# COMPUTABILITY OF PROBABILITY DISTRIBUTIONS AND CHARACTERISTIC FUNCTIONS

TAKAKAZU MORI<sup>*a*</sup>, YOSHIKI TSUJII<sup>*b*</sup>, AND MARIKO YASUGI<sup>*c*</sup>

<sup>a,b</sup> Faculty of Science, Kyoto Sangyo University
 *e-mail address*: {morita,tsujiiy}@cc.kyoto-su.ac.jp

<sup>c</sup> Kyoto Sangyo University and Kyoto University e-mail address: yasugi@cc.kyoto-su.ac.jp

ABSTRACT. As a part of our works on effective properties of probability distributions, we deal with the corresponding characteristic functions. A sequence of probability distributions is computable if and only if the corresponding sequence of characteristic functions is computable. As for the onvergence problem, the effectivized Glivenko's theorem holds. Effectivizations of Bochner's theorem and de Moivre-Laplace central limit theorem are also proved.

# 1. INTRODUCTION

We are concerned with mutual relationships between computability of probability distributions (Borel probability measures on the real line  $\mathbb{R}$ ) and that of the corresponding probability distribution functions as well as that of the corresponding characteristic functions. We are also concerned with mutual relationship between effective convergence of probability distributions and that of the corresponding probability distribution functions as well as that of the corresponding characteristic functions.

We have no general characterization of probability distribution functions which correspond to computable probability distributions. What we can do is to start with computable (hence continuous) probability distribution functions ([7]) and examine what will happen with wider classes of probability distribution functions. In [7], it has been proved that, for a sequence of probability distributions with effectively bounded densities, it is computable if and only if the corresponding sequence of probability distribution functions is computable, and that it converges effectively to a (computable) probability distribution if and only if the corresponding sequence of probability distribution functions behaves similarly.

DOI:10.2168/LMCS-9(3:9)2013

<sup>2012</sup> ACM CCS: [Theory of computation]: Models of computation—Computability—Recursive functions.

Key words and phrases: computable probability distribution, effective convergence of probability distributions, characteristic function, computable function, effective convergence of functions.

<sup>&</sup>lt;sup>a</sup> This work has been supported in part by Scientific Foundations of JSPS No. 21540152.

<sup>&</sup>lt;sup>b</sup> This work has been supported in part by Scientific Foundations of JSPS No. 23540170.

However, the Dirac probability distribution has no density function. Furthermore, its probability distribution function is discontinuous. We have then treated a wider class of probability distribution functions, namely Fine computable ones (cf. [8], [9]), as the first step to deal with cases where probability distributions may not have bounded density functions.

Here in this article, we will work on the relationship between computability of probability distributions and that of the corresponding characteristic functions as well as between their effective convergences. Effective versions of some classical results regarding probability distributions and the corresponding characteristic functions such as Glivenko's theorem and de Moivre-Laplace central limit theorem will be proved.

Let us explain some facts on probability distributions and related objects.

Many theorems in probability theory concerning convergence of random variables, such as *central limit theorems*, are formulated in terms of convergence of probability distributions, and convergence of probability distributions is proved in terms of convergence of the corresponding characteristic functions.

For a probability distribution  $\mu$ , its probability distribution function is defined by  $F(x) = \mu((-\infty, x])$ , and characterized by the following properties:

(Fi) monotonically increasing,

(Fii) right continuous,

(Fiii)  $\lim_{x\to\infty} F(x) = 0$  and  $\lim_{x\to\infty} F(x) = 1$ .

On the other hand, the characteristic function of  $\mu$  is defined as the Fourier transformation, that is,  $\varphi(t) = \int_{\mathbb{R}} e^{itx} \mu(dx) = \mu(e^{it})$ , and characterized by the following:

(Ci)  $\varphi$  is positive definite.

(Cii)  $\varphi(t)$  is continuous at 0.

(Ciii)  $\varphi(0) = 1$ .

To obtain a probability distribution from a probability distribution function amounts to a construction of a measure. To obtain a probability distribution from a characteristic function corresponds to Bochner's Theorem.

In this article, we investigate relations between computability of probability distributions and computability of the corresponding characteristic functions as well as relations between effective convergence of probability distributions and effective convergence of the corresponding characteristic functions.

In Section 2, we review briefly the previous results in [7] and [9].

In Section 3, we treat effective integrability on  $\mathbb{R}$  and generalize some theorems in [4], [5] and [8] to turn them to account for our purpose.

Section 4 contains most of our main results. We will work on some relations between effective properties of a sequence of probability distributions  $\{\mu_m\}$  and the corresponding sequence of characteristic functions  $\{\varphi_m\}$ . We have the following results: equivalence of computability of  $\{\mu_m\}$  and that of the corresponding  $\{\varphi_m\}$  (Theorems 4.1 and 4.5); effective Glivenko's theorem on convergence (Theorems 4.2 and 4.3) and effective Bochner's theorem (Theorem 4.7).

In section 5, we apply Theorem 4.2 to prove the effective de Moivre-Laplace central limit theorem.

#### 2. Preliminaries

We follow the notions of computability developed by Pour-El and Richards ([10]). Computability of a sequence of functions and effective convergence of a sequence of functions are formulated as follows.

**Definition 2.1.** (*Computability of a sequence of functions on*  $\mathbb{R}$  [10]) A sequence of functions  $\{f_n\}$  on  $\mathbb{R}$  is said to be computable if it satisfies the following two properties.

- (i) Sequential computability: the double sequence  $\{f_n(x_m)\}$  is computable for any computable sequence  $\{x_m\}$ .
- (ii) Effective continuity: there exists a recursive function  $\alpha(n, L, k)$  such that  $|x y| < 2^{-\alpha(n,L,k)}$  and  $|x|, |y| \le L$  imply  $|f_n(x) f_n(y)| < 2^{-k}$ .

**Definition 2.2.** (Effective convergence of a sequence of functions on  $\mathbb{R}$  [10]) A double sequence of functions  $\{f_{n,m}\}$  converges effectively to a sequence  $\{f_n\}$  if there exists a recursive function  $\alpha(n, L, k)$  such that  $m \geq \alpha(n, L, k)$  and  $|x| \leq L$  imply  $|f_{n,m}(x) - f_n(x)| < 2^{-k}$ .

We say that a sequence of functions  $\{f_n\}$  is computable with compact support L(n), if  $\{f_n\}$  is (uniformly) computable and L(n) is a recursive function which satisfies that  $f_n(x) = 0$  for all x with  $|x| \ge L(n)$ .

A sequence of functions  $\{f_n\}$  is said to be effectively bounded if there exists a recursive function M(n) which satisfies  $|f_n(x)| \leq M(n)$  for all x.

We denote  $\int_{\mathbb{R}} f(x)\mu(dx)$  with  $\mu(f)$ .

**Definition 2.3.** (Computability of probability distributions [7]) A sequence of probability distributions  $\{\mu_m\}$  is said to be computable, if  $\{\mu_m(f_n)\}$  is computable for any computable  $\{f_n\}$  with compact support L(n).

**Definition 2.4.** (*Effective convergence of probability distributions* [7]) A sequence of probability distributions  $\{\mu_m\}$  is said to converge effectively to a probability distribution  $\mu$  if, for any computable sequence of functions  $\{f_n\}$  with compact support, there exists a recursive function  $\alpha(n,k)$  such that

$$|\mu_m(f_n) - \mu(f_n)| < 2^{-k} \text{ if } m \ge \alpha(n,k)$$
(2.1)

holds.

If the condition (2.1) holds, we say that  $\{\mu_m(f_n)\}\$  converges effectively to  $\mu(f_n)$  as m tends to infinity effectively in n.

The following proposition holds.

**Proposition 2.5.** ([7]) If a computable sequence of probability distributions  $\{\mu_m\}$  converges effectively to a probability distribution  $\mu$ , then  $\mu$  is computable.

**Lemma 2.6.** (Monotone Lemma [10]) Let  $\{x_{n,k}\}$  be a computable sequence of reals which converges monotonically to  $\{x_n\}$  as k tends to infinity for each n. Then  $\{x_n\}$  is computable if and only if the convergence is effective in n.

We will use the following functions  $w_n$  and  $w_n^c$ .

$$w_n(x) := \begin{cases} 0 & \text{if } x \le -n - 1 \\ (x+n)+1 & -n-1 \le x \le -n \\ 1 & \text{if } -n \le x \le n \\ -(x-n)+1 & \text{if } n \le x \le n+1 \\ 0 & \text{if } x \ge n+1 \end{cases} \qquad 1 \underbrace{w_n \\ 0 \underbrace{-n-1 - n \\ -n-1 -n \\ -n-1 -n$$

 $w_n$  and  $w_n^c$  satisfy the following properties.

- $w_n(x) + w_n^c(x) \equiv 1.$
- $\{w_n\}$  is monotonically increasing in n and  $\{w_n^c\}$  is monotonically decreasing.
- $\chi_{[-n,n]}(x) \le w_n(x) \le \chi_{[-(n+1),n+1]}(x).$
- $\chi_{(-\infty,-(n+1)]}(x) + \chi_{[n+1,\infty)}(x) \le w_n^c(x) \le \chi_{(-\infty,-n]}(x) + \chi_{[n,\infty)}(x).$

**Lemma 2.7.** If a sequence of probability distributions  $\{\mu_m\}$  is computable, then there exists a recursive function L(m,k) such that  $\mu_m(w_n) > 1 - 2^{-k}$ , or equivalently  $\mu_m(w_n^c) < 2^{-k}$ , for all  $n \geq L(m, k)$ .

Observe that for all  $n \ge L(m, k)$ ,

$$\mu_m([-(n+1), n+1]) > 1 - 2^{-k}$$
 and  $\mu_m((-\infty, -(n+1)]) + \mu_m([n+1, \infty)) < 2^{-k}$ 

However, these quantities may not be computable.

*Proof of Lemma 2.7.* By the definition,  $\{\mu_m(w_n)\}\$  is a computable sequence of reals and converges monotonically to  $\mu_m(\mathbb{R})$  (= 1) from below as n tends to infinity. By Lemma 2.6, this convergence is effective. 

**Lemma 2.8.** (Effective tightness of an effectively convergent sequence, Effectivization of Lemma 15.4 in [11]) If a computable sequence of probability distributions  $\{\mu_m\}$  effectively converges to a probability distribution  $\mu$ , then there exists a recursive function  $\alpha(k)$  such that  $\mu(w_{\alpha(k)}^c) < 2^{-k}$  and  $\mu_m(w_{\alpha(k)}^c) < 2^{-k}$  for all m. It also holds that

$$\mu_m([-\alpha(k) - 1, \alpha(k) + 1]^c) < 2^{-k} \quad for \ all \ m \ and \quad \mu([-\alpha(k) - 1, \alpha(k) + 1]^c) < 2^{-k}$$

*Proof.* Let L(m,k) be the recursive function in Lemma 2.7, and let L(k) be the one for a single  $\mu$ .

By definition,  $\{w_n\}$  is a uniformly computable sequence of functions with compact support. So,  $\{\mu_m(w_n)\}$  converges effectively to  $\{\mu(w_n)\}$ , that is, there exists a recursive function  $\gamma(n,k)$  such that  $m \ge \gamma(n,k)$  implies  $|\mu_m(w_n) - \mu(w_n)| < 2^{-k}$ . It holds that  $\mu(w_n^c) < 2^{-k}$  for all  $n \ge L(k)$  by Lemma 2.7.

If  $m \ge \gamma(L(k), k)$ , then  $|\mu_m(w_{L(k)}^c) - \mu(w_{L(k)}^c)| = |\mu_m(w_{L(k)}) - \mu(w_{L(k)})| < 2^{-k}$  and hence  $\mu_m(w_{L(k)}^c) < 2 \cdot 2^{-k}$ . For each  $\ell \leq \gamma(L(k), k) - 1$ ,  $\mu_\ell(w_{L(\ell,k)}^c) < 2^{-k}$  by Lemma 2.7. If we put

 $\bar{\alpha(k)} = \max\{L(k+1), L(1, k+1), L(2, k+1), \dots, L(\gamma(L(k+1), k+1) - 1, k+1)\},$ then  $\mu(w_{\alpha(k)}^c) < 2^{-k}$  and  $\mu_m(w_{\alpha(k)}^c) < 2^{-k}$  holds for all m. 

**Proposition 2.9.** ([7]) If  $\{\mu_m\}$  is computable, then it is weakly sequentially computable, that is,  $\{\mu_m(f_n)\}\$  is a computable sequence of reals for all effectively bounded computable sequence of functions  $\{f_n\}$ .

**Proposition 2.10.** ([7]) Let  $\{\mu_m\}$  and  $\mu$  be computable probability distributions. Then the effective convergence of  $\{\mu_m\}$  to  $\mu$  is equivalent to the effective weak convergence of  $\{\mu_m\}$  to  $\mu$ , that is,  $\{\mu_m(f_n)\}$  converges to  $\mu(f_n)$  effectively for all effectively bounded computable sequence of functions.

For characteristic functions, the following classical Bochner's Theorem holds.

**Theorem 2.11.** (Bochner's Theorem [1]) In order for  $\varphi(t)$  to be a characteristic function, it is necessary and sufficient that the following three conditions hold.

(i)  $\varphi$  is positive definite. (ii)  $\varphi(t)$  is continuous at 0. (iii)  $\varphi(0) = 1$ .

### 3. Effective Integrability

Effective integrability of Fine computable functions on [0,1) and  $[0,1)^2$  with respect to the Lebesgue measure dx and  $dx \times dy$  respectively has been treated in [3], [4], [5], [6]. We need to extend some of these results to  $(\mathbb{R}, dx)$ ,  $(\mathbb{R}, \mu)$  and  $(\mathbb{R}^2, \mu(dx)dy)$  for computable integrands, where  $\mu(dx)dy$  is the product measure of a probability distribution  $\mu$  and the Lebesgue measure dx. This product measure is defined by assigning the measure  $\mu((a,b]) \times (d-c)$  to the product set  $(a,b] \times (c,d]$ .

A bounded measurable function is integrable on any finite interval. A positive (nonnegative) measurable function f is called integrable if  $\{\int_{[-n,n]} (f \wedge m)(x) dx\}$  converges to a finite limit as m and n tend to infinity, where  $(f \wedge m)(x) = \min\{f(x), m\}$ . We denote this limit with  $\int_{\mathbb{R}} f(x) dx$ .

A measurable f is called integrable if  $f^+$  and  $f^-$  are integrable, where  $f^+(x) = f(x) \lor 0$ and  $f^-(x) = (-f(x)) \lor 0$ .  $\int_{\mathbb{R}} f(x) dx$  is then defined to be  $\int_{\mathbb{R}} f^+(x) dx - \int_{\mathbb{R}} f^-(x) dx$ .

If f is integrable, then  $\int_{\mathbb{R}} |f(x)| w_n^c(x) dx$  converges to zero.

 $\int_E f(x) dx$  is defined to be  $\int_{\mathbb{R}} \chi_E(x) f(x) dx$ .

A continuous function on a finite closed interval is Riemann integrable, and it is also Lebesgue integrable. The two integrals coincide.

About effective integrability of computable functions, we have the following theorem.

**Theorem 3.1.** ([10]) Let  $\{f_n\}$  be a computable sequence of functions.

- (i)  $\{\int_{[a_m,b_m]} f_n(x)dx\}$  is a computable double sequence of reals for computable sequences  $\{a_m\}$  and  $\{b_m\}$ .
- (ii) If  $\{f_n\}$  converges effectively to f, then  $\{\int_{[a_m,b_m]} f_n(x)dx\}$  converges effectively to  $\{\int_{[a_m,b_m]} f(x)dx\}$  effectively in m.

**Definition 3.2.** (*Effective integrability*) A sequence of computable functions  $\{f_n\}$  is said to be effectively integrable (on  $\mathbb{R}$ ) if the sequences  $\{\int_{[-m,m]} f_n^+(x)dx\}$  and  $\{\int_{[-m,m]} f_n^-(x)dx\}$  converge effectively as m tends to infinity effectively in n.

We denote these limits with  $\int_{\mathbb{R}} f_n^+(x) dx$  and  $\int_{\mathbb{R}} f_n^-(x) dx$ .  $\int_{\mathbb{R}} f_n(x) dx$  is defined to be  $\int_{\mathbb{R}} f_n^+(x) dx - \int_{\mathbb{R}} f_n^-(x) dx$ .

We can easily obtain the following proposition.

**Proposition 3.3.** A computable sequence of functions  $\{f_n\}$  is effectively integrable if and only if  $\int_{\mathbb{R}} |f_n(x)| w_m^c(x) dx$  converges effectively to 0 as m tends to infinity effectively in n.

By Proposition 3.3 and Theorem 3.1, we can prove the following theorem.

**Theorem 3.4.** (Effective dominated convergence theorem for dx) Let  $\{g_{m,n}\}$  be a computable sequence of functions which converges effectively to  $\{f_m\}$ . Assume that there exists an effectively integrable computable sequence of functions  $\{h_m\}$  such that  $|g_{m,n}(x)| \leq h_m(x)$ .

Then  $\{g_{m,n}\}$  is effectively integrable and  $\{\int_{\mathbb{R}} g_{m,n}(x)dx\}$  converges effectively to

 $\{\int_{\mathbb{R}} f_m(x) dx\}$  as n tends to infinity effectively in m.

For the proof of the theorems in the next section, we treat  $\int_{\mathbb{R}} f(x, y) dy$  and  $\int_{\mathbb{R}} f(x, y) \mu(dy)$  for a computable binary function f(x, y).

We say that a sequence of binary functions  $\{f_n\}$  is computable with compact support  $\{L(n)\}$ , if  $\{f_n\}$  is (uniformly) computable and L(n) is a recursive function satisfying  $f_n(x, y) = 0$  for all x, y with  $\min\{|x|, |y|\} \ge L(n)$ .

**Theorem 3.5.** Let  $\{f_n(x,y)\}$  be a computable sequence of binary functions and let  $\{\mu_m\}$  be a computable sequence of probability distributions.

- (1) If  $\{f_n(x,y)\}$  is effectively bounded, then, as a function of x,  $\{\int_{\mathbb{R}} f_n(x,y)\mu_m(dy)\}$  is an effectively bounded computable double sequence of functions.
- (2) If there exists an effectively integrable computable function g(y) such that  $|f_n(x,y)| \le g(y)$ , then  $\{\int_{\mathbb{R}} f_n(x,y) dy\}$  is a computable sequence of functions on the real line.

*Proof.* (1) Sequential computability: Let  $\{x_\ell\}$  be a computable sequence of reals. Then,  $\{f_n(x_\ell, y)\}$  is a computable (double) sequence of functions of y. By the computability of  $\mu_m$  and Proposition 2.9,  $\{\int_{\mathbb{R}} f_n(x_\ell, y)\mu_m(dy)\}$  is computable.

Effective continuity: Let  $\alpha(n, H, k)$  be a recursive modulus of continuity of  $\{f_n\}$ , L(m, k) be a recursive function in Lemma 2.7 and let M(n) be a recursive bound of  $\{f_n\}$ . Then,  $\mu_m(w_{L(m,k)}^c) < 2^{-k}$ . If  $|x|, |z|, |y|, |w| \leq H$  and  $|x - y|, |z - w| < 2^{-\alpha(n,H,k)}$ , then  $|f_n(x, z) - f_n(y, w)| < 2^{-k}$ .

Put  $\ell = \ell(m, k, H) = \max\{H, L(m, k)\}$  and assume  $|x - y| < 2^{-\alpha(n, \ell, k)}$ . Then,  $\mu_m(w_\ell^c) \le \mu_m(w_{L(m,k)}^c) < 2^{-k}$  and

$$\begin{aligned} &|\int_{\mathbb{R}} f_n(x,z)\mu_m(dz) - \int_{\mathbb{R}} f_n(y,z)\mu_m(dz)| \\ &\leq \int_{\mathbb{R}} |f_n(x,z) - f_n(y,z)|w_\ell(z)\mu_m(dz) + \int_{\mathbb{R}} |f_n(x,z) - f_n(y,z)|w_\ell^c(z)\mu_m(dz) \\ &\leq \int_{\mathbb{R}} |f_n(x,z) - f_n(y,z)|w_\ell(z)\mu_m(dz) + 2M(n)2^{-k} \\ &\leq (1+2M(n))2^{-k}. \end{aligned}$$

Hence  $\{\int_{\mathbb{R}} f_n(x, z) \mu_m(dz)\}$  is effectively continuous with respect to

$$\gamma(m, n, H, k) = \alpha(n, \ell(m, k, H), k + 2M(n) + 3) .$$

(2) can be proved similarly by means of Theorem 3.1 and Proposition 3.3.

We explain at this point the computability of a complex-valued function. *i* denotes  $\sqrt{-1}$ , Re(z) denotes the real part of z and Im(z) denotes the imaginary part of z.

A sequence of complex numbers  $\{r_n + is_n\}$  is called recursive if  $\{r_n\}$  and  $\{s_n\}$  are recursive sequences of rationals numbers. Computability of a sequence of complex numbers  $\{z_n\}$  is defined in terms of recursive sequences of complex numbers and it is equivalent to computability of  $\{\operatorname{Re}(z_n)\}$  and  $\{\operatorname{Im}(z_n)\}$ .

Computability of a complex-valued function on  $\mathbb{R}$  is defined by viewing it as a mapping from the metric space  $\mathbb{R}$  to the metric space  $\mathbb{C}$  ([2]). Computability of a sequence of complexvalued functions  $\{f_n\}$  is defined similarly and is equivalent to computability of  $\{\operatorname{Re}(f_n)\}$ 

and  $\{\text{Im}(f_n)\}$ .  $\{f_n\}$  is uniformly computable if and only if  $\{\text{Re}(f_n)\}\)$  and  $\{\text{Im}(f_n)\}\)$  are uniformly computable.

A complex-valued function is called integrable if  $\{\operatorname{Re}(f)\}\$  and  $\{\operatorname{Im}(f)\}\$  are integrable and  $\int_{\mathbb{R}} f(x) dx$  is defined to be  $\int_{\mathbb{R}} \operatorname{Re}(f)(x) dx + i \int_{\mathbb{R}} \operatorname{Im}(f)(x) dx$ .

A sequence of complex-valued functions  $\{f_n\}$  is said to be effectively integrable if each  $f_n$  is integrable and  $\{\int_{\mathbb{R}} \operatorname{Re}(f_n)(x)dx\}$  and  $\{\int_{\mathbb{R}} \operatorname{Im}(f_n)(x)dx\}$  are computable sequences of reals.

Theorem 3.5 can be generalized easily to complex-valued functions.

### 4. CHARACTERISTIC FUNCTIONS

**Theorem 4.1.** If a sequence of probability distributions  $\{\mu_m\}$  is computable, then the corresponding sequence of characteristic functions  $\{\varphi_m\}$  is uniformly computable.

*Proof. Sequential computability:* Let  $\{t_n\}$  be a computable sequence of reals. Then  $\{e^{it_nx}\}$  is a bounded computable sequence of functions of x. Hence the extension of Proposition 2.9 to the complex case shows that  $\{\mu_n(e^{it_nx})\}$  is a computable sequence of complex numbers.

Effective uniform continuity: It is well known that  $|\varphi_m(t+h) - \varphi_m(t)| \le \mu_m(|e^{ihx} - 1|)$ and  $|e^{iz} - 1| \le |z|$  for any real z.

Take a recursive function L(m, k) in Lemma 2.7. Then, we obtain

$$\mu_m(|e^{ihx} - 1|) = \mu(w_{L(m,k)}|e^{ihx} - 1|) + \mu(w_{L(m,k)}^c|e^{ihx} - 1|)$$

$$\leq \int_{\mathbb{R}} w_{L(m,k)}(x)|hx|\mu_m(dx) + 2 \cdot 2^{-k} \leq |h|(L(m,k) + 1)\mu_m(w_{L(m,k)}) + 2 \cdot 2^{-k}$$

$$\leq |h|(L(m,k)+1)+2\cdot 2^{-k}$$

So, if  $|h| < \frac{1}{(L(m,k+2)+1)2^{k+2}}$ , then  $|\varphi_m(t+h) - \varphi_m(t)| < 2^{-k}$ .

**Theorem 4.2.** (Effective Glivenko's theorem, cf. Theorem 2.6.4 in [1]) Let  $\{\varphi_m\}$  and  $\varphi$  be a computable sequence of characteristic functions and a characteristic function respectively, and let  $\{\mu_m\}$  and  $\mu$  be the corresponding probability distributions. Then,  $\{\mu_m\}$  converges effectively to  $\mu$  if  $\{\varphi_m\}$  converges effectively to  $\varphi$ .

*Proof.* We follow the proof of Theorem 2.6.4 in [1] and prove that  $\{\mu_m(f)\}$  converges effectively to  $\mu(f)$  for any computable function f with compact support. Notice that

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{izx} g(z) dz \right) \mu(dx) = \iint_{\mathbb{R} \times \mathbb{R}} g(z) e^{izx} \mu(dx) dz = \int_{\mathbb{R}} \varphi(z) g(z) dz$$
(4.1)

holds for an integrable function g and any pair of a probability distribution  $\mu$  and the corresponding characteristic function  $\varphi$  by virtue of the classical Fubini Theorem. Assume that g is an effectively integrable computable function. Then  $\int_{\mathbb{R}} e^{izx}g(z)dz$  is a bounded uniformly computable function.

Effective convergence of  $\{\varphi_m\}$  to  $\varphi$  implies effective convergence of  $\{\int_{\mathbb{R}} \varphi_m(z)g(z)dz\}$  to  $\int_{\mathbb{R}} \varphi(z)g(z)dz$  by Theorem 3.4. This proves that  $\{\mu_m(h)\}$  converges effectively to  $\mu(h)$  for functions of type  $h(x) = \int_{\mathbb{R}} e^{izx}g(z)dz$  with effectively integrable computable g.

To complete the proof, it is sufficient to prove that any computable function f with compact support can be approximated effectively by a sequence of functions  $\{h_n\}$  of type of h.

Now, define  $g_n(z) = \frac{1}{2\pi} e^{-\frac{z^2}{2n}} \int_{\mathbb{R}} e^{-izy} f(y) dy$  and  $h_n(x) = \int_{\mathbb{R}} e^{izx} g_n(z) dz$ . Then,  $\{g_n\}$  is a computable sequence of functions, and since  $|g_n(z)| \leq \frac{1}{2\pi} e^{-\frac{z^2}{2n}} \int_{\mathbb{R}} |f(y)| dy$ , it is effectively

integrable. We have

$$\begin{aligned} h_n(x) &= \int_{\mathbb{R}} e^{izx} \frac{1}{2\pi} e^{-\frac{z^2}{2n}} dz \int_{\mathbb{R}} e^{-izy} f(y) dy = \frac{1}{2\pi} \int_{\mathbb{R}} f(y) dy \int_{\mathbb{R}} e^{iz(x-y)} e^{-\frac{z^2}{2n}} dz \\ &= \sqrt{\frac{n}{2\pi}} \int_{\mathbb{R}} f(y) dy \int_{\mathbb{R}} e^{iz(x-y)} \frac{1}{\sqrt{2\pi n}} e^{-\frac{z^2}{2n}} dz = \sqrt{\frac{n}{2\pi}} \int_{\mathbb{R}} e^{-\frac{n(x-y)^2}{2}} f(y) dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} f(x + \frac{t}{\sqrt{n}}) dt. \end{aligned}$$

Hence, we obtain the following estimate:

$$\begin{aligned} |h_n(x) - f(x)| &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} |f(x + \frac{t}{\sqrt{n}}) - f(x)| dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{|t| \le L} e^{-\frac{t^2}{2}} |f(x + \frac{t}{\sqrt{n}}) - f(x)| dt + \frac{1}{\sqrt{2\pi}} \int_{|t| > L} e^{-\frac{t^2}{2}} |f(x + \frac{t}{\sqrt{n}}) - f(x)| dt \\ &\le \sup_{|t| \le L} |f(x + \frac{t}{\sqrt{n}}) - f(x)| + \frac{2}{\sqrt{2\pi}} M_f \int_{|t| > L} e^{-\frac{t^2}{2}} dt, \end{aligned}$$

for any L, where  $M_f$  denotes the maximum of f, which is computable. Furthermore,

$$\int_{|t|>L} e^{-\frac{t^2}{2}} dt \le e^{-\frac{L^2}{2}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi} e^{-\frac{L^2}{2}}$$

Finally, let  $\alpha(k)$  be a recursive modulus of continuity of f. For each k, take L = L(k) so large that  $2M_f e^{-\frac{L^2}{2}} < \frac{1}{2^{k+1}}$ , and take  $n > L^2 2^{2\alpha(k+1)}$ . Then, we have  $|h_n(x) - f(x)| < \frac{1}{2^k}$ .

**Theorem 4.3.** Assume that a sequence of probability distributions  $\{\mu_m\}$  converges effectively to a probability distribution  $\mu$ . Then the corresponding sequence of characteristic functions  $\{\varphi_m\}$  converges effectively to  $\varphi$ , which is the corresponding characteristic function of  $\mu$ .

If a sequence of probability distributions  $\{\mu_m\}$  converges effectively to a probability distribution  $\mu$ , then we can replace L(m, k) with  $\alpha(k)$  in the proof of Theorem 4.1. Hence, we obtain the following lemma.

**Lemma 4.4.** Assume that a sequence of probability distributions  $\{\mu_m\}$  converges effectively to a probability distribution  $\mu$ . Then the corresponding characteristic functions  $\{\varphi_m\}$  and  $\varphi$  are effectively uniformly equi-continuous, that is, there exists a recursive function  $\beta(k)$ such that  $|t-s| < 2^{-\beta(k)}$  implies  $|\varphi_m(t) - \varphi_m(s)| < 2^{-k}$  for any m and  $|\varphi(t) - \varphi(s)| < 2^{-k}$ .

Proof of Theorem 4.3. Let  $\{e_{\ell}\}$  be an effective enumaration of all dyadic rationals and  $\beta(k)$  be an effective modulus of uniform equi-continuity in Lemma 4.4.

 $\{e^{ie_{\ell}x}\}$  is an effectively bounded computable sequence of functions. Hence,  $\{\mu_m(e^{ie_{\ell}x})\}$  converges effectively to  $\mu(e^{ie_{\ell}x})$  by the extension of Proposition 2.10 to the complex case, that is, there exists a recursive function  $\gamma(\ell, k)$  such that  $m \geq \gamma(\ell, k)$  implies  $|\mu_m(e^{ie_{\ell}x}) - \mu(e^{ie_{\ell}x})| < 2^{-k}$ .

Let us assume  $|t| \leq M$ . Put  $t_j = -M + j 2^{-\beta(k)}$  (where  $0 \leq j \leq 2M2^{\beta(k)}$ ). Then  $t \in [t_j, t_{j+1})$  for some j. We can find a recursive function  $\xi(j)$  such that  $t_j = e_{\xi(j)}$ . Define

 $\eta(M,k) = \max\{\gamma(\xi(0),\beta(k)),\gamma(\xi(1),\beta(k)),\ldots,\gamma(\xi(2M2^{\beta(k)}),\beta(k))\}.$ 

Suppose that  $|t| \leq M$  and  $m \geq \eta(M, k)$ . Then

 $|\varphi_m(t) - \varphi(t)| \le |\varphi_m(t) - \varphi_m(t_j)| + |\varphi_m(t_j) - \varphi(t_j)| + |\varphi(t_j) - \varphi(t)| < 3 \cdot 2^{-k}.$ Therefore  $\{\varphi_m(t)\}$  converges effectively to  $\varphi(t)$ . Modifying the proof of Bochner's Theorem in [1], we can prove the following two theorems.

**Theorem 4.5.** Let  $\{\mu_m\}$  be probability distributions, and let  $\{\varphi_m\}$  be the corresponding characteristic functions. Then,  $\{\mu_m\}$  is computable if  $\{\varphi_m\}$  is computable.

*Proof.* We prove the theorem for a single  $\mu$  and the corresponding characteristic function  $\varphi$ . First, we note that the following is proved in [1]: If we put

$$f_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t) e^{-\frac{t^2}{n}} e^{-ixt} dt,$$

then  $f_n(x) \ge 0$  and  $\int_{\mathbb{R}} f_n(x) dx = 1$ . By computability of  $\varphi$  and Theorem 3.5 (2),  $\{f_n\}$  is a computable sequence of functions.

Define  $\nu_n(dx) = f_n(x)dx$ . Then,  $\{\nu_n\}$  is a computable sequence of probability distributions.  $\psi_n(z) = \varphi(z)e^{-\frac{z^2}{n}}$  is the characteristic function of  $\nu_n$ .

Computability of  $\{\psi_n\}$  and effective convergence of  $\{\psi_n\}$  to  $\varphi$  are obvious.

Therefore,  $\nu_n$  converges effectively to  $\mu$  by Theorem 4.2 and hence  $\mu$  is computable by Proposition 2.5.

The argument above goes through for a sequence  $\{\mu_m\}$ .

Combining Theorems 4.1 and 4.5, we obtain the following theorem.

**Theorem 4.6.** Let  $\{\mu_m\}$  be probability distributions, and let  $\{\varphi_m\}$  be the corresponding characteristic functions.

Then,  $\{\mu_m\}$  is computable if and only if  $\{\varphi_m\}$  is computable.

**Theorem 4.7.** (Effective Bochner's theorem) In order for  $\varphi(t)$  to be the characteristic function of a computable probability distribution, it is necessary and sufficient that the following three conditions hold.

(i)  $\varphi$  is positive definite. (ii')  $\varphi$  is computable. (iii)  $\varphi(0) = 1$ .

*Proof.*  $\varphi$  is a characteristic function of a probability distribution if and only if the conditions (i), (ii), and (iii) of Theorem 2.11 hold. If (ii) is replaced by (ii'), the corresponding probability distribution is computable by Theorem 4.5.

The converse is a special case of Theorem 4.1.

**Example 4.8.** For  $\delta_0$  and  $\mu_m(dx) = 2^{m-1} \chi_{[-2^{-m}, 2^{-m}]} dx$ ,  $\{\mu_m\}$  converges effectively to  $\delta_0$ .  $\varphi_{\delta_0}(t) \equiv 1$ .

 $\varphi_m(t) = 2^{m-1} \int_{-2^{-m}}^{2^{-m}} e^{itx} dx = 2^m \int_0^{2^{-m}} \cos tx dx = \frac{\sin t 2^{-m}}{t 2^{-m}} \text{ if } t \neq 0, \text{ and } \varphi_m(0) = 1.$  $\{\varphi_m(t)\}$  converges effectively to 1.

# 5. DE MOIVRE-LAPLACE CENTRAL LIMIT THEOREM

Let  $(\Omega, \mathcal{B}, \mathbb{P}, \{X_n\})$  be a realization of Coin Tossing with success probability p, that is,  $(\Omega, \mathcal{B}, \mathbb{P})$  is a probability space and  $\{X_n\}$  is a sequence of independent  $\{0, 1\}$ -valued random variables with the same probability distribution  $\mathbb{P}(X_n = 1) = p$  and  $\mathbb{P}(X_n = 0) = q = 1 - p$ .

The probability distribution of  $S_m = X_1 + \cdots + X_m$  is the binomial distribution  $\mu_m = \sum_{\ell=0}^m {m \choose \ell} p^\ell (1-p)^{m-\ell} \delta_\ell$  and its characteristic function  $\varphi_m(t)$  is equal to  $(pe^{it} + q)^m$ .

**Theorem 5.1.** (Effective de Moivre-Laplace central limit theorem) If p is a computable real number, then the sequence of probability distributions of  $Y_m = \frac{X_1 + \cdots + X_m - mp}{\sqrt{mp(1-p)}} = \sum_{\ell=1}^m \frac{X_\ell - p}{\sqrt{mpq}}$  converges effectively to the standard Gaussian distribution.

Lemma 5.2. (Ito [1]) (1) Put  $R_n(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} i^n e^{it_n} dt_n \cdots dt_1$ . Then  $e^{it} = \sum_{k=0}^{n-1} \frac{(it)^k}{k!} + R_n(t)$  and  $|R_n(t)| \le \frac{|t|^n}{(n)!}$  for any  $t \in \mathbb{R}$ . (2) If  $|z| \le \frac{1}{2}$ , then  $|\log(1+z) - z| \le |z|^2$ .

Outline of the proof of Theorem 5.1: By virtue of Theorem 4.2, it is sufficient to prove effective convergence of  $\log \varphi_m(t)$  to  $-\frac{t^2}{2}$ , since then the corresponding districution is the standard Gaussian. We follow Ito [1] (p. 192). The following holds by definition and Lemma 5.2 (1);

$$\begin{split} \psi_{m}(t) &= \mathbb{E}(e^{itY_{m}}) = \prod_{\ell=1}^{m} \mathbb{E}(e^{it\frac{X_{\ell}-p}{\sqrt{mpq}}}) = (pe^{\frac{it\sqrt{q}}{\sqrt{mp}}} + qe^{-\frac{it\sqrt{p}}{\sqrt{mq}}})^{m} \\ e^{\frac{it\sqrt{q}}{\sqrt{mp}}} &= 1 + it\frac{\sqrt{q}}{\sqrt{mp}} - t^{2}\frac{q}{2mp} + R_{3}(\frac{it\sqrt{q}}{\sqrt{mp}}). \\ e^{-\frac{it\sqrt{p}}{\sqrt{mq}}} &= 1 - it\frac{\sqrt{p}}{\sqrt{mq}} - t^{2}\frac{p}{2mq} + R_{3}(-\frac{it\sqrt{p}}{\sqrt{mq}}). \\ \log\psi_{m}(t) &= m\log(1 - \frac{t^{2}}{2m} + R_{3}(\frac{it\sqrt{q}}{\sqrt{mp}}) + R_{3}(-\frac{it\sqrt{p}}{\sqrt{mq}})). \\ |R_{3}(\frac{it\sqrt{q}}{\sqrt{mp}})| &\leq |t|^{3}(\frac{q}{mp})^{\frac{3}{2}}. \\ R_{3}(-\frac{it\sqrt{p}}{\sqrt{mq}})| &\leq |t|^{3}(\frac{p}{mq})^{\frac{3}{2}}. \end{split}$$

If  $\frac{t^2}{2m} + |t|^3 \left(\frac{q}{mp}\right)^{\frac{3}{2}} + |t|^3 \left(\frac{p}{mq}\right)^{\frac{3}{2}} < \frac{1}{2}$ , then by Lemma 5.2 (2) and the above facts,  $|\log \psi_m(t) - (-\frac{t^2}{2})| \le |t|^3 \left\{ \left(\frac{q}{p}\right)^{\frac{3}{2}} + \left(\frac{p}{q}\right)^{\frac{3}{2}} \right\} \frac{1}{\sqrt{m}} + \left(\frac{t^2}{2m} + |t|^3 \left(\frac{q}{mp}\right)^{\frac{3}{2}} + |t|^3 \left(\frac{p}{mq}\right)^{\frac{3}{2}} \right)^2$ .

The last term converges to zero effectively and uniformly in  $|t| \leq K$  as m tends to infinity.

**Acknowledgment:** The authors would like to express their gratitude to the referees for their valuable suggestions. Improvement of Theorem 4.3 and its proof especially owe to one of the referees.

#### References

- Ito, K. An Introduction to Probability Theory. Cambridge University Press, 1978. (*Probability Theory*. Iwanami Shotenn, 1991, in Japanese.)
- [2] Mori, T., Y. Tsujii and M. Yasugi. Computability Structures on Metric Spaces. Combinatorics, Complexity and Logic (Proceedings of DMTCS'96), ed. by Bridges et al., 351-362. Springer, 1996.
- [3] Mori, T., Y. Tsujii and M. Yasugi. Integral of Fine Computable functions and Walsh Fourier series. ENTCS 202:279-293, 2008.
- Mori, T., M. Yasugi and Y. Tsujii. Effective Fine-convergence of Walsh-Fourier series. MLQ 54, 519-534, 2008.
- [5] Mori, T., M. Yasugi and Y. Tsujii. Integral of Two-dimensional Fine-computable Functions. ENTCS 221, 141-152, 2008.
- Mori, T., M. Yasugi and Y. Tsujii. Fine-computable Functions on the Unit Square and their Integral. JUCS 15, no. 6, 1264-1279, 2009.
- [7] Mori, T., Y. Tsujii and M. Yasugi. Computability of probability distributions and probability distribution functions. Proceedings of the Sixth International Conference on Computability and Complexity in Analysis (DROPS 20), 185-196, 2009.
- [8] Mori, T., Y. Tsujii and M. Yasugi. Fine convergence of functions and its effectivization. Automata, Formal Languages and Algebraic Systems, World Scientific, 2010.

 $|R_3|$ 

#### COMPUTABILITY OF PROBABILITY DISTRIBUTIONS AND CHARACTERISTIC FUNCTIONS 11

- [9] Mori, T., Y. Tsujii and M. Yasugi. Fine Computability of Probability Distribution Functions and Computability of Probability Distributions on the Real Line, communicated at CCA2011.
- [10] Pour-El, M.B. and J. I. Richards. Computability in Analysis and Physics. Springer, 1988.
- [11] Tsurumi, S., Probability Theory. Shibunndou, 1964 (in Japanese).

This work is licensed under the Creative Commons Attribution-NoDerivs License. To view a copy of this license, visit http://creativecommons.org/licenses/by-nd/2.0/ or send a letter to Creative Commons, 171 Second St, Suite 300, San Francisco, CA 94105, USA, or Eisenacher Strasse 2, 10777 Berlin, Germany