a_n, q_2) \right] + \\
& + \sum_{2 \leq i < j \leq n-1} \sum_{\substack{q_3, q_4, q_5, \\ q_6, q_7, q_8 \in Q}} \delta_{\text{call}}(q_1, a_1, q_3) \cdot \sum_{\substack{nw_1 \in \text{NW}(\Delta) \\ \pi(nw_1) = a_2 \dots a_{i-1}}} \sum_{r_1: q_3 \xrightarrow{nw_1} q_4} \text{wgt}_{\mathcal{A}}(r_1) \cdot \delta_{\text{ret}}(q_4, q_1, a_i, q_5) \cdot \\
& \quad \cdot \sum_{\substack{nw_2 \in \text{NW}(\Delta) \\ \pi(nw_2) = a_{i+1} \dots a_{j-1}}} \sum_{r_2: q_5 \xrightarrow{nw_2} q_6} \text{wgt}_{\mathcal{A}}(r_2) \cdot \\
& \quad \cdot \delta_{\text{call}}(q_6, a_j, q_7) \cdot \sum_{\substack{nw_3 \in \text{NW}(\Delta) \\ \pi(nw_3) = a_{j+1} \dots a_{n-1}}} \sum_{r_3: q_7 \xrightarrow{nw_3} q_8} \text{wgt}_{\mathcal{A}}(r_3) \cdot \delta_{\text{ret}}(q_8, q_6, a_n, q_2) \\
& = \sum_{\pi(nw) = w} \sum_{r: q_1 \xrightarrow{nw} q_2} \text{wgt}_{\mathcal{A}}(r).
\end{aligned}$$

Now, let X be a fresh variable and extend $(P_{(q_1, q_2)})_{q_1, q_2 \in Q}$ by adding the new polynomial $P_X = \sum_{q_1, q_2 \in Q} \iota(q_1) \cdot \kappa(q_2) \cdot P_{(q_1, q_2)}$. Clearly, the unique solution of this extended system is obtained by adding $S_X = \sum_{q_1, q_2 \in Q} \iota(q_1) \cdot \kappa(q_2) \cdot S_{(q_1, q_2)}$ to $(S_{(q_1, q_2)})_{q_1, q_2 \in Q}$. The quasiregular part of S_X equals $\pi(\|\mathcal{A}\|)$ which is thus algebraic by our considerations after Definition 6.2. \square

Given an algebraic system $(P_X)_{X \in \mathcal{X}}$ over $(\Delta \cup \mathcal{X})$ and some $X \in \mathcal{X}$, we define the *underlying grammar* $G_X = (\Delta, \mathcal{X}, X, F)$ where the set $F \subseteq \mathcal{X} \times (\mathcal{X} \cup \Delta)^*$ of productions is given by letting $(Y, w) \in F$ iff $(P_Y, w) \neq 0$. Let $u \in \Delta^*$. A *derivation tree* of u under G_X is a finite tree t such that the following holds:

- (a) The root is labeled with (X, w) for some $w \in \text{supp}(P_X)$.
- (b) For each inner node v with label (Y, w) the first component of the labels of the children of v from left to right yield w .
- (c) The labels of the leaves from left to right yield u .

We collect all derivation trees t of u under G_X in $\text{Der}(G_X, u)$. Clearly, if $(P_X)_{X \in \mathcal{X}}$ is proper, then each inner node of t either has a single leaf attached or branches at least binarily. Hence, in this case $\text{Der}(G_X, u)$ is a finite set. Let v be a node of t . If v is an inner node and (Y, w) its label, then we let $\text{wgt}(t, v) = (P_Y, w)$. If v is a leaf, we let $\text{wgt}(t, v) = 1$. Now we define the weight $\text{wgt}(t)$ of t by $\text{wgt}(t) = \prod_{v \text{ node of } t} \text{wgt}(t, v)$. The following lemma seems to belong to what is sometimes called folklore, it can easily be shown by induction on the length of w . A proof of a similar but weaker result can be found in [37, Theorem IV.1.5].

Lemma 6.4. *Let $(P_X)_{X \in \mathcal{X}}$ be a proper algebraic system and let $(S_X)_{X \in \mathcal{X}}$ be its unique quasiregular solution. Then*

$$(S_X, w) = \sum_{t \in \text{Der}(G_X, w)} \text{wgt}(t) \text{ for all } X \in \mathcal{X} \text{ and } w \in \Delta^*.$$

We now show the converse of Proposition 6.3.

Proposition 6.5. *Let $R : \Delta^* \rightarrow \mathbb{K}$ be an algebraic formal power series. Then there is a regular nested word series $S : \text{NW}(\Delta) \rightarrow \mathbb{K}$ such that $\pi(S) = R$.*

Proof. Let $(P_X)_{X \in \mathcal{X}}$ be a proper algebraic system with quasiregular solution $(S_X)_{X \in \mathcal{X}}$ and let $Y \in \mathcal{X}$ such that $R = S_Y$. We construct a WNWA $\mathcal{A} = (Q, \iota, \delta, \kappa)$ such that $\pi(\|\mathcal{A}\|) = S_Y$. Any element in the support of some P_X will define a transition in the automaton. In order not to produce ε -transitions, we require that each word in the support of some P_X contains an element of Δ , and in order to produce at most one call for each transition, each word in the support of some P_Y contains at most two elements of \mathcal{X} . Therefore we assume the algebraic system $(P_X)_{X \in \mathcal{X}}$ to be in Greibach normal form [26], i.e. we require that $\text{supp}(P_X) \subseteq \Delta \cup \Delta \mathcal{X} \cup \Delta \mathcal{X} \mathcal{X}$ for all $X \in \mathcal{X}$. Elements of $\Delta \mathcal{X} \mathcal{X}$ produce call transitions, elements in $\Delta \mathcal{X}$ produce internal transitions and elements in Δ produce return transitions. More precisely, let $Q = (\mathcal{X} \cup \{\perp\}) \times (\mathcal{X} \cup \{\perp\})$ for some fresh symbol \perp , and for all $X_1, X_3, X_4 \in \mathcal{X}$ and $X_2 \in \mathcal{X} \cup \{\perp\}$ let

$$\begin{aligned} \delta_{\text{call}}((X_1, X_4), a, (X_3, X_2)) &= (P_{X_1}, aX_3X_4) \\ \delta_{\text{int}}((X_1, X_2), a, (X_3, X_2)) &= (P_{X_1}, aX_3) \\ \delta_{\text{ret}}((X_1, X_2), (X_3, X_4), a, (X_4, X_2)) &= (P_{X_1}, a). \end{aligned}$$

Moreover, let $\delta_{\text{int}}((X_1, \perp), a, (\perp, \perp)) = (P_{X_1}, a)$. Any other transition gets weight 0. Furthermore, for all $X, Z \in \mathcal{X} \cup \{\perp\}$ we let

$$\iota(X, Z) = \begin{cases} 1 & \text{if } X = Y \\ 0 & \text{otherwise} \end{cases} \quad \kappa(X, Z) = \begin{cases} 1 & \text{if } X = Z = \perp \\ 0 & \text{otherwise.} \end{cases}$$

The idea is to simulate a derivation tree of the underlying grammar G_Y traversed from the left to the right. More precisely, when processing a production (X_1, aX_3X_2) in a derivation tree, then a call transition is executed and we continue in a state with first component X_3 . At the return position the automaton changes to X_2 . Since the automaton looks back to the state in which the automaton was *before* the corresponding call position, it has to guess X_2 in advance. This is stored in the second component which was introduced for this reason. One can show by induction on $|w|$ that for all $w \in \Delta^*$, $X \in \mathcal{X}$ and $Z' \in \mathcal{X} \cup \{\perp\}$ we have

$$(S_X, wa) = \sum_{\nu \in \text{Nest}_{|w|}} \sum_{\substack{X' \in \mathcal{X}, Z \in \mathcal{X} \cup \{\perp\} \\ r: (X, Z) \xrightarrow{(w, \nu)} (X', Z')}} \text{wgt}_{\mathcal{A}}(r) \cdot (P_{X'}, a) \quad (6.2)$$

where we make the convention that there is a run $r : (X, Z) \xrightarrow{(\varepsilon, \emptyset)} (X', Z')$ iff $X = X'$ and $Z = Z'$. Moreover, for this run we let $\text{wgt}_{\mathcal{A}}(r) = 1$. Now the result follows easily from the observation that by the definition of δ the last transition of a run $r : (Y, Z) \xrightarrow{(w, \nu)} (\perp, \perp)$ with $\text{wgt}(r) \neq 0$ must be an internal transition. \square

Subsequently we make use of the following well known result [26]. We just indicate how it can be obtained in this context using Propositions 6.3 and 6.5, but note that a more elementary proof and more general results can be found in [26, Chapter 15].

Corollary 6.6 (Kuich & Salomaa [26, Lemma 15.2]). *Let $S : \Delta^* \rightarrow \mathbb{K}$ be an algebraic formal power series. Then there is an algebraic system $(P_X)_{X \in \mathcal{X}}$ such that $\text{supp}(P_X) \subseteq \Delta \cup \Delta(\Delta \cup \mathcal{X})^* \Delta$ for all $X \in \mathcal{X}$ and $S = S_X$ for some $X \in \mathcal{X}$.*

Proof. By Proposition 6.5, S is the projection of some regular nested word series $R : \text{NW}(\Delta) \rightarrow \mathbb{K}$. Now let \mathcal{A} be a WNWA and Q its set of states such that $\|\mathcal{A}\| = R$. Consider the weakly strict algebraic system $(P_X, (P_{(q_1, q_2)})_{q_1, q_2 \in Q})$ of the proof of Proposition 6.3 and its unique solution $(S_X, (S_{(q_1, q_2)})_{q_1, q_2 \in Q})$. Using the manipulations given after Definition 6.2 we can transform this system into a system of the required form having as a solution the quasiregular part of S_X which equals S . \square

6.2. A Logical Characterization of Algebraic Formal Power Series. Our aim is to give a logical characterization of algebraic formal power series in the spirit of Lautemann, Schwentick and Thérien [27]. They showed that the context-free languages are precisely the languages which can be defined by second-order sentences over words of the form $\exists \nu. \varphi$ where φ is a first-order formula and ν a binary predicate ranging over nesting relations³. We identify a word $a_1 \dots a_n \in \Delta^*$ with the structure $([n], \leq, \lambda)$, where \leq is the canonical order of $[n]$ and $\lambda : [n] \rightarrow \Delta$ is given by $\lambda(i) = a_i$ for all $i \in [n]$. Let φ be a weighted second-order formula over words containing, apart from a single 2-ary relation variable ν ,

³In [27] nesting relations were named *matchings*.

only 1-ary relation variables. In other words let $\varphi \in \text{MSO}(\mathbb{K}, \Delta, \leq, \nu)$. Let $\text{Free}(\varphi) \subseteq \mathcal{V}$, $w \in \Delta^*$ and γ a (\mathcal{V}, w) -assignment. We define the semantics $\llbracket \exists \nu. \varphi \rrbracket^{\text{nest}} : \Delta^* \rightarrow \mathbb{K}$ by letting

$$(\llbracket \exists \nu. \varphi \rrbracket^{\text{nest}}, (w, \gamma)) = \sum_{\nu \in \text{Nest}_{|w|}} (\llbracket \varphi \rrbracket, ((w, \nu), \gamma)).$$

Using our characterization of nested word automata by means of weighted logics (Theorem 3.6), we may reformulate Proposition 6.3 as follows:

Corollary 6.7. *Let $\varphi \in \text{sRMSO}(\mathbb{K}, \Delta, \leq, \nu)$ be a sentence. Then $\llbracket \exists \nu. \varphi \rrbracket^{\text{nest}} : \Delta^* \rightarrow \mathbb{K}$ is an algebraic formal power series.*

Next we show a result which sharpens Proposition 6.5. For this we follow the proof of Lautemann, Schwentick and Thérien [27, Theorem 2.1] with small changes in the details.

Proposition 6.8. *Let S be an algebraic formal power series. Then there is a sentence $\varphi \in \text{sRFO}(\mathbb{K}, \Delta, \leq, \nu)$ such that $S = \llbracket \exists \nu. \varphi \rrbracket^{\text{nest}}$.*

Proof. We use an idea of Lautemann, Schwentick and Thérien [27] and adapt it to the weighted setting. This requires that we have to be more careful in order not to count weights twice.

A normal form. By Corollary 6.6 we may assume that S is the component of the solution of an algebraic system with variables in \mathcal{X} having all supports in $\Delta \cup \Delta(\Delta \cup \mathcal{X})^* \Delta$. By the transformations discussed after Definition 6.2 we obtain from this a proper algebraic system $(P'_X)_{X \in \mathcal{X}}$ with solution $(S_X)_{X \in \mathcal{X}}$ such that for all $X \in \mathcal{X}$, $\text{supp}(P'_X)$ does not contain elements of $\Delta \cup \{\varepsilon\}$ and $\mathbb{1}_{\{w \in \Delta^* \mid |w| > 1\}} \odot S$ is a component of the solution. Clearly, it suffices to show the proposition for the latter series instead of S .

Now we proceed as in [27] and transform the system $(P'_X)_{X \in \mathcal{X}}$ into an equivalent system $(P_X)_{X \in \mathcal{X}}$. Let $w \in \text{supp}(P'_X)$ for some $X \in \mathcal{X}$. The image of w under the homomorphism which is the identity on Δ and maps any $Y \in \mathcal{X}$ to the fresh symbol $|$ is called the *pattern* $\text{patt}(w)$ of w . Let us now fix a strict linear order $<$ on \mathcal{X} . Similarly to [27], we proceed along this linear order. Let X be the current symbol. In order to obtain P_X we substitute iteratively some $Z \in \mathcal{X}$ in some $w \in \text{supp}(P'_X)$ by P'_Z (cf. considerations after Definition 6.2) until for all $Y \in \mathcal{X}$, with $Y < X$, $\text{patt}(w) \neq \text{patt}(w')$ for all $w' \in \text{supp}(P_Y) \setminus \Delta^*$ and $w \in \text{supp}(P_X) \setminus \Delta^*$. This is possible since by our considerations after Definition 6.2 we can ensure that all elements in $\text{supp}(P_X) \setminus \Delta^*$ are longer than all elements in $\text{supp}(P_Y) \setminus \Delta^*$ for all $Y < X$. We finally obtain a proper algebraic system $(P_X)_{X \in \mathcal{X}}$ equivalent to $(P'_X)_{X \in \mathcal{X}}$ having the following properties:

- (1) $\text{supp}(P_X) \subseteq \Delta(\Delta \cup \mathcal{X})^+ \Delta$ for all $X \in \mathcal{X}$.
- (2) For all $X, Y \in \mathcal{X}$, if $\text{patt}(w) = \text{patt}(w')$ for some $w \in \text{supp}(P_X) \setminus \Delta^*$ and $w' \in \text{supp}(P_Y) \setminus \Delta^*$, then $X = Y$.

Let us fix $Y \in \mathcal{X}$. We now proceed by giving a sentence $\varphi_Y \in \text{sRFO}(\mathbb{K})$ such that $\pi(\llbracket \varphi_Y \rrbracket) = S_Y$. This will conclude the proof.

Some macros. Let G_Y be the underlying grammar (see the definition after the proof of Proposition 6.3) and let $u \in \Delta^*$. The basic idea now is to assign to each derivation tree $t \in \text{Der}(G_Y, u)$ a nesting relation ν_t of width $|u|$. This is done by letting $(i, j) \in \nu_t$ if there is an inner node of t such that the leaves of the subtree rooted at this node are exactly the leaves between the i th and the j th leaf of t (in lexicographic order including the i th and the j th leaf). Clearly, due to the special form of $(P_X)_{X \in \mathcal{X}}$ this binary relation is indeed a nesting relation. Let us now define some macros for nested words. Let $nw =$

$(u, \nu) = (a_1 \dots a_k, \nu) \in \text{NW}(\Delta)$. Then let $\min(x)$ and $\max(y)$ express that x is assigned the first position and y the last position. Furthermore, the formula $\text{inchild}(x, y)$ express that $(x, y) \in \nu$ corresponds to an inner node of t which has an inner node as a child.

$$\text{inchild}(x, y) = \nu(x, y) \wedge \exists z, z'. (x < z < y) \wedge \nu(z, z')$$

The macro $\text{surf}(x, y, x_1, y_1)$ says that (x_1, y_1) is a surface arch of $nw[x, y]$:

$$\begin{aligned} \text{surf}(x, y, x_1, y_1) &= (x < x_1 < y_1 < y) \wedge \nu(x_1, y_1) \wedge \\ &\quad \wedge \forall z, z'. (x < z < x_1 < y_1 < z' < y) \rightarrow \neg \nu(z, z'). \end{aligned}$$

As in [27], for $v \in \Delta^*$ let $\psi_v(x, y)$ be a first-order formula that expresses there is no call strictly between positions x and y and that the substring given by the positions strictly between position x and y equals v . For a word $w = avb \in \Delta^+$ define $\vartheta_w(x, y)$ as follows.

$$\vartheta_w(x, y) = \text{Lab}_a(x) \wedge \text{Lab}_b(y) \wedge \psi_v(x, y).$$

Now we will need the notion of a pattern also for nested words [27]. Let $(i_1, j_1), \dots, (i_s, j_s)$ be the sequence of all *surface arches* of nw . The *pattern* $\text{patt}(nw)$ of nw is the string $a_1 \dots a_{i_1-1} \mid a_{j_1+1} \dots a_{i_s-1} \mid a_{j_s+1} \dots a_k$. Now, let $X \in \mathcal{X}$, let $w = av_0X_1v_1 \dots v_{s-1}X_s v_s b \in \text{supp}(P_X) \setminus \Delta^+$ and let $p = \text{patt}(w) = av_0 \mid v_1 \dots v_{s-1} \mid v_s b$. We define the formula $\chi_p(x)$ (cf. [27]) which states that x is a call position with return position y and $\text{patt}(nw[x, y]) = p$.

$$\begin{aligned} \chi_p(x) &= \exists y. \nu(x, y) \wedge \text{Lab}_a(x) \wedge \text{Lab}_b(y) \wedge \\ &\quad \wedge \exists x_1, y_1, \dots, x_s, y_s. \left[(x < x_1 < y_1 \dots < y_s < y) \wedge \psi_{v_0}(x, x_1) \wedge \dots \wedge \psi_{v_s}(y_s, y) \wedge \right. \\ &\quad \left. \wedge (\text{surf}(x, y, x_1, y_1) \wedge \dots \wedge \text{surf}(x, y, x_s, y_s)) \right] \end{aligned}$$

Now let $\tilde{\chi}_X(x)$ be the disjunction of all $\chi_p(x)$ over all patterns p of words $w \in \text{supp}(P_X) \setminus \Delta^+$ and let $\tilde{\vartheta}_X(x, y)$ be the disjunction of all $\vartheta_w(x, y)$ over $w \in \text{supp}(P_X) \cap \Delta^+$. Let again $w = av_0X_1v_1 \dots v_{s-1}X_s v_s b \in \text{supp}(P_X) \setminus \Delta^+$. Similarly to [27] we define now the formula $\tilde{\chi}_w(x, y)$:

$$\begin{aligned} \tilde{\chi}_w(x, y) &= \exists y. \nu(x, y) \wedge \text{Lab}_a(x) \wedge \text{Lab}_b(y) \wedge \\ &\quad \wedge \exists x_1, y_1, \dots, x_s, y_s. \left[(x < x_1 < y_1 \dots < y_s < y) \wedge \psi_{v_0}(x, x_1) \wedge \dots \wedge \psi_{v_s}(y_s, y) \wedge \right. \\ &\quad \wedge (\text{surf}(x, y, x_1, y_1) \wedge \dots \wedge \text{surf}(x, y, x_s, y_s)) \wedge \\ &\quad \left. \wedge \left(\tilde{\chi}_{X_1}(x_1) \vee \vartheta_{X_1}(x_1, y_1) \right) \wedge \dots \wedge \left(\tilde{\chi}_{X_s}(x_s) \vee \vartheta_{X_s}(x_s, y_s) \right) \right]. \end{aligned}$$

We show in the next paragraph that there is a bijective correspondence between the set of derivation trees $t \in \text{Der}(G_Y, u)$ and the nested words (u, ν) satisfying the following formula

$$\begin{aligned} \psi_Y &= \exists x, y. \min(x) \wedge \max(y) \wedge \nu(x, y) \wedge \left(\tilde{\chi}_Y(x, y) \vee \tilde{\vartheta}_Y(x, y) \right) \wedge \\ &\quad \wedge \forall z, z'. \text{inchild}(z, z') \rightarrow \bigvee_{\substack{X \in \mathcal{X} \\ w \in \text{supp}(P_X) \setminus \Delta^+}} \tilde{\chi}_w(z, z'). \end{aligned}$$

The formula. Given a derivation tree $t \in \text{Der}(G_Y, u)$ we assign to it a nesting relation ν_t as described above. Clearly, $(1, n) \in \nu_t$ and either $(u, \nu_t) \models \tilde{\vartheta}_Y[1, n]$ or $(u, \nu_t) \models \tilde{\chi}_Y[1, n]$. Furthermore, if $1 \leq i < j \leq n$ and $(u, \nu_t) \models \text{inchild}[i, j]$, then there is an inner node of t such that the leaves of the subtree rooted at this node are exactly the leaves between the i th and the j th leaf of t . Let (X, w) be the label of this inner node, then $(u, \nu_t) \models \tilde{\chi}_w[i, j]$

by construction and hence $(u, \nu_t) \models \psi$. Conversely, let ν be a nesting relation such that $(u, \nu) \models \psi$. We define a derivation tree t_ν inductively as follows. If $\{(1, n)\} = \nu$, then t_ν consists of a single inner node, the root, labeled by (Y, u) . In this case we must have $(u, \nu) \models \tilde{\vartheta}_Y[1, n]$ and hence t_ν is a derivation tree. Otherwise, let $(i_1, j_1), \dots, (i_s, j_s)$ be the sequence of surface arches of $(u, \nu \setminus \{(1, n)\})$ and let $a_1^1 \dots a_{n_1}^1 \mid \dots \mid a_1^s \dots a_{n_s}^s \mid a_1^{s+1} \dots a_{n_{s+1}}^{s+1}$ be the pattern of $(u, \nu \setminus \{(1, n)\})$. Moreover, for $1 \leq k \leq s$ let $u[i_k, j_k]$ be the subword of u from the i_k th position to the j_k th position. Then we must have

$$(u, \nu) \models \bigvee_{\substack{X \in \mathcal{X} \\ w \in \text{supp}(P_X) \setminus \Delta^+}} \tilde{\chi}_w[1, n]$$

and hence for all $1 \leq k \leq s$ we have $(u, \nu)[i_k, j_k] \models \psi_{X_k}$ for some $X_k \in \mathcal{X}$. Thus by induction hypothesis there are $t_k \in \text{Der}(G_{X_k}, u[i_k, j_k])$. We define t_ν to be the tree whose root is labeled $(Y, a_1^1 \dots a_{n_1}^1 X_1 \dots X_s a_1^{s+1} \dots a_{n_{s+1}}^{s+1})$ and where the trees rooted at the children of the root are as follows from left to right: $a_1^1, \dots, a_{n_1}^1, t_1, \dots, t_s, a_1^{s+1}, \dots, a_{n_{s+1}}^{s+1}$. We conclude that t_ν is a derivation tree, since $(u, \nu) \models \tilde{\chi}_Y[1, n]$.

Now we can give the formula φ_Y .

$$\begin{aligned} \varphi_Y = \psi_Y^+ \wedge \forall x, y. \nu(x, y) \rightarrow & \bigvee_{\substack{X \in \mathcal{X} \\ w \in \text{supp}(P_X)}} \left(\text{inchild}(x, y) \xrightarrow{+} (\tilde{\chi}_w(x, y)^+ \wedge (P_X, w)) \wedge \right. \\ & \left. \wedge \text{-inchild}(x, y) \xrightarrow{+} (\vartheta_w(x, y)^+ \wedge (P_X, w)) \right) \end{aligned}$$

Let $t \in \text{Der}(G_Y, u)$ and let ν_t be the corresponding nesting relation. By construction $(\llbracket \varphi_Y \rrbracket, (u, \nu_t)) = \text{wgt}(t)$ and thus $\llbracket \exists \nu. \varphi_Y \rrbracket^{\text{nest}} = S_Y$ by Lemma 6.4. \square

Let us summarize our results of this section so far.

Theorem 6.9. *Let \mathbb{K} be a commutative semiring and let $S : \Delta^* \rightarrow \mathbb{K}$ be a formal power series. Then the following are equivalent:*

- (1) *S is an algebraic formal power series.*
- (2) *$S = \pi(R)$ for some regular $R : \text{NW}(\Delta) \rightarrow \mathbb{K}$.*
- (3) *There is a sentence $\varphi \in \text{sRFO}(\mathbb{K}, \Delta, \leq, \nu)$ such that $\llbracket \exists \nu. \varphi \rrbracket^{\text{nest}} = S$.*

Proof. (1) \Rightarrow (3). This is Proposition 6.8.

(3) \Rightarrow (2). Follows from Theorem 3.6(a) and the definition of π .

(2) \Rightarrow (1). This is Proposition 6.3. \square

Let $\mathbb{K} = \mathbb{N}$ and let $S : \{a\}^+ \rightarrow \mathbb{N}$ be an algebraic series. As $S = \pi(R)$ for some regular nested word series R , it is not hard to see that $(S, a^n) \leq 2^{n^2} \cdot c^n$ for some constant c and all $n \in \mathbb{N}$. Using weighted pushdown automata (cf. [26]) one can even show that $(S, a^n) \leq c^n$ for some constant c and all $n \in \mathbb{N}$. Thus in item 3 of the last result we may not replace $\text{sRFO}(\mathbb{K})$ by $\text{FO}(\mathbb{K})$ since $(\llbracket \forall x. \exists y. 1 \rrbracket, a^n) = n^n$.

Again we note that all proofs are effective and given a proper algebraic system $(P_X)_{X \in \mathcal{X}}$ with solution $(S_X)_{X \in \mathcal{X}}$ and an effectively given semiring \mathbb{K} , we can compute an $\text{sRFO}(\mathbb{K})$ sentence φ_Y for all $Y \in \mathcal{X}$ such that $S_Y = \llbracket \exists \nu. \varphi_Y \rrbracket^{\text{nest}}$ and vice versa.

6.3. Yet Another Characterization of Algebraic Formal Power Series. Even though our logical characterization of regular nested word series (Theorem 3.6) might also be obtained by structural induction, the connection between alternating texts and nested words we established enables us now to also obtain a generalization of the second main result of [27]. In this paper another logical characterization of context-free languages was given where quantification over nesting relations is now replaced by quantification over tree-definable orders. In [27] a linear order \leq on $[n]$ was called tree-definable if there is a binary tree t with n leaves which are labeled $1, \dots, n$ in lexicographic order and whose internal nodes are labeled with $\{\swarrow, \searrow\}$ such that $i \leq j$ iff i is visited before j in the depth-first traversal of t in which, at every node with label \swarrow , first the left, and at every node with label \searrow , first the right child is visited. We will give a slightly different definition which is easily seen to be equivalent by simply replacing \swarrow by \bullet and \searrow by \circ .

Definition 6.10. Let $n \in \mathbb{N}_+$ and let \leq_1 be the canonical order of $[n]$. Moreover, let $\lambda : [n] \rightarrow \Delta$ be a labeling. A linear order \leq_2 of $[n]$ is *tree-definable* iff $([n], \lambda, \leq_1, \leq_2)$ is an alternating text.

We collect all tree-definable orders of $[n]$ in TDO_n . Our aim is now to extend the above mentioned result of [27] and to show, using the connection between nested words and alternating texts, that a formal power series is an algebraic formal power series iff it can be defined by a second-order sentence over words of the form $\exists \leq_2 .\varphi$ where φ is a first-order formula and \leq_2 a binary relation symbol ranging over tree-definable orders. Note that like matchings, tree-definable orders are first-order definable relations [22, 30]. First, we start by defining the projection $\pi(\tau)$ of an alternating text $\tau = ([n], \leq_1, \leq_2, \lambda) \in \text{TXT}(\Delta)$ to be the word $([n], \leq_1, \lambda)$, i.e. we forget the second order. As for nested words, this projection is canonically generalized to languages $L \subseteq \text{TXT}(\Delta)$ by setting $\pi(L) = \{\pi(\tau) \mid \tau \in L\}$ and to series $S : \text{TXT}(\Delta) \rightarrow \mathbb{K}$ by letting

$$\begin{aligned} \pi(S) : \Delta^* &\rightarrow \mathbb{K} \\ w &\mapsto \sum_{\substack{\tau \in \text{TXT}(\Delta) \\ w = \pi(\tau)}} (S, \tau). \end{aligned}$$

Proposition 6.11. *Let $S : \text{TXT}(\Delta) \rightarrow \mathbb{K}$ be regular. Then $\pi(S) : \Delta^* \rightarrow \mathbb{K}$ is an algebraic formal power series.*

Proof. Consider a WPA $\mathcal{A} = (\mathcal{H}, \mathcal{V}, \Omega, \mu, \mu_{\text{op}}, \mu_{\text{cl}}, \lambda, \gamma)$ such that $\|\mathcal{A}\| = S$. Let $\mathcal{X} = (\mathcal{H}^2 \times \{0, 1\}) \cup (\mathcal{V}^2 \times \{0, 1\})$. We define an algebraic system $(P_X)_{X \in \mathcal{X}}$ as follows: For all $h_1, h_2 \in \mathcal{H}$ and $v_1, v_2 \in \mathcal{V}$ we let

$$\begin{aligned} (P_{(h_1, h_2, 1)}, w) &= \sum_{a \in \Delta} \mu(h_1, a, h_2).a + \sum_{v, v' \in \mathcal{V}} \sum_{s \in \Omega} \mu_{\text{op}}(h_1, (s, v) \cdot \mu_{\text{cl}}(v',)_s, h_2).(v, v', 0) \\ (P_{(h_1, h_2, 0)}, w) &= \sum_{h_3 \in \mathcal{H}} (h_1, h_3, 1)(h_3, h_2, 1) + (h_1, h_3, 1)(h_3, h_2, 0) \\ (P_{(v_1, v_2, 1)}, w) &= \sum_{a \in \Delta} \mu(v_1, a, v_2).a + \sum_{h, h' \in \mathcal{H}} \sum_{s \in \Omega} \mu_{\text{op}}(v_1, (s, h) \cdot \mu_{\text{cl}}(h',)_s, v_2).(h, h', 0) \\ (P_{(v_1, v_2, 0)}, w) &= \sum_{v_3 \in \mathcal{H}} (v_1, v_3, 1)(v_3, v_2, 1) + (v_1, v_3, 1)(v_3, v_2, 0) \end{aligned}$$

We claim that this algebraic system has a unique quasiregular solution $(S_X)_{X \in \mathcal{X}}$ which consists of algebraic formal power series. Indeed, if we replace the polynomial $P_{(h_1, h_2, 1)}$ by the polynomial

$$\sum_{a \in \Delta} \mu(h_1, a, h_2) \cdot a + \sum_{v, v' \in \mathcal{V}} \sum_{s \in \Omega} \mu_{\text{op}}(h_1, (s, v) \cdot \mu_{\text{cl}}(v',)_s, h_2) \cdot P_{(v, v', 0)}$$

and the polynomial $P_{(v_1, v_2, 1)}$ by the polynomial

$$\sum_{a \in \Delta} \mu(v_1, a, v_2) \cdot a + \sum_{h, h' \in \mathcal{H}} \sum_{s \in \Omega} \mu_{\text{op}}(v_1, (s, h) \cdot \mu_{\text{cl}}(h',)_s, v_2) \cdot P_{(h, h', 0)},$$

we obtain an equivalent system (cf. manipulations after Definition 6.2) which is proper and has thus the unique quasiregular solution $(S_X)_{X \in \mathcal{X}}$ which consists of algebraic formal power series. Let $\text{TXT}^\circ \subseteq \text{TXT}(\Delta)$ be the set of all alternating texts which are either singletons or \circ -products. Analogously let $\text{TXT}^\bullet \subseteq \text{TXT}(\Delta)$ be the set of all alternating texts which are either singletons or \bullet -products. We will show by induction that we have for all $w \in \Delta^*$ with $|w| \geq 1$

$$(S_{(h_1, h_2, 1)}, w) = \sum_{\substack{\tau \in \text{TXT}^\bullet \\ \pi(\tau) = w}} \sum_{r: h_1 \xrightarrow{\tau} h_2} \text{wgt}_{\mathcal{A}}(r) \text{ and} \quad (6.3)$$

$$(S_{(v_1, v_2, 1)}, w) = \sum_{\substack{\tau \in \text{TXT}^\circ \\ \pi(\tau) = w}} \sum_{r: v_1 \xrightarrow{\tau} v_2} \text{wgt}_{\mathcal{A}}(r) \quad (6.4)$$

as well as

$$(S_{(h_1, h_2, 1)}, w) + (S_{(h_1, h_2, 0)}, w) = \sum_{\substack{\tau \in \text{TXT}(\Delta) \\ \pi(\tau) = w}} \sum_{r: h_1 \xrightarrow{\tau} h_2} \text{wgt}_{\mathcal{A}}(r) \text{ and} \quad (6.5)$$

$$(S_{(v_1, v_2, 1)}, w) + (S_{(v_1, v_2, 0)}, w) = \sum_{\substack{\tau \in \text{TXT}(\Delta) \\ \pi(\tau) = w}} \sum_{r: v_1 \xrightarrow{\tau} v_2} \text{wgt}_{\mathcal{A}}(r). \quad (6.6)$$

The result then follows immediately from the fact that algebraic formal power series are closed under pointwise sum and scalar multiplication. Let $w = a$ for some $a \in \Delta$. Since the series $S_{(h_1, h_2, 1)}$ and $S_{(v_1, v_2, 1)}$ are quasiregular, we obtain that $(S_{(v_1, v_2, 0)}, a) = (S_{(h_1, h_2, 0)}, a) = 0$. From this it is easy to deduce the induction base. Let now $|w| > 1$. Then

$$\begin{aligned} (S_{(h_1, h_2, 1)}, w) &= \sum_{v, v' \in \mathcal{V}} \sum_{s \in \Omega} \mu_{\text{op}}(h_1, (s, v) \cdot \mu_{\text{cl}}(v',)_s, h_2) \cdot (S_{(v, v', 0)}, w) \\ &= \sum_{v, v' \in \mathcal{V}} \sum_{s \in \Omega} \mu_{\text{op}}(h_1, (s, v) \cdot \\ &\quad \left(\sum_{v_3 \in \mathcal{V}} \sum_{w = w_1 w_2} (S_{(v, v_3, 1)}, w_1) \cdot (S_{(v_3, v', 1)}, w_2) + (S_{(v, v_3, 1)}, w_1) \cdot (S_{(v_3, v', 0)}, w_2) \right) \cdot \mu_{\text{cl}}(v',)_s, h_2) \\ &= \sum_{v, v' \in \mathcal{V}} \sum_{s \in \Omega} \mu_{\text{op}}(h_1, (s, v) \cdot \\ &\quad \left(\sum_{v_3 \in \mathcal{V}} \sum_{w = w_1 w_2} (S_{(v, v_3, 1)}, w_1) \cdot ((S_{(v_3, v', 1)}, w_2) + (S_{(v_3, v', 0)}, w_2)) \right) \cdot \mu_{\text{cl}}(v',)_s, h_2) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{v, v' \in \mathcal{V}} \sum_{s \in \Omega} \mu_{\text{op}}(h_1, (s, v)) \cdot \\
 &\quad \left(\sum_{v_3 \in \mathcal{V}} \sum_{w=w_1 w_2} \sum_{\substack{\tau_1 \in \text{TXT}^\circ \\ \pi(\tau_1)=w_1}} \sum_{r_1: v \xrightarrow{\tau_1} v_3} \text{wgt}_{\mathcal{A}}(r_1) \cdot \sum_{\substack{\tau_2 \in \text{TXT}(\Delta) \\ \pi(\tau_2)=w_2}} \sum_{r_1: v_3 \xrightarrow{\tau_2} v'} \text{wgt}_{\mathcal{A}}(r_1) \right) \cdot \mu_{\text{cl}}(v',)_s, h_2
 \end{aligned}$$

Since $\text{TXT}(\Delta)$ is the free bisemigroup, given some w of length at least two, each $\tau \in \text{TXT}^\bullet$ with $\pi(\tau) = w$ decomposes uniquely into $\tau = \tau_1 \bullet \tau_2$ with $\tau_1 \in \text{TXT}^\circ$ and $\tau_2 \in \text{TXT}(\Delta)$. Hence we can continue

$$= \sum_{\substack{\tau \in \text{TXT}^\bullet \\ \pi(\tau)=w}} \sum_{r: h_1 \xrightarrow{\tau} h_2} \text{wgt}_{\mathcal{A}}(r).$$

Analogously we get Equation (6.4). Similarly we get:

$$\begin{aligned}
 &(S_{(h_1, h_2, 1)}, w) + (S_{(h_1, h_2, 0)}, w) = \\
 &= \sum_{\substack{\tau \in \text{TXT}^\bullet \\ \pi(\tau)=w}} \sum_{r: h_1 \xrightarrow{\tau} h_2} \text{wgt}_{\mathcal{A}}(r) + \sum_{h_3 \in \mathcal{H}} \sum_{w=w_1 w_2} (S_{(h_1, h_3, 1)}, w_1) \cdot ((S_{(h_3, h_2, 1)}, w_2) + (S_{(h_3, h_2, 0)}, w_2)) \\
 &= \sum_{\substack{\tau \in \text{TXT}^\bullet \\ \pi(\tau)=w}} \sum_{r: h_1 \xrightarrow{\tau} h_2} \text{wgt}_{\mathcal{A}}(r) + \\
 &\quad \sum_{h_3 \in \mathcal{H}} \sum_{w=w_1 w_2} \sum_{\substack{\tau_1 \in \text{TXT}^\bullet \\ \pi(\tau_1)=w_1}} \sum_{r_1: h_1 \xrightarrow{\tau_1} h_3} \text{wgt}_{\mathcal{A}}(r_1) \cdot \sum_{\substack{\tau_2 \in \text{TXT}(\Delta) \\ \pi(\tau_2)=w_2}} \sum_{r_1: h_3 \xrightarrow{\tau_2} h_2} \text{wgt}_{\mathcal{A}}(r_1) \\
 &= \sum_{\substack{\tau \in \text{TXT}^\bullet \\ \pi(\tau)=w}} \sum_{r: h_1 \xrightarrow{\tau} h_2} \text{wgt}_{\mathcal{A}}(r) + \sum_{\substack{\tau \in \text{TXT}^\circ \\ \pi(\tau)=w}} \sum_{r: h_1 \xrightarrow{\tau} h_2} \text{wgt}_{\mathcal{A}}(r) \\
 &= \sum_{\substack{\tau \in \text{TXT}(\Delta) \\ \pi(\tau)=w}} \sum_{r: h_1 \xrightarrow{\tau} h_2} \text{wgt}_{\mathcal{A}}(r).
 \end{aligned}$$

Again Equation (6.6) can be shown analogously, which concludes the proof. \square

Now we get our second characterization of algebraic formal power series. For this, we proceed as follows: Let φ be a weighted second-order formula over words containing, apart from a single 2-ary relation variable \leq_2 , only 1-ary relation variables. In other words, let $\varphi \in \text{MSO}(\mathbb{K}, \Delta, \leq_1, \leq_2)$. Let $\text{Free}(\varphi) \subseteq \mathcal{V}$, $w = ([n], \leq_1, \lambda) \in \Delta^*$ and γ a (\mathcal{V}, w) -assignment. We define the semantics $\llbracket \exists \leq_2 . \varphi \rrbracket^{\text{tdo}} : \Delta^* \rightarrow \mathbb{K}$ by letting

$$(\llbracket \exists \leq_2 . \varphi \rrbracket^{\text{tdo}}, (w, \gamma)) = \sum_{\leq_2 \in \text{TDO}_n} (\llbracket \varphi \rrbracket, (([n], \leq_1, \leq_2, \lambda), \gamma)).$$

Theorem 6.12. *Let \mathbb{K} be a commutative semiring and let $S : \Delta^* \rightarrow \mathbb{K}$ be a formal power series. Then the following are equivalent:*

- (1) S is an algebraic formal power series.
- (2) $S = \pi(R)$ for some regular $R : \text{TXT}(\Delta) \rightarrow \mathbb{K}$.
- (3) There is a sentence $\varphi \in \text{sRFO}(\mathbb{K}, \Delta, \leq_1, \leq_2)$ such that $\llbracket \exists \leq_2 . \varphi \rrbracket^{\text{tdo}} = S$.

Proof. (1) \Rightarrow (3). Let $S : \Delta^* \rightarrow \mathbb{K}$ be an algebraic formal power series. By Theorem 6.9 there is an sRFO(\mathbb{K}) sentence over nested words such that $\llbracket \exists \nu. \varphi \rrbracket^{\text{nest}} = S$. By Corollary 5.6 the partial function Φ_{\circ}^{-1} is FO-definable without parameter. Similar to Proposition 5.2 one can show that there is thus an sRFO(\mathbb{K}) sentence φ' over texts such that $\Phi_{\circ}(\llbracket \varphi \rrbracket) = \llbracket \varphi' \rrbracket$. Now we can calculate using observations (a) and (b) before Lemma 5.3 as follows.

$$\begin{aligned} (\llbracket \exists \nu. \varphi \rrbracket^{\text{nest}}, w) &= \sum_{\substack{nw \in \text{NW}(\Delta) \\ \pi(nw) = w}} (\llbracket \varphi \rrbracket, nw) = \sum_{\substack{nw \in \text{NW}(\Delta) \\ \pi(\Phi_{\circ}(nw)) = w}} (\llbracket \varphi \rrbracket, nw) \\ &= \sum_{\substack{\tau \in \Phi_{\circ}(\text{NW}(\Delta)) \\ \pi(\tau) = w}} (\llbracket \varphi \rrbracket, \Phi_{\circ}^{-1}(\tau)) = \sum_{\substack{\tau \in \Phi_{\circ}(\text{NW}(\Delta)) \\ \pi(\tau) = w}} (\llbracket \varphi' \rrbracket, \tau) = (\llbracket \exists \leq_2. \varphi' \rrbracket^{\text{tdo}}, w). \end{aligned}$$

(3) \Rightarrow (2). Follows from Theorem 4.2 and the definition of π .

(2) \Rightarrow (1). This is Proposition 6.11. □

7. CONCLUDING REMARKS AND FUTURE WORK

We introduced a quantitative automaton model and a quantitative logic for nested words and showed that they are equally expressive. This generalizes the logical characterization of the unweighted case as given in [4]. Moreover, we established a new connection between nested words and alternating texts. Applying the result, we obtained a characterization of algebraic formal power series in terms of weighted logics. Presumably, the logical characterization of regular nested word series could also be obtained by structural induction. However, the connection between alternating texts and nested words enabled us to also obtain a second characterization of algebraic formal power series. Note that even though the characterizations of algebraic formal power series are generalizations of the results of [27] to a weighted setting, in contrast to the latter paper we gave a different proof using this connection as well as (weighted) nested word automata and (weighted) parenthesizing automata. Also note that weighted nested word automata and weighted parenthesizing automata were characterized algebraically in [31].

Let us remark that regular formal power series also fall into the pattern of our characterizations (Theorem 6.9 and Theorem 6.12) of algebraic formal power series. In fact, Thomas showed that a single existential monadic second-order quantifier suffices to characterize finite automata [39, Theorem 5.2]. That is, in the pattern of the last results we can formulate that $L \subseteq \Delta^*$ is regular iff $L = \llbracket \exists M. \varphi \rrbracket^{\text{set}}$ for some first-order formula φ (where $\llbracket \exists M. \varphi \rrbracket^{\text{set}}$ means that we sum over all subsets M of the domain of a given structure). Let us explain the idea of the proof with an example. Given an automaton \mathcal{A} with set of states $Q = \{0, 1\}^k$ for L and some word $a_1 \dots a_{2k} \in L$, the idea is to think of the interpretation of M as a word $u_1 u_2$ where $u_1, u_2 \in \{0, 1\}^k = Q$ and to express by φ that u_1 is an initial state, that there is a run from u_1 to u_2 on $a_1 \dots a_k$ and that there is a run from u_2 into a final state on $a_{k+1} \dots a_{2k}$. Alternatively, one can prove the result similarly to Proposition 6.8 where one starts with a right-regular system and applies a similar transformation. Then a set M suffices to encode a derivation tree since any inner node has at most one non-terminal child whose position is collected in M . In any way, it is not hard to see that the proof can be adapted to a weighted setting. So, also in the weighted case we can restrict ourselves to a single existential monadic second-order quantifier.

Following these pattern it might be interesting to further investigate whether other important classes of formal power series can be characterized in this manner. Again, the work of Lautemann, Schwentick and Thérien [27] can be used as a starting point where the so-called k -linear languages were considered.

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