

COMPACT MANIFOLDS WITH COMPUTABLE BOUNDARIES

ZVONKO ILJAZOVIĆ

Department of Mathematics, Faculty of Science, University of Zagreb, Croatia
e-mail address: zilj@math.hr

ABSTRACT. We investigate conditions under which a co-computably enumerable closed set in a computable metric space is computable and prove that in each locally computable metric space each co-computably enumerable compact manifold with computable boundary is computable. In fact, we examine the notion of a semi-computable compact set and we prove a more general result: in any computable metric space each semi-computable compact manifold with computable boundary is computable. In particular, each semi-computable compact (boundaryless) manifold is computable.

1. INTRODUCTION

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a computable function and if for some $a < b$ we have $f(a) < 0$ and $f(b) > 0$, then there exists a computable number c between a and b such that $f(c) = 0$ [13]. In general, however, we cannot conclude that a computable function which has a zero-point has a computable zero-point: there exists a computable function $f : \mathbb{R} \rightarrow \mathbb{R}$ which has zero-points, but none of them is computable [12]. However, it is known that if a computable function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, $m \geq 1$, has an isolated zero-point, then that point must be computable. This raises the following question: under what conditions a computable function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ has a computable zero-point, or, even better, under what conditions the set $f^{-1}(\{0\})$ of all zero points of f is computable?

A closed subset of \mathbb{R}^m is computable if it can be effectively approximated by a finite set of points with rational coordinates with arbitrary precision on an arbitrary bounded region of \mathbb{R}^m . Each nonempty computable set contains computable points, in fact they are dense in it. On the other hand, if $S \subseteq \mathbb{R}^m$, then S is equal to $f^{-1}(\{0\})$ for some computable function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ if and only if the complement of S can be effectively covered by open balls. A closed subset of \mathbb{R}^m is called co-computably enumerable (co-c.e.) if its complement can be effectively covered by open balls. So the question under what conditions a set of the form $f^{-1}(\{0\})$ is computable can be restated in the following way: under what conditions a co-c.e. set is computable, i.e. under what conditions the implication

$$S \text{ co-computably enumerable closed set} \Rightarrow S \text{ computable closed set} \quad (1.1)$$

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holds?

The implication (1.1) does not hold in general, moreover, as mentioned, there exists a nonempty co-c.e. subset of \mathbb{R} which contains no computable points. Such a set cannot be connected (otherwise it would consist of a single point or it would contain an open interval). It is easy to construct a connected co-c.e. subset of \mathbb{R}^2 which contains no computable points, but it is an interesting question is it possible to achieve that such a set is even simply connected [11]. Kihara [8] recently showed that such a set does exist.

Regarding the implication (1.1), it fails to be true even for sets which are very simple from the topological viewpoint: in each \mathbb{R}^m there exists a line segment which is co-c.e., but not computable [9]. However, there are certain conditions under which this implication holds. In [6] it was proved that (1.1) holds when S is a circularly chainable continuum or a continuum chainable from a to b , where a and b are computable points. Brattka [1] gave some results concerning (1.1) in the case when S is the graph of a continuous function. Miller [9] proved that (1.1) holds whenever $S \subseteq \mathbb{R}^m$ is a topological sphere (i.e. homeomorphic to the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ for some n) or S is homeomorphic to the closed unit ball $B^n \subseteq \mathbb{R}^n$ for some n (i.e. S is an n -cell) by a homeomorphism $f : B^n \rightarrow S$ such that $f(S^{n-1})$ is co-c.e. set in \mathbb{R}^m . Furthermore, by [7], these results hold not just in \mathbb{R}^m , but also in any computable metric space which is locally computable.

This, for example, means that each co-c.e. arc in \mathbb{R}^m is computable if its endpoints are computable. Hence the computability of endpoints implies the computability of the whole arc. And if we have a co-c.e. cell, then the computability of its boundary sphere implies the computability of the whole cell. Since arcs and cells are examples of spaces known as manifolds with boundary, the question which arises here is what can be said about computability of such spaces, i.e. if we have a co-c.e. manifold with boundary, does the computability of its boundary imply the computability of the whole manifold?

In this paper we answer this question. We prove that if (X, d, α) is a locally computable computable metric space and if S is a co-c.e. closed set in (X, d, α) which is, as a subspace of (X, d) , a compact manifold with boundary, then the computability of ∂S implies the computability of S . This in particular means that a co-c.e. set S is computable if it is a (boundaryless) manifold. A manifold with boundary is a topological space in which each point is contained in some open set which is homeomorphic to Euclidean space \mathbb{R}^n or Euclidean half-space $\{(x_1, \dots, x_n) \mid x_n \geq 0\}$. If X is a manifold with boundary, then the boundary ∂X of X consists of all points $x \in X$ such that none of the open sets containing x is homeomorphic to Euclidean space. If X is a manifold with boundary and $\partial X = \emptyset$, than we simply say that X is a manifold.

For example, the unit sphere S^n is a manifold and the unit ball B^n is a manifold with boundary, its boundary is S^{n-1} . Moreover, each topological sphere is a manifold and if $f : B^n \rightarrow X$ is a homeomorphism, then X is a manifold with boundary and $\partial X = f(S^{n-1})$.

In fact, we will prove a result which turns out to be more general: in any computable metric space (X, d, α) the implication

$$S \text{ semi-computable compact set} \Rightarrow S \text{ computable compact set} \quad (1.2)$$

holds if S is a compact manifold with computable boundary. That S is a semi-computable compact set means that S is compact and we can effectively enumerate all rational open sets which cover S . In other words, we will prove that if S is semi-computable compact manifold with boundary in (X, d, α) then the following implication holds:

$$\partial S \text{ computable} \Rightarrow S \text{ computable.}$$

In order to prove this, the central notion will be the notion of computability up to some set. Using techniques from [6] and [7] (in particular using n -dimensional chains) we will see that in each semi-computable compact manifold with computable boundary each point has a neighborhood which is computable up to the manifold. This fact will then imply that a semi-computable compact manifold with computable boundary is computable.

In Section 2 we give some basic notions and in Section 3 we examine semi-computable compact sets. In Section 4 we introduce the notion of computability up to a set. In Section 5 we examine n -chains and prove that in each semi-computable compact manifold with computable boundary each point has a neighborhood which is computable up to the manifold (Theorem 5.6). In Section 6 we prove that semi-computable compact manifolds with computable boundaries are computable.

2. BASIC NOTIONS AND TECHNIQUES

If X is a set, let $\mathcal{P}(X)$ denote the set of all subsets of X .

For $m \in \mathbb{N}$ let $\mathbb{N}_m = \{0, \dots, m\}$. For $n \geq 1$ let

$$\mathbb{N}_m^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{N}_m\}.$$

We say that a function $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ is **computable** if the function $\bar{\Phi} : \mathbb{N}^{k+n} \rightarrow \mathbb{N}$ defined by

$$\bar{\Phi}(x, y) = \chi_{\Phi(x)}(y),$$

$x \in \mathbb{N}^k$, $y \in \mathbb{N}^n$, is computable (i.e. recursive). Here $\chi_S : \mathbb{N}^n \rightarrow \{0, 1\}$ denotes the characteristic function of $S \subseteq \mathbb{N}^n$. A function $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ is said to be **computably bounded** if there exists a computable function $\varphi : \mathbb{N}^k \rightarrow \mathbb{N}$ such that $\Phi(x) \subseteq \mathbb{N}_{\varphi(x)}^n$ for all $x \in \mathbb{N}^k$.

We say that a function $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ is **c.c.b.** if Φ is computable and computably bounded.

Proposition 2.1.

- (1) If $\Phi, \Psi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ are c.c.b. functions, then the function $\mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$, $x \mapsto \Phi(x) \cup \Psi(x)$ is c.c.b.
- (2) If $\Phi, \Psi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ are c.c.b. functions, then the sets $\{x \in \mathbb{N}^k \mid \Phi(x) = \Psi(x)\}$ and $\{x \in \mathbb{N}^k \mid \Phi(x) \subseteq \Psi(x)\}$ are decidable.
- (3) Let $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ and $\Psi : \mathbb{N}^n \rightarrow \mathcal{P}(\mathbb{N}^m)$ be c.c.b. functions. Let $\Lambda : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^m)$ be defined by

$$\Lambda(x) = \bigcup_{z \in \Phi(x)} \Psi(z),$$

$x \in \mathbb{N}^k$. Then Λ is a c.c.b. function.

- (4) Let $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ be c.c.b. and let $T \subseteq \mathbb{N}^n$ be c.e. Then the set $S = \{x \in \mathbb{N}^k \mid \Phi(x) \subseteq T\}$ is c.e. □

A function $F : \mathbb{N}^k \rightarrow \mathbb{Q}$ is called **computable** if there exist computable functions $a, b, c : \mathbb{N}^k \rightarrow \mathbb{N}$ such that

$$F(x) = (-1)^{c(x)} \frac{a(x)}{b(x) + 1}$$

for each $x \in \mathbb{N}^k$. A number $x \in \mathbb{R}$ is said to be **computable** if there exists a computable function $g : \mathbb{N} \rightarrow \mathbb{Q}$ such that $|x - g(i)| < 2^{-i}$ for each $i \in \mathbb{N}$ [14].

By a **computable** function $\mathbb{N}^k \rightarrow \mathbb{R}$ we mean a function $f : \mathbb{N}^k \rightarrow \mathbb{R}$ for which there exists a computable function $F : \mathbb{N}^{k+1} \rightarrow \mathbb{Q}$ such that

$$|f(x) - F(x, i)| < 2^{-i}$$

for all $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$.

In the following proposition we state some facts about computable functions $\mathbb{N}^k \rightarrow \mathbb{R}$.

Proposition 2.2.

- (1) If $f, g : \mathbb{N}^k \rightarrow \mathbb{R}$ are computable, then $f + g, f - g, \sqrt{|f|} : \mathbb{N}^k \rightarrow \mathbb{R}$ are computable.
- (2) If $f : \mathbb{N}^k \rightarrow \mathbb{R}$ and $F : \mathbb{N}^{k+1} \rightarrow \mathbb{R}$ are functions such that F is computable and $|f(x) - F(x, i)| < 2^{-i}$ for each $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$, then f is computable.
- (3) If $f, g : \mathbb{N}^k \rightarrow \mathbb{R}$ are computable functions, then the set $\{x \in \mathbb{N}^k \mid f(x) > g(x)\}$ is c.e.
- (4) If $f : \mathbb{N}^k \rightarrow \mathbb{R}$ is a computable function and $\Phi : \mathbb{N}^n \rightarrow \mathcal{P}(\mathbb{N}^k)$ is a c.c.b. function such that $\Phi(x) \neq \emptyset$ for each $x \in \mathbb{N}^n$, then the functions $g, h : \mathbb{N}^n \rightarrow \mathbb{R}$ defined by

$$g(x) = \max_{y \in \Phi(x)} f(y), \quad h(x) = \min_{y \in \Phi(x)} f(y)$$

are computable. □

2.1. Computable metric spaces. A tuple (X, d, α) is said to be a **computable metric space** if (X, d) is a metric space and $\alpha : \mathbb{N} \rightarrow X$ is a sequence dense in (X, d) (i.e. a sequence whose range is dense in (X, d)) such that the function $\mathbb{N}^2 \rightarrow \mathbb{R}$,

$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$

is computable (we use notation $\alpha = (\alpha_i)$).

If (X, d, α) is a computable metric space, then a sequence (x_i) in X is said to be **computable** in (X, d, α) if there exists a computable function $F : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that

$$d(x_i, \alpha_{F(i,k)}) < 2^{-k}$$

for all $i, k \in \mathbb{N}$. A point $a \in X$ is said to be **computable** in (X, d, α) if there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $d(a, \alpha_{f(k)}) < 2^{-k}$ for each $k \in \mathbb{N}$.

The points $\alpha_0, \alpha_1, \dots$ are called **rational points**. If $i \in \mathbb{N}$ and $q \in \mathbb{Q}$, $q > 0$, then we say that $B(\alpha_i, q)$ is an (open) **rational ball** and $\widehat{B}(\alpha_i, q)$ is a **closed rational ball**. Here, for $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball of radius r centered at x and by $\widehat{B}(x, r)$ the corresponding closed ball, i.e. $B(x, r) = \{y \in X \mid d(x, y) < r\}$, $\widehat{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}$.

If B_1, \dots, B_n , $n \geq 1$, are open rational balls, then the union $B_1 \cup \dots \cup B_n$ will be called a **rational open set**.

Example 2.3. If $\alpha : \mathbb{N} \rightarrow \mathbb{R}^n$ is a computable function (in the sense that the component functions of α are computable) whose image is dense in \mathbb{R}^n and d is the Euclidean metric on \mathbb{R}^n , then $(\mathbb{R}^n, d, \alpha)$ is a computable metric space (Proposition 2.2). A sequence (x_i) is computable in this computable metric space if and only if (x_i) is a computable sequence in \mathbb{R}^n and $(x_1, \dots, x_n) \in \mathbb{R}^n$ is a computable point in this space if and only if x_1, \dots, x_n are computable numbers.

2.2. Effective enumerations. Let (X, d, α) be a computable metric space. Let $q : \mathbb{N} \rightarrow \mathbb{Q}$ be some fixed computable function whose image is $\mathbb{Q} \cap \langle 0, \infty \rangle$ and let $\tau_1, \tau_2 : \mathbb{N} \rightarrow \mathbb{N}$ be some fixed computable functions such that $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2$. Let $(\lambda_i)_{i \in \mathbb{N}}$ be the sequence of points in X defined by $\lambda_i = \alpha_{\tau_1(i)}$ and let $(\rho_i)_{i \in \mathbb{N}}$ be the sequence of rational numbers defined by $\rho_i = q_{\tau_2(i)}$. For $i \in \mathbb{N}$ we define

$$I_i = B(\lambda_i, \rho_i), \quad \widehat{I}_i = \widehat{B}(\lambda_i, \rho_i).$$

The sequences (I_i) and (\widehat{I}_i) represent effective enumerations of all open rational balls and all closed rational balls.

Now we want to define some effective enumeration of all rational open sets.

Let $\sigma : \mathbb{N}^2 \rightarrow \mathbb{N}$ and $\eta : \mathbb{N} \rightarrow \mathbb{N}$ be some fixed computable functions with the following property: $\{(\sigma(j, 0), \dots, \sigma(j, \eta(j))) \mid j \in \mathbb{N}\}$ is the set of all finite sequences in \mathbb{N} (excluding the empty sequence), i.e. the set $\{(a_0, \dots, a_n) \mid n \in \mathbb{N}, a_0, \dots, a_n \in \mathbb{N}\}$. Such functions, for instance, can be defined using the Cantor pairing function. We use the following notation: $(j)_i$ instead of $\sigma(j, i)$ and \bar{j} instead of $\eta(j)$. Hence

$$\{((j)_0, \dots, (j)_{\bar{j}}) \mid j \in \mathbb{N}\}$$

is the set of all finite sequences in \mathbb{N} . For $j \in \mathbb{N}$ let $[j]$ be defined by

$$[j] = \{(j)_i \mid 0 \leq i \leq \bar{j}\}. \quad (2.1)$$

Note that the function $\mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$, $j \mapsto [j]$, is c.c.b.

For $j \in \mathbb{N}$ we define

$$J_j = \bigcup_{i \in [j]} I_i.$$

Then (J_j) is an effective enumeration of all rational open sets.

For $i \in \mathbb{N}$ we define $\Lambda_i = \alpha([i])$, hence

$$\Lambda_i = \{\alpha_{(i)_0}, \dots, \alpha_{(i)_{\bar{i}}}\}.$$

The sequence (Λ_i) is an effective enumeration of all finite nonempty sets of rational points.

2.3. Formal diameter and formal disjointness. In Euclidean space \mathbb{R}^n we can effectively calculate the diameter of the finite union of rational balls. However, in a general computable metric space the function $\mathbb{N} \rightarrow \mathbb{R}$, $j \mapsto \text{diam}(J_j)$, need not be computable. For that reason we are going to use the notion of the formal diameter. Let (X, d) be a metric space and $x_0, \dots, x_k \in X$, $r_0, \dots, r_k \in \mathbb{R}_+$. The **formal diameter** associated to the finite sequence $(x_0, r_0), \dots, (x_k, r_k)$ is the number $D \in \mathbb{R}$ defined by

$$D = \max_{0 \leq v, w \leq k} d(x_v, x_w) + 2 \max_{0 \leq v \leq k} r_v.$$

It follows from this definition that $\text{diam}(B(x_0, r_0) \cup \dots \cup B(x_k, r_k)) \leq D$.

Let (X, d, α) be a computable metric space. We define the function $\text{fdiam} : \mathbb{N} \rightarrow \mathbb{R}$ in the following way. For $j \in \mathbb{N}$ the number $\text{fdiam}(j)$ is the formal diameter associated to the finite sequence

$$(\lambda_{(j)_0}, \rho_{(j)_0}), \dots, (\lambda_{(j)_{\bar{j}}}, \rho_{(j)_{\bar{j}}}).$$

Clearly $\text{diam}(J_j) \leq \text{fdiam}(j)$ for each $j \in \mathbb{N}$.

Let $i, j \in \mathbb{N}$. We say that I_i and I_j are **formally disjoint** if

$$d(\lambda_i, \lambda_j) > \rho_i + \rho_j.$$

Note that we define this as a relation between the numbers i and j , not the sets I_i and I_j (it is possible that for some i, i', j, j' we have that I_i and I_j are formally disjoint and $I_i = I_{i'}$, $I_j = I_{j'}$, but $I_{i'}$ and $I_{j'}$ are not formally disjoint, for example such a situation can occur if d is the discrete metric on X). If I_i and I_j are formally disjoint, then $I_i \cap I_j = \emptyset$.

Let $i, j \in \mathbb{N}$. We say that J_i and J_j are **formally disjoint** if I_k and I_l are formally disjoint for all $k \in [i]$ and $l \in [j]$. Clearly, if J_i and J_j are formally disjoint, then $J_i \cap J_j = \emptyset$.

Proposition 2.4.

- (1) *The function $\text{fdiam} : \mathbb{N} \rightarrow \mathbb{R}$ is computable.*
- (2) *The set $\{(i, j) \in \mathbb{N}^2 \mid J_i \text{ and } J_j \text{ are formally disjoint}\}$ is c.e.*

Proof. This follows from Proposition 2.1 and Proposition 2.2 (for details see [6], Proposition 13 and Proposition 8). □

2.4. Computable sets. Let (X, d, α) be a computable metric space. We say that S is a **computably enumerable closed set** in (X, d, α) if S is a closed subset of (X, d) and if

$$\{i \in \mathbb{N} \mid S \cap I_i \neq \emptyset\}$$

is a c.e. subset of \mathbb{N} . A closed subset of (X, d) is said to be **co-computably enumerable closed set** in (X, d, α) if there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$X \setminus S = \bigcup_{i \in \mathbb{N}} I_{f(i)}.$$

It is easy to see that these definitions do not depend on functions τ_1 , τ_2 and q . We say that S is a **computable closed set** in (X, d, α) if S is a computably enumerable closed set and a co-computably enumerable closed set [2, 15].

We say that K is a **computable compact set** in (X, d, α) if K is a compact set in (X, d) , the set $\{j \in \mathbb{N} \mid K \subseteq J_j\}$ is c.e. and K is a c.e. closed set in (X, d, α) .

2.5. Hausdorff metric. Let (X, d, α) be a computable metric space. Let $A, B \subseteq X$ and $\varepsilon > 0$. We write $A \prec_\varepsilon B$ if for each $a \in A$ there exists $b \in B$ such that $d(a, b) < \varepsilon$. We write

$$A \approx_\varepsilon B$$

if $A \prec_\varepsilon B$ and $B \prec_\varepsilon A$. Note that $A \prec_\varepsilon B$ and $B \prec_{\varepsilon'} C$ imply $A \prec_{\varepsilon+\varepsilon'} C$. Also note that $A \prec_\varepsilon B$ and $A' \prec_\varepsilon B'$ imply $A \cup A' \prec_\varepsilon B \cup B'$.

Let \mathcal{H} be the set of all nonempty compact subsets of (X, d) . The **Hausdorff metric** on \mathcal{H} is the metric ϱ on \mathcal{H} defined by

$$\varrho(A, B) = \inf\{\varepsilon > 0 \mid A \approx_\varepsilon B\}.$$

If $A, B \in \mathcal{H}$ and $\varepsilon > 0$, then

$$A \prec_\varepsilon B \Leftrightarrow d(a, B) < \varepsilon \text{ for each } a \in A \Leftrightarrow \max_{a \in A} d(a, B) < \varepsilon$$

(the last equivalence follows from the fact that the continuous function $X \rightarrow \mathbb{R}$, $x \mapsto d(x, B)$ attains its maximum on A). Therefore, for all $A, B \in \mathcal{H}$ we have

$$\varrho(A, B) = \max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(b, A) \right\}. \quad (2.2)$$

Also note that for $A, B \in \mathcal{H}$ and $\varepsilon > 0$ we have $\varrho(A, B) < \varepsilon$ if and only if $A \approx_\varepsilon B$.

Proposition 2.5. *Let (X, d, α) be a computable metric space. The function $\mathbb{N}^2 \rightarrow \mathbb{R}$, $(i, j) \mapsto \varrho(\Lambda_i, \Lambda_j)$ is computable.*

Proof. This follows from (2.2) and Proposition 2.2(4). \square

It is easy to conclude that the sequence (Λ_i) is dense in the metric space (\mathcal{H}, ϱ) . Therefore, the triple $(\mathcal{H}, \varrho, \Lambda)$ is a computable metric space. The following proposition says that computable points in this space are exactly nonempty computable compact sets in (X, d, α) (see also Theorem 4.12. in [2]).

Proposition 2.6. *Let (X, d, α) be a computable metric space and let K be a nonempty compact set in (X, d) . Then K is a computable compact set in (X, d, α) if and only if there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that*

$$\varrho(K, \Lambda_{f(k)}) < 2^{-k} \quad (2.3)$$

for each $k \in \mathbb{N}$.

Proof. Suppose K is a computable compact set. Then the set $\{i \in \mathbb{N} \mid I_i \cap K \neq \emptyset\}$ is c.e. and we conclude from Proposition 2.1(4) that the set $\{j \in \mathbb{N} \mid I_i \cap K \neq \emptyset \text{ for each } i \in [j]\}$ is c.e. Similarly, using also Proposition 2.2(3), we get that the set

$$\{(j, k) \in \mathbb{N}^2 \mid \rho_i < 2^{-k} \text{ for each } i \in [j]\}$$

is c.e. Therefore the set

$$\Omega = \{(j, k) \in \mathbb{N}^2 \mid K \subseteq J_j, I_i \cap K \neq \emptyset \text{ and } \rho_i < 2^{-k} \text{ for each } i \in [j]\}$$

is c.e. Since for each $k \in \mathbb{N}$ there exist $j \in \mathbb{N}$ such that $(j, k) \in \Omega$, there exists a computable function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that $(\varphi(k), k) \in \Omega$ for each $k \in \mathbb{N}$. Since the function $k \mapsto [k]$ is c.c.b. and each nonempty subset of \mathbb{N} is equal to $[k]$ for some $k \in \mathbb{N}$, Proposition 2.1 implies that there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $[f(k)] = \{\tau_1(i) \mid i \in [\varphi(k)]\}$ for each $k \in \mathbb{N}$. Then

$$\Lambda_{f(k)} = \{\lambda_i \mid i \in [\varphi(k)]\}$$

and it follows that $\varrho(K, \Lambda_{f(k)}) < 2^{-k}$.

Conversely, suppose that (2.3) holds for some computable function $f : \mathbb{N} \rightarrow \mathbb{N}$. We want to prove that K is a computable compact set.

If $x, y \in X$ and $r, s > 0$, we will say that (y, s) is formally contained in (x, r) and write $(y, s) \subseteq_F (x, r)$ if $d(x, y) + s < r$. Note that $(y, s) \subseteq_F (x, r)$ implies $B(y, s) \subseteq B(x, r)$.

Suppose that $K \subseteq J_j$ for some $j \in \mathbb{N}$. Using compactness of K it is not hard to conclude (see Lemma 19 in [6]) that there exists $\mu > 0$ with property that for each $x \in K$ there exists $i \in [j]$ such that $(x, 2\mu) \subseteq_F (\lambda_i, \rho_i)$. Choose $k \in \mathbb{N}$ so that $2^{-k} < \mu$. Let $l \in [f(k)]$. Since

(2.3) holds, there exists $x \in K$ such that $d(\alpha_l, x) < \mu$. On the other hand, there exists $i \in [j]$ such that $(x, 2\mu) \subseteq_F (\lambda_i, \rho_i)$. It follows $(\alpha_l, \mu) \subseteq_F (\lambda_i, \rho_i)$. Hence

$$(\forall l \in [f(k)]) (\exists i \in [j]) (\alpha_l, 2^{-k}) \subseteq_F (\lambda_i, \rho_i). \quad (2.4)$$

However, if $j, k \in \mathbb{N}$ are such that (2.4) holds, then we have $K \subseteq J_j$. Namely, for each $x \in K$ by (2.3) there exists $l \in [f(k)]$ such that $d(x, \alpha_l) < 2^{-k}$ and, by (2.4), there exists $i \in [j]$ such that $d(\alpha_l, \lambda_i) + 2^{-k} < \rho_i$ from which we get $d(x, \lambda_i) < \rho_i$. Hence $x \in J_j$.

Therefore $K \subseteq J_j$ if and only if there exists $k \in \mathbb{N}$ such that (2.4) holds. Since the set of all $(j, k) \in \mathbb{N}^2$ such that (2.4) holds is c.e. (which follows from Proposition 2.2 and Proposition 2.1), the set $\{j \in \mathbb{N} \mid K \subseteq J_j\}$ is c.e.

Let $i \in \mathbb{N}$. Then using (2.3) we conclude that

$$K \cap I_i \neq \emptyset \Leftrightarrow (\exists x \in K) d(x, \lambda_i) < \rho_i \Leftrightarrow (\exists k \in \mathbb{N}) (\exists l \in [f(k)]) d(\alpha_l, \lambda_i) + 2^{-k} < \rho_i$$

and it follows from Proposition 2.2 that K is a c.e. closed set. \square

3. SEMI-COMPUTABLE COMPACT SETS

Let (X, d, α) be a computable metric space. We say that K is a **semi-computable compact set** in (X, d, α) if K is a compact set in (X, d) and if the set $\{j \in \mathbb{N} \mid K \subseteq J_j\}$ is c.e. Hence K is a computable compact set if and only if K is a semi-computable compact set and K is a c.e. closed set.

Proposition 3.1. *Let (X, d, α) be a computable metric space. If S is a semi-computable compact set in (X, d, α) , then S is a co-c.e. closed set in (X, d, α) .*

Proof. Let L be the set of all $l \in \mathbb{N}$ for which there exists $j \in \mathbb{N}$ such that $S \subseteq J_j$ and such that I_l is formally disjoint with I_i for each $i \in [j]$. Since the set $\{(l, i) \in \mathbb{N}^2 \mid I_l \text{ and } I_i \text{ are formally disjoint}\}$ is c.e. by Proposition 2.2, L is c.e. Clearly $\bigcup_{l \in L} I_l \subseteq X \setminus S$.

On the other hand, let $x \in X \setminus S$. Then, since S is compact, there exists $\mu > 0$ such that $d(x, s) > 4\mu$ for each $s \in S$. Compactness of S furthermore implies that there exists $j \in \mathbb{N}$ such that $S \subseteq J_j$ and such that $\rho_i < \mu$ and $I_i \cap S \neq \emptyset$ for each $i \in [j]$. Choose $l \in \mathbb{N}$ so that $x \in I_l$ and $\rho_l < \mu$. It is straightforward to check that I_l and I_i are formally disjoint for each $i \in [j]$. Hence $l \in L$ and this proves that $X \setminus S \subseteq \bigcup_{l \in L} I_l$, hence $X \setminus S = \bigcup_{l \in L} I_l$. \square

The preceding proposition implies that each computable compact set is computable closed set.

A computable metric space (X, d, α) is **locally computable** [1] if for each compact set A in (X, d) there exists a computable compact set K in (X, d, α) such that $A \subseteq K$. For example, the computable metric space $(\mathbb{R}^n, d, \alpha)$ from Example 2.3 is locally computable (this can be deduced from Proposition 2.6).

Proposition 3.2. *Let (X, d, α) be a computable metric space which is locally computable. Let S be a co-c.e. closed set in (X, d, α) which is compact. Then S is a semi-computable compact set in (X, d, α) .*

Proof. Since S is co-c.e., there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$X \setminus S = \bigcup_{n \in \mathbb{N}} J_{f(n)}$$

and $J_{f(n)} \subseteq J_{f(n+1)}$ for each $n \in \mathbb{N}$. Furthermore, there exists a computable compact set K such that $S \subseteq K$.

Let $j \in \mathbb{N}$. If $S \subseteq J_j$, then $K \setminus J_j$ is a compact subset of $X \setminus S$ and therefore there exists $n \in \mathbb{N}$ such that $K \setminus J_j \subseteq J_{f(n)}$. We have the following conclusion:

$$S \subseteq J_j \Leftrightarrow (\exists n \in \mathbb{N}) K \subseteq J_j \cup J_{f(n)}. \quad (3.1)$$

By Proposition 2.1(1) the function $\mathbb{N}^2 \rightarrow \mathcal{P}(\mathbb{N})$, $(j, n) \mapsto [j] \cup [f(n)]$ is c.c.b. and therefore by claim (2) of the same proposition there exists a computable function $g : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $[j] \cup [f(n)] = [g(j, n)]$ for all $j, n \in \mathbb{N}$. Hence $J_j \cup J_{f(n)} = J_{g(j, n)}$ and the set of all $(j, n) \in \mathbb{N}^2$ such that $K \subseteq J_{g(j, n)}$ is c.e. since K is semi-computable. Now (3.1) implies that S is semi-computable. \square

In this paper we will prove that in every computable metric space (X, d, α) the implication

$$S \text{ semi-computable compact set} \Rightarrow S \text{ computable compact set}$$

holds for each compact manifold S with computable boundary. Using this and Proposition 3.2 we conclude that if (X, d, α) is locally computable, then for every compact manifold S with computable boundary the following implication holds:

$$S \text{ co-c.e. closed} \Rightarrow S \text{ computable closed}. \quad (3.2)$$

Lemma 3.3. *Let (X, d, α) be a computable metric space and let S be a semi-computable compact set in this space. Let $m \in \mathbb{N}$. Then $S \setminus J_m$ is a semi-computable compact set.* \square

Proof. Let $j \in \mathbb{N}$. Then $S \setminus J_m \subseteq J_j$ if and only if $S \subseteq J_j \cup J_m$ and it follows that $S \setminus J_m$ is semi-computable (similarly as in the proof of the previous proposition). \square

4. COMPUTABILITY UP TO A SET

Let (X, d, α) be a computable metric space. Let $A, B \subseteq X$, $A \subseteq B$. We say that A is **computable up to B** if there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$A \prec_{2^{-k}} \Lambda_{f(k)} \text{ and } \Lambda_{f(k)} \prec_{2^{-k}} B$$

for each $k \in \mathbb{N}$. Using Proposition 2.6 we conclude that for each nonempty compact subset A of (X, d) the following equivalence holds:

$$A \text{ is a computable compact set} \Leftrightarrow A \text{ is computable up to } A.$$

Proposition 4.1. *Let (X, d, α) be a computable metric space. Suppose A, B and S are subsets of X such that $A \subseteq S$ and $B \subseteq S$ and such that A and B are computable up to S . Then $A \cup B$ is computable up to S .*

Proof. Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be computable functions such that $A \prec_{2^{-k}} \Lambda_{f(k)} \prec_{2^{-k}} S$ and $B \prec_{2^{-k}} \Lambda_{g(k)} \prec_{2^{-k}} S$ for each $k \in \mathbb{N}$. Then

$$A \cup B \prec_{2^{-k}} (\Lambda_{f(k)} \cup \Lambda_{g(k)}) \prec_{2^{-k}} S$$

for each $k \in \mathbb{N}$. Let $h : \mathbb{N} \rightarrow \mathbb{N}$ be a computable function such that $[h(k)] = [f(k)] \cup [g(k)]$ for each $k \in \mathbb{N}$. Then $\Lambda_{f(k)} \cup \Lambda_{g(k)} = \alpha([f(k)] \cup [g(k)]) = \Lambda_{h(k)}$ for each $k \in \mathbb{N}$ and the claim follows. \square

Proposition 4.1 implies: if $A_1, \dots, A_n \subseteq S$ and A_i is computable up to S for each $i \in \{1, \dots, n\}$, then $A_1 \cup \dots \cup A_n$ is computable up to S . Therefore, we have the following statement.

Proposition 4.2. *Let (X, d, α) be a computable metric space. Let S be a compact subset of (X, d) and suppose A_1, \dots, A_n are subsets of S such that $S = A_1 \cup \dots \cup A_n$ and such that A_i is computable up to S for each $i \in \{1, \dots, n\}$. Then S is a computable compact set. \square*

5. LOCALIZATION OF COMPUTABILITY

That S is a semi-computable compact set in a computable metric space means that we can effectively enumerate all rational open sets which contain S . However we don't know, in general, how close U is to S for a given rational open set U containing S if S is not computable. On the other hand, if we can, for a given $\varepsilon > 0$, effectively find a rational open set U which contains S and such that $U \prec_{\varepsilon} S$, then we will have $S \approx_{\varepsilon} U$ and it will follow from Proposition 2.6 that S is a computable compact set.

In order to effectively find, for a given $\varepsilon > 0$, such a rational open set U , it would be enough to effectively find finitely many sets C_0, \dots, C_m whose union contains S , such that the diameter of each of these sets is less than ε and such that each of these sets intersects S . Then $C_0 \cup \dots \cup C_m \prec_{\varepsilon} S$ and the computability of S would follow.

This can be done in certain situations. For example, let us observe the case when S is an arc. Then for each $\varepsilon > 0$ there exists a finite sequence of rational open sets C_0, \dots, C_m whose union contains S such that $C_i \cap C_j = \emptyset$ whenever $|i - j| > 1$ and such that $\text{diam}(C_i) < \varepsilon$ for each $i \in \{0, \dots, m\}$. And if S is a semi-computable compact set, a sequence C_0, \dots, C_m with these properties can be found effectively. Still, this is not enough to conclude that each of these sets intersects S . However, if the arc S has computable endpoints a and b , then the sequence C_0, \dots, C_m can be found so that $a \in C_0$ and $b \in C_m$. Now we can conclude that each C_i intersects S , otherwise $C_0 \cup \dots \cup C_{i-1}$ and $C_{i+1} \cup \dots \cup C_m$ would be disjoint open sets, each of them would intersect S and S would be contained in their union which is impossible since S is connected.

Suppose now that S is homeomorphic to $[0, 1] \times [0, 1]$. We can proceed in a similar way as in the case of an arc. For a given $\varepsilon > 0$ we want sets $C_{i,j}$, $0 \leq i, j \leq m$ whose union contains S , whose diameters are less than ε and such that $C_{i,j}$ and $C_{i',j'}$ are disjoint when $|i - i'| > 1$ or $|j - j'| > 1$. It turns out that, under some additional assumptions, we can do this in a satisfactory way [7], however when at the end we want to conclude that each of these sets intersects S , the fact that S is connected is not enough for this conclusion and here we need some deeper topological facts about the space $[0, 1]^n$ (see the discussion on pages 5, 6 and 7 in [7]).

For $n \geq 1$ and $i \in \{1, \dots, n\}$ let

$$A_i^n = \{(x_1, \dots, x_n) \in [-2, 2]^n \mid x_i = -2\}, \quad (5.1)$$

$$B_i^n = \{(x_1, \dots, x_n) \in [-2, 2]^n \mid x_i = 2\}. \quad (5.2)$$

(Here $[-2, 2]^n$ denotes the set $\{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in [-2, 2]\}$.) When the context is clear, we write A_i and B_i instead of A_i^n and B_i^n .

The following topological fact will be crucial in the proof of our main result (see [4], Theorem 1.8.1, and [7], Corollary 3.3.).

Theorem 5.1. *Let $n \geq 1$. Suppose U_1, \dots, U_n and V_1, \dots, V_n are open subsets of \mathbb{R}^n such that*

$$U_i \cap A_i = \emptyset, \quad V_i \cap B_i = \emptyset \quad \text{and} \quad U_i \cap V_i = \emptyset$$

for all $i \in \{1, \dots, n\}$. Then $[-2, 2]^n$ is not contained in the union $U_1 \cup \dots \cup U_n \cup V_1 \cup \dots \cup V_n$. \square

Let X be a set, $n \geq 1$ and $m \in \mathbb{N}$. A function

$$C : \mathbb{N}_m^n \rightarrow \mathcal{P}(X)$$

is called an n -**chain** in X (of length m) if

$$C_{i_1, \dots, i_n} \cap C_{j_1, \dots, j_n} = \emptyset \tag{5.3}$$

for all $(i_1, \dots, i_n), (j_1, \dots, j_n) \in \mathbb{N}_m^n$ such that $|i_l - j_l| > 1$ for some $l \in \{1, \dots, n\}$. Here we use C_{i_1, \dots, i_n} to denote $C(i_1, \dots, i_n)$.

For $a, b \in \mathbb{N}_m^n$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, we define the number $p(a, b)$ by

$$p(a, b) = \max_{1 \leq i \leq n} |a_i - b_i|.$$

With this notation, we have that $C : \mathbb{N}_m^n \rightarrow \mathcal{P}(X)$ is an n -chain if and only if $C_a \cap C_b = \emptyset$ for all $a, b \in \mathbb{N}_m^n$ such that $p(a, b) > 1$. If $a \in \mathbb{N}_m^n$, we say that C_a is a **link** of the chain C .

If (X, d) is a metric space, then we say that an n -chain $C = (C_{i_1, \dots, i_n})_{0 \leq i_1, \dots, i_n \leq m}$ in X is **open** if C_{i_1, \dots, i_n} is an open set in (X, d) for all $i_1, \dots, i_n \in \mathbb{N}_m$. We similarly define the notion of a **compact** n -chain in (X, d) .

If $C : \mathbb{N}_m^n \rightarrow \mathcal{P}(X)$, $C = (C_{i_1, \dots, i_n})_{0 \leq i_1, \dots, i_n \leq m}$ is a function, then the function

$$(C_{i_1, \dots, i_{n-1}, 0})_{0 \leq i_1, \dots, i_{n-1} \leq m}$$

is called the **lower boundary** of C . If C is an n -chain in (X, d) , then its lower boundary is an $n - 1$ -chain in (X, d) .

In general, if A is a set and $f : A \rightarrow \mathcal{P}(X)$ a function, we will denote by $\bigcup f$ the union $\bigcup_{a \in A} f(a)$ and we will say that f **covers** S , where $S \subseteq X$, if $S \subseteq \bigcup f$. If (X, d) is a metric space, A a nonempty finite set and $f(a)$ a nonempty bounded set for each $a \in A$, then we define $\text{mesh}(f)$ as the number

$$\text{mesh}(f) = \max_{a \in A} (\text{diam } f(a)).$$

Let $\varepsilon > 0$. An n -chain C in a metric space (X, d) is said to be an ε - n -**chain** if $\text{mesh}(C) < \varepsilon$.

Let $n \geq 1$. A **finite n -sequence** in \mathbb{N} is any function of the form

$$\{0, \dots, m\}^n \rightarrow \mathbb{N}.$$

Recall that any finite sequence i_0, \dots, i_m in \mathbb{N} is of the form $(j)_0, \dots, (j)_m$ for some $j \in \mathbb{N}$. Let $\nu : \mathbb{N}^n \rightarrow \mathbb{N}$ be some fixed computable injection and let τ_1 and τ_2 be the functions from the section 2. We define $\Sigma : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ by

$$\Sigma(i, j_1, \dots, j_n) = (\tau_1(i))_{\nu(j_1, \dots, j_n)}.$$

Then for any finite n -sequence a in \mathbb{N} there exists $i \in \mathbb{N}$ such that a equals the function

$$\begin{aligned} &\{0, \dots, \tau_2(i)\}^n \rightarrow \mathbb{N}, \\ &(j_1, \dots, j_n) \mapsto \Sigma(i, j_1, \dots, j_n). \end{aligned}$$

We will use the following notation: \widehat{i} instead of $\tau_2(i)$ and, for $n \geq 2$, $(i)_{j_1, \dots, j_n}$ instead of $\Sigma(i, j_1, \dots, j_n)$.

Let (X, d, α) be a computable metric space. For $l \in \mathbb{N}$ let \mathcal{H}_l be the finite n -sequence of sets in X defined by

$$\mathcal{H}_l = \left(J_{(l)_{j_1, \dots, j_n}} \right)_{0 \leq j_1, \dots, j_n \leq \widehat{l}}$$

(i.e. \mathcal{H}_l is the function $\{0, \dots, \widehat{l}\}^n \rightarrow \mathcal{P}(X)$ which maps (j_1, \dots, j_n) to $J_{(l)_{j_1, \dots, j_n}}$).

Lemma 5.2. *Let (X, d, α) be a computable metric space. Then there exist computable functions $\zeta, \zeta' : \mathbb{N} \rightarrow \mathbb{N}$ such that, for each $l \in \mathbb{N}$, the set $J_{\zeta(l)}$ is the union of \mathcal{H}_l and $J_{\zeta'(l)}$ is the union of the lower boundary of \mathcal{H}_l .*

Proof. See Lemma 4.3. in [7]. □

Proposition 5.3. *Let (X, d, α) be a computable metric space and let S be a semi-computable compact set in this space.*

- (i) *The set $\{l \in \mathbb{N} \mid \mathcal{H}_l \text{ covers } S\}$ is c.e.*
- (ii) *The set $\{l \in \mathbb{N} \mid \text{the lower boundary of } \mathcal{H}_l \text{ covers } S\}$ is c.e.*

Proof. Let ζ, ζ' be the functions from Lemma 5.2. Then \mathcal{H}_l covers S if and only if $S \subseteq J_{\zeta(l)}$ and the lower boundary of \mathcal{H}_l covers S if and only if $S \subseteq J_{\zeta'(l)}$. Claims (i) and (ii) follow. □

We say that \mathcal{H}_l is a **formal** n -chain if $J_{(l)_a}$ and $J_{(l)_b}$ are formally disjoint for all $a, b \in \mathbb{N}_l^n$ such that $p(a, b) > 1$.

Let the function $\text{fmesh} : \mathbb{N} \rightarrow \mathbb{R}$ be defined by

$$\text{fmesh}(l) = \max_{0 \leq j_1, \dots, j_n \leq \widehat{l}} \text{fdiam}((l)_{j_1, \dots, j_n}).$$

Proposition 5.4.

- (i) *The function fmesh is computable.*
- (ii) *The set $\{l \in \mathbb{N} \mid \mathcal{H}_l \text{ is a formal } n\text{-chain}\}$ is computably enumerable.*

Proof. (i) follows from Proposition 2.4(1) and Proposition 2.2(4).

Let us prove (ii). Let $\Phi : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N}^{2n+1})$ be defined by $\Phi(l) = \{(l, v_1, \dots, v_n, w_1, \dots, w_n) \mid 0 \leq v_1, \dots, v_n, w_1, \dots, w_n \leq \widehat{l}, |v_i - w_i| > 1 \text{ for some } i \in \{1, \dots, n\}\}$. Then Φ is c.c.b. Let $\Psi : \mathbb{N}^{2n+1} \rightarrow \mathcal{P}(\mathbb{N}^2)$ be defined by

$$\Psi(l, v_1, \dots, v_n, w_1, \dots, w_n) = \{((l)_{v_1, \dots, v_n}, (l)_{w_1, \dots, w_n})\}.$$

Clearly, Ψ is c.c.b. The set Ω of all $(i, j) \in \mathbb{N}^2$ such that J_i and J_j are formally disjoint is c.e. by Proposition 2.4(2). Finally, the function $\Lambda : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N}^2)$ defined by Φ and Ψ as in Proposition 2.1(3) is c.c.b. and we have that \mathcal{H}_l is a formal n -chain if and only if $\Lambda(l) \subseteq \Omega$. It follows now from Proposition 2.1(4) that the set $\{l \in \mathbb{N} \mid \mathcal{H}_l \text{ is a formal } n\text{-chain}\}$ is c.e. □

If (X, d) is a metric space, then for nonempty subsets S and T of X we denote the number $\inf\{d(x, y) \mid x \in S, y \in T\}$ by $d(S, T)$. The proof of the following lemma is straightforward.

Lemma 5.5. *Let (X, d) be a metric space and let K and L be nonempty compact sets in this space. Let $r = \frac{d(K, L)}{4}$. Then, if $x, y \in X$ and $s, t \in \mathbb{R}$ are such that $0 < s, t < r$ and $B(x, s) \cap K \neq \emptyset$ and $B(y, t) \cap L \neq \emptyset$, then $d(x, y) > s + t$. □*

Let us suppose that S is a semi-computable compact set in a computable metric space (X, d, α) . Suppose S is homeomorphic to $[0, 1] \times [0, 1]$. To show that S is a computable compact set, it would be enough to effectively find, for a given $\varepsilon > 0$, an $\varepsilon - 2$ -chain which covers S and such that each of its links intersects S . The question whether a chain covers S or not is semi-decidable since S is a semi-computable compact set. But the problem is how to ensure that each link of a chain intersects S . In fact, this is not possible in general since S needs not be computable. The idea we used to tackle this problem is to find some effective procedure which will, for a given $\varepsilon > 0$, give an $\varepsilon - 2$ -chain which covers S in such a way that we can be sure that certain links (which we can effectively determine) do intersect S . This idea is used in the proof of the following theorem.

Theorem 5.6. *Let (X, d, α) be a computable metric space. Let S be a semi-computable compact set in this space and suppose $x \in S$ is a point which has a neighborhood in S homeomorphic to \mathbb{R}^n for some $n \in \mathbb{N}$. Then there exists a neighborhood of x in S which is computable up to S .*

Proof. Let U be a neighborhood of x in S (i.e. $x \in \text{Int}_S U$, where $\text{Int}_S U$ is the interior of U in S) and $n \in \mathbb{N}$ so that U is homeomorphic to \mathbb{R}^n . We may assume that U is open in S (certainly $\text{Int}_S U$ is homeomorphic to an open subset of \mathbb{R}^n which implies that some open subset of $\text{Int}_S U$ which contains x is homeomorphic to an open ball in \mathbb{R}^n ; however every open ball in \mathbb{R}^n is homeomorphic to \mathbb{R}^n). Let $f : \mathbb{R}^n \rightarrow U$ be a homeomorphism. We may assume $f(0) = x$ (otherwise we take the composition of f and some translation $\mathbb{R}^n \rightarrow \mathbb{R}^n$). The set $f(\langle -4, 4 \rangle^n)$ is open in U and therefore in S which implies that $S \setminus f(\langle -4, 4 \rangle^n)$ is closed in S . (Here, for $a, b \in \mathbb{R}$, we use $\langle a, b \rangle$ to denote the open interval $\{x \in \mathbb{R} \mid a < x < b\}$.) Hence $S \setminus f(\langle -4, 4 \rangle^n)$ is compact and since it is disjoint with $f([-2, 2]^n)$, which is clearly compact, there exists $m_0 \in \mathbb{N}$ such that the rational open set J_{m_0} satisfies the following:

$$S \setminus f(\langle -4, 4 \rangle^n) \subseteq J_{m_0} \text{ and } J_{m_0} \cap f([-2, 2]^n) = \emptyset.$$

Let $S' = S \setminus J_{m_0}$. Then, by Lemma 3.3, S' is a semi-computable compact set in (X, d, α) and

$$f([-2, 2]^n) \subseteq S' \subseteq f([-4, 4]^n). \quad (5.4)$$

See Figure 1. The areas bounded by the inside, the middle and the outside curve are $f([-1, 1]^n)$, $f([-2, 2]^n)$ and $f([-4, 4]^n)$ respectively. The grey area is S' .

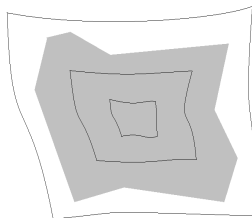


Figure 1.

For $i \in \{1, \dots, n\}$ let

$$C_i = \{(x_1, \dots, x_n) \in [-4, 4]^n \mid x_i \geq -1\}, \quad D_i = \{(x_1, \dots, x_n) \in [-4, 4]^n \mid x_i \leq 1\}.$$

Note that for each $i \in \{1, \dots, n\}$ the sets $f(A_i)$ and $f(C_i)$ are compact and disjoint and the sets $f(B_i)$ and $f(D_i)$ are compact and disjoint (recall that A_i and B_i are defined by (5.1)

and (5.2)). On Figure 2 the blue area is $f(A_i)$ and the green area is $f(C_i)$. On Figure 3 the blue area is $f(B_i)$ and the green area is $f(D_i)$.

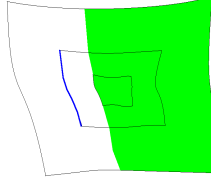


Figure 2.

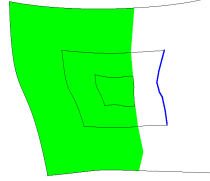


Figure 3.

Also, $\{x\}$ and $f([-4, 4]^n \setminus \langle -1, 1 \rangle^n)$ are compact and disjoint. Choose a positive rational number γ such that

$$\gamma < \frac{d(f(A_i)), f(C_i))}{4}, \quad \gamma < \frac{d(f(B_i)), f(D_i))}{4} \quad \text{and} \quad \gamma < \frac{d(\{x\}, f([-4, 4]^n \setminus \langle -1, 1 \rangle^n))}{4} \quad (5.5)$$

for each $i \in \{1, \dots, n\}$.

Let $i \in \{1, \dots, n\}$. Let $\mathcal{U} = \{B(\alpha_l, \gamma) \mid l \in \mathbb{N}\}$. Then \mathcal{U} is an open cover for $f(A_i)$ in (X, d) and therefore there exists a finite subcover \mathcal{U}' of \mathcal{U} for $f(A_i)$. Let \mathcal{U}'' be the subset of \mathcal{U}' consisting of those members of \mathcal{U}' which intersect $f(A_i)$. We have

$$\mathcal{U}'' = \{B(\alpha_{l_0}, \gamma), \dots, B(\alpha_{l_k}, \gamma)\},$$

where $l_0, \dots, l_k \in \mathbb{N}$. Choose $\omega \in \mathbb{N}$ such that $\gamma = q_\omega$. Let $v_0, \dots, v_k \in \mathbb{N}$ be such that $\tau_1(v_j) = l_j$ and $\tau_2(v_j) = \omega$ for each $j \in \{0, \dots, k\}$. Finally, choose $a_i \in \mathbb{N}$ so that

$$[a_i] = \{v_0, \dots, v_k\}.$$

In this way we have defined the numbers a_1, \dots, a_n . For each $i \in \{1, \dots, n\}$ clearly $f(A_i) \subseteq J_{a_i}$ and if $j \in \mathbb{N}$ is such that $I_j \cap f(C_i) \neq \emptyset$ and $\rho_j < \gamma$, then by (5.5) and Lemma 5.5 we have that I_j and J_{a_i} are formally disjoint.

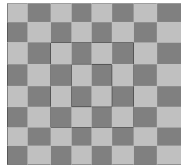
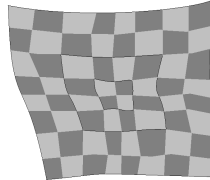
In the same way we define numbers b_1, \dots, b_n with the following property: if $i \in \{1, \dots, n\}$, then $f(B_i) \subseteq J_{b_i}$ and if $j \in \mathbb{N}$ is such that $I_j \cap f(D_i) \neq \emptyset$ and $\rho_j < \gamma$, then I_j and J_{b_i} are formally disjoint.

Similarly, there exists $x' \in \mathbb{N}$ with the property that $x \in J_{x'}$ and if $j \in \mathbb{N}$ is such that $I_j \cap f([-4, 4]^n \setminus \langle -1, 1 \rangle^n) \neq \emptyset$ and $\rho_j < \gamma$, then I_j and $J_{x'}$ are formally disjoint.

For $m \in \mathbb{N}$ let $E^m : \mathbb{N}_{8m+7}^n \rightarrow \mathcal{P}([-4, 4]^n)$ be defined by

$$E_{i_1, \dots, i_n}^m = \left[-4 + \frac{i_1}{m+1}, -4 + \frac{i_1+1}{m+1} \right] \times \dots \times \left[-4 + \frac{i_n}{m+1}, -4 + \frac{i_n+1}{m+1} \right].$$

Then E^m is a compact n -chain in $[-4, 4]^n$ which covers $[-4, 4]^n$. Let $F^m : \mathbb{N}_{8m+7}^n \rightarrow \mathcal{P}(X)$ be defined by $F^m(v) = f(E^m(v))$, $v \in \mathbb{N}_{8m+7}^n$. Then F^m is a compact n -chain in (X, d) which covers $f([-4, 4]^n)$.

Figure 4. Chain E^m Figure 5. Chain F^m

Let $\varepsilon > 0$. Since the restriction of f to $[-4, 4]^n$ is uniformly continuous, there exists $m \in \mathbb{N}$ such that $\text{mesh}(F^m) < \frac{\varepsilon}{6}$. Let δ be a positive rational number such that $\delta < \gamma$, $\delta < \frac{\varepsilon}{6}$ and

$$\delta < \frac{d(F^m(v), F^m(w))}{4}$$

for all $v, w \in \mathbb{N}_{8m+7}^n$ such that $p(v, w) > 1$.

Now we define numbers j_v for $v \in \mathbb{N}_{8m+7}$ in the following way. Let $v \in \mathbb{N}_{8m+7}^n$. As before, we conclude that there exist finitely many numbers i_0, \dots, i_k such that $F^m(v)$ is contained in the union of the balls $B(\alpha_{i_0}, \delta), \dots, B(\alpha_{i_k}, \delta)$ and each of these balls intersects $F^m(v)$. Let $p \in \mathbb{N}$ be such that $q_p = \delta$. Now, let w_0, \dots, w_k be such that $\tau_1(w_l) = i_l$, $\tau_2(w_l) = p$ for each $l \in \{0, \dots, k\}$. Finally, choose $j_v \in \mathbb{N}$ so that $[j_v] = \{w_0, \dots, w_m\}$.

We conclude readily that $\text{fdiam}(j_v) < \varepsilon$ for each $v \in \mathbb{N}_{8m+7}^n$ and it also follows, using Lemma 5.5, that J_{j_v} and J_{j_w} are formally disjoint for all $v, w \in \mathbb{N}_{8m+7}^n$ such that $p(v, w) > 1$. Furthermore, let $v \in \mathbb{N}_{8m+7}^n$, $v = (v_1, \dots, v_n)$ and let $i \in \{1, \dots, n\}$. If $v_i \geq 3m + 3$, then $F^m(v) \subseteq f(C_i)$ and by the construction of j_v and a_i we have that J_{j_v} and J_{a_i} are formally disjoint. Similarly, if $v_i \leq 5m + 4$, then $F^m(v) \subseteq f(D_i)$ and we get that J_{j_v} and J_{b_i} are formally disjoint. Also note that $v_i < 3m + 3$ or $v_i > 5m + 4$ implies that J_{j_v} and $J_{x'}$ are formally disjoint. Note that S' (in fact $f([-4, 4]^n)$) is contained in the union of the sets J_{j_v} , $v \in \mathbb{N}_{8m+7}^n$.

Choose $l \in \mathbb{N}$ such that $\widehat{l} = 8m + 7$ and such that $j_v = (l)_v$ for each $v \in \mathbb{N}_{8m+7}^n$. Put $e = 3m + 3$, $h = 5m + 4$. The preceding construction can be made for $\varepsilon = 2^{-k}$ for each $k \in \mathbb{N}$. We have the following conclusions:

- (1) \mathcal{H}_l covers S' ;
- (2) \mathcal{H}_l is a formal n -chain;
- (3) $J_{(l)_{v_1, \dots, v_n}}$ and J_{a_i} are formally disjoint if $1 \leq i \leq n$, $0 \leq v_1, \dots, v_n \leq \widehat{l}$ and $v_i \geq e$;
- (4) $J_{(l)_{v_1, \dots, v_n}}$ and J_{b_i} are formally disjoint if $1 \leq i \leq n$, $0 \leq v_1, \dots, v_n \leq \widehat{l}$ and $v_i \leq h$;
- (5) $J_{(l)_{v_1, \dots, v_n}}$ and $J_{x'}$ are formally disjoint if $1 \leq i \leq n$, $0 \leq v_1, \dots, v_n \leq \widehat{l}$ and $(v_i < e$ or $v_i > h)$;
- (6) $\text{fmesh}(l) < 2^{-k}$.

These conditions are semi-decidable, i.e. the set Ω of all $(k, l, e, h) \in \mathbb{N}^4$ such that (1)–(6) hold is c.e. (Proposition 5.3, Proposition 5.4 and Proposition 2.2(3) imply that conditions (1), (2) and (6) are semi-decidable; that (3)–(5) are semi-decidable can be seen as in the proof of Proposition 5.4). Since for each $k \in \mathbb{N}$ there exist $l, e, h \in \mathbb{N}$ such that $(k, l, e, h) \in \Omega$, there exist computable functions $\tilde{l}, \tilde{e}, \tilde{h} : \mathbb{N} \rightarrow \mathbb{N}$ such that $(k, \tilde{l}(k), \tilde{e}(k), \tilde{h}(k)) \in \Omega$ for each $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ and let $l = \tilde{l}(k)$, $e = \tilde{e}(k)$ and $h = \tilde{h}(k)$. Let Γ be the set of all $(v_1, \dots, v_n) \in \mathbb{N}_l^n$ such that $e \leq v_i \leq h$ for each $i \in \{1, \dots, n\}$. We claim that

$$J_{(l)_v} \cap f([-2, 2]^n) \neq \emptyset \text{ for all } v \in \Gamma, \quad (5.6)$$

and

$$S' \cap J_{x'} \subseteq \bigcup_{v \in \Gamma} J_{(l)_v}. \quad (5.7)$$

Suppose $J_{(l)_v} \cap f([-2, 2]^n) = \emptyset$ for some $v = (v_1, \dots, v_n) \in \Gamma$.

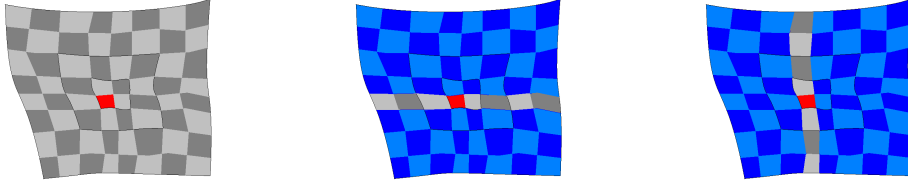


Figure 6. The red link is $J_{(l)v}$.

For each $i \in \{1, \dots, n\}$ we define U_i as the union of all $J_{(l)w_1, \dots, w_n}$ such that $(w_1, \dots, w_n) \in \mathbb{N}_l^n$ and $w_i < v_i$ and we define V_i as the union of all $J_{(l)w_1, \dots, w_n}$ such that $(w_1, \dots, w_n) \in \mathbb{N}_l^n$ and $w_i > v_i$. (In Figure 6 the sets U_i are represented by the bottom and the left blue set and the sets V_i by the upper and the right blue set.) Then

$$f([-2, 2]^n) \subseteq (U_1 \cup \dots \cup U_n) \cup (V_1 \cup \dots \cup V_n).$$

Since \mathcal{H}_l is a chain, $U_i \cap V_i = \emptyset$ for each $i \in \{1, \dots, n\}$. Since $v_i \leq h$, by property (4) we have $U_i \cap J_{b_i} = \emptyset$, hence $U_i \cap f(B_i) = \emptyset$. Similarly, $v_i \geq e$ implies $V_i \cap J_{a_i} = \emptyset$ and this gives $V_i \cap f(A_i) = \emptyset$. Therefore

$$f^{-1}(U_i) \cap B_i = \emptyset, \quad f^{-1}(V_i) \cap A_i = \emptyset \quad \text{and} \quad f^{-1}(U_i) \cap f^{-1}(V_i) = \emptyset$$

for each $i \in \{1, \dots, n\}$ and

$$[-2, 2]^n \subseteq f^{-1}(U_1) \cup \dots \cup f^{-1}(U_n) \cup f^{-1}(V_1) \cup \dots \cup f^{-1}(V_n).$$

This is impossible by Theorem 5.1. So (5.6) holds.

That (5.7) holds, it follows readily from property (5). Note that (5.7) implies $\Gamma \neq \emptyset$. Hence $\tilde{e}(k) \leq \tilde{h}(k)$ for each $k \in \mathbb{N}$.

Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be some computable function such that

$$\Lambda_{g(k)} \approx_{2^{-k}} \bigcup_{\tilde{e}(k) \leq v_1, \dots, v_n \leq \tilde{h}(k)} J_{(\tilde{l}(k))v_1, \dots, v_n}.$$

Such a function can be obtained by defining $\Phi : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$, $\Phi(k) = \{\tau_1(((l)_{v_1, \dots, v_n})_0) \mid \tilde{e}(k) \leq v_1, \dots, v_n \leq \tilde{h}(k)\}$ and taking a computable $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $\Phi(k) = [g(k)]$ for each $k \in \mathbb{N}$.

By (5.6) and (5.7)

$$S' \cap J_{x'} \prec_{2^{-k}} \Lambda_{g(k)} \quad \text{and} \quad \Lambda_{g(k)} \prec_{2 \cdot 2^{-k}} S \tag{5.8}$$

for each $k \in \mathbb{N}$. Since $f(\langle -1, 1 \rangle^n)$ is an open set in U and U is open in S , $f(\langle -1, 1 \rangle^n)$ is open in S and by (5.4) we have $f(\langle -1, 1 \rangle^n) \subseteq S'$. Hence S' is a neighborhood of x in S and therefore $S' \cap J_{x'}$ is a neighborhood of x in S . By (5.8) $S' \cap J_{x'}$ is computable up to S . \square

For $n \in \mathbb{N}$, $n \geq 1$, let

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

The subset of \mathbb{H}^n consisting of those points whose last coordinate is zero is denoted by $\text{Bd } \mathbb{H}^n$.

Theorem 5.7. *Let (X, d, α) be a computable metric space and let S and T be semi-computable compact sets in this space such that $T \subseteq S$. Suppose $x \in S$ is a point which has a neighborhood U (in S) with the following property: there exists a homeomorphism*

$f : \mathbb{H}^n \rightarrow U$ such that $f(\text{Bd } \mathbb{H}^n) = U \cap T$. Then there exists a neighborhood of x in S which is computable up to S .

Proof. If $f^{-1}(x) \in \mathbb{H}^n \setminus (\text{Bd } \mathbb{H}^n)$, then x has a neighborhood homeomorphic to some open ball in \mathbb{R}^n , hence homeomorphic to \mathbb{R}^n and the claim of the theorem follows from Theorem 5.6.

Suppose now that $f^{-1}(x) \in \text{Bd } \mathbb{H}^n$. As in the proof of Theorem 5.6, we may assume that U is open in S and $f(0) = x$. As before we conclude that there exists $m_0 \in \mathbb{N}$ such that

$$S \setminus f([-4, 4]^n \cap \mathbb{H}^n) \subseteq J_{m_0} \text{ and } J_{m_0} \cap f([-2, 2]^n \cap \mathbb{H}^n) = \emptyset.$$

Let $S' = S \setminus J_{m_0}$. By Lemma 3.3 S' is a semi-computable compact set in (X, d, α) and

$$f([-2, 2]^n \cap \mathbb{H}^n) \subseteq S' \subseteq f([-4, 4]^n \cap \mathbb{H}^n). \quad (5.9)$$

Let $T' = T \setminus J_{m_0}$. Then T' is semi-computable and

$$f([-2, 2]^n \cap (\text{Bd } \mathbb{H}^n)) \subseteq T' \subseteq f([-4, 4]^n \cap (\text{Bd } \mathbb{H}^n)). \quad (5.10)$$

For $i \in \{1, \dots, n-1\}$ let

$$C_i = \{(x_1, \dots, x_n) \in [-4, 4]^n \cap \mathbb{H}^n \mid x_i \geq -1\}, \quad D_i = \{(x_1, \dots, x_n) \in [-4, 4]^n \cap \mathbb{H}^n \mid x_i \leq 1\}.$$

Let

$$D_n = \{(x_1, \dots, x_n) \in [-4, 4]^n \cap \mathbb{H}^n \mid x_n \leq 1\}.$$

For $i \in \{1, \dots, n-1\}$ let

$$\tilde{A}_i = \{(x_1, \dots, x_n) \in [-2, 2]^n \cap \mathbb{H}^n \mid x_i = -2\},$$

$$\tilde{B}_i = \{(x_1, \dots, x_n) \in [-2, 2]^n \cap \mathbb{H}^n \mid x_i = 2\}.$$

Let

$$\tilde{A}_n = \{(x_1, \dots, x_n) \in [-2, 2]^n \cap \mathbb{H}^n \mid x_n = 0\}, \quad \tilde{B}_n = \{(x_1, \dots, x_n) \in [-2, 2]^n \cap \mathbb{H}^n \mid x_n = 2\}.$$

For $m \in \mathbb{N}$ let $E^m : \mathbb{N}_{8m+7}^n \rightarrow \mathcal{P}([-4, 4]^n \cap \mathbb{H}^n)$ be defined so that E_{i_1, \dots, i_n}^m is equal to

$$\left[-4 + \frac{i_1}{m+1}, -4 + \frac{i_1+1}{m+1}\right] \times \dots \times \left[-4 + \frac{i_{n-1}}{m+1}, -4 + \frac{i_{n-1}+1}{m+1}\right] \times \left[\frac{i_n}{2m+2}, \frac{i_n+1}{2m+2}\right].$$

Then E^m is a compact n -chain in $[-4, 4]^n \cap \mathbb{H}^n$ which covers $[-4, 4]^n \cap \mathbb{H}^n$. Let $F^m : \mathbb{N}_{8m+7}^n \rightarrow \mathcal{P}(X)$ be defined by $F^m(v) = f(E^m(v))$, $v \in \mathbb{N}_{8m+7}^n$. Then F^m is a compact n -chain in (X, d) which covers $f([-4, 4]^n \cap \mathbb{H}^n)$. Note that the lower boundary of F^m covers $f([-4, 4]^n \cap (\text{Bd } \mathbb{H}^n))$. Therefore by (5.10) the lower boundary of F^m covers T' .

Exactly as in the proof of Theorem 5.6 we get numbers $a_1, \dots, a_n, b_1, \dots, b_n, x' \in \mathbb{N}$ so that, for each $i \in \{1, \dots, n\}$, $f(\tilde{A}_i) \subseteq J_{a_i}$, $f(\tilde{B}_i) \subseteq J_{b_i}$, $x \in J_{x'}$ and so that for each $k \in \mathbb{N}$ there exist $l, e, h, u \in \mathbb{N}$ with the following properties:

- (1) \mathcal{H}_l covers S' ;
- (2) the lower boundary of \mathcal{H}_l covers T' ;
- (3) \mathcal{H}_l is a formal n -chain;
- (4) $J_{(l)v_1, \dots, v_n}$ and J_{a_i} are formally disjoint if $1 \leq i \leq n-1$, $0 \leq v_1, \dots, v_n \leq \hat{l}$ and $v_i \geq e$;
- (5) $J_{(l)v_1, \dots, v_n}$ and J_{b_i} are formally disjoint if $1 \leq i \leq n-1$, $0 \leq v_1, \dots, v_n \leq \hat{l}$ and $v_i \leq h$;
- (6) $J_{(l)v_1, \dots, v_n}$ and J_{b_n} are formally disjoint if $0 \leq v_1, \dots, v_n \leq \hat{l}$ and $v_n \leq u$;
- (7) $J_{(l)v_1, \dots, v_n}$ and $J_{x'}$ are formally disjoint if $1 \leq i \leq n-1$, $0 \leq v_1, \dots, v_n \leq \hat{l}$ and $(v_i < e$ or $v_i > h)$;

- (8) $J_{(l)_{v_1, \dots, v_n}}$ and $J_{x'}$ are formally disjoint if $0 \leq v_1, \dots, v_n \leq \hat{l}$ and $v_n > u$;
(9) $\text{fmesh}(l) < 2^{-k}$.

The set Ω of all $(k, l, e, h, u) \in \mathbb{N}^5$ such that (1)-(9) hold is c.e. Therefore there exist computable functions $\tilde{l}, \tilde{e}, \tilde{h}, \tilde{u} : \mathbb{N} \rightarrow \mathbb{N}$ such that $(k, \tilde{l}(k), \tilde{e}(k), \tilde{h}(k), \tilde{u}(k)) \in \Omega$ for each $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ and let $l = \tilde{l}(k)$, $e = \tilde{e}(k)$, $h = \tilde{h}(k)$ and $u = \tilde{u}(k)$. Let Γ be the set of all $(v_1, \dots, v_n) \in \mathbb{N}_l^n$ such that $v_n \leq u$ and $e \leq v_i \leq h$ for each $i \in \{1, \dots, n-1\}$. We claim that

$$J_{(l)_v} \cap f([-2, 2]^n \cap \mathbb{H}^n) \neq \emptyset \text{ for all } v \in \Gamma, \quad (5.11)$$

and

$$S' \cap J_{x'} \subseteq \bigcup_{v \in \Gamma} J_{(l)_v}. \quad (5.12)$$

Suppose $J_{(l)_v} \cap f([-2, 2]^n) = \emptyset$ for some $v = (v_1, \dots, v_n) \in \Gamma$ such that $v_n = 0$. For each $i \in \{1, \dots, n-1\}$ let U_i be the union of all $J_{(l)_{w_1, \dots, w_n}}$ such that $(w_1, \dots, w_n) \in \mathbb{N}_l^n$, $w_n = 0$ and $w_i < v_i$ and let V_i be the union of all $J_{(l)_{w_1, \dots, w_n}}$ such that $(w_1, \dots, w_n) \in \mathbb{N}_l^n$, $w_n = 0$ and $w_i > v_i$. It follows from (2) and (5.10) that

$$f([-2, 2]^n \cap (\text{Bd } \mathbb{H}^n)) \subseteq (U_1 \cup \dots \cup U_{n-1}) \cup (V_1 \cup \dots \cup V_{n-1}).$$

Since \mathcal{H}_l is a chain, $U_i \cap V_i = \emptyset$ for each $i \in \{1, \dots, n-1\}$. Since $e \leq v_i \leq h$, by properties (4) and (5) we have $U_i \cap J_{b_i} = \emptyset$ and $V_i \cap J_{a_i} = \emptyset$. Hence $U_i \cap f(\tilde{B}_i) = \emptyset$ and $V_i \cap f(\tilde{A}_i) = \emptyset$. Therefore

$$f^{-1}(U_i) \cap \tilde{B}_i = \emptyset, \quad f^{-1}(V_i) \cap \tilde{A}_i = \emptyset \text{ and } f^{-1}(U_i) \cap f^{-1}(V_i) = \emptyset \quad (5.13)$$

for each $i \in \{1, \dots, n-1\}$ and

$$[-2, 2]^n \cap (\text{Bd } \mathbb{H}^n) \subseteq f^{-1}(U_1) \cup \dots \cup f^{-1}(U_{n-1}) \cup f^{-1}(V_1) \cup \dots \cup f^{-1}(V_{n-1}).$$

This is impossible by Theorem 5.1 (formally, we take the function $\gamma : \mathbb{R}^{n-1} \rightarrow \mathbb{H}^n$, $\gamma(z) = (z, 0)$ and then for each $i \in \{1, \dots, n-1\}$ we get from (5.13) that

$$\gamma^{-1}(f^{-1}(U_i)) \cap B_i^{n-1} = \emptyset, \quad \gamma^{-1}(f^{-1}(V_i)) \cap A_i^{n-1} = \emptyset, \quad \gamma^{-1}(f^{-1}(U_i)) \cap \gamma^{-1}(f^{-1}(V_i)) = \emptyset$$

which is impossible since the sets $\gamma^{-1}(f^{-1}(U_1)), \dots, \gamma^{-1}(f^{-1}(U_{n-1})), \gamma^{-1}(f^{-1}(V_1)), \dots, \gamma^{-1}(f^{-1}(V_{n-1}))$ are open in \mathbb{R}^{n-1} and cover $[-2, 2]^{n-1} = I^{n-1}$.)

So $J_{(l)_v} \cap f([-2, 2]^n) \neq \emptyset$ for each $v = (v_1, \dots, v_n) \in \Gamma$ such that $v_n = 0$.

Let $v = (v_1, \dots, v_n) \in \Gamma$ be such that $v_n > 0$. Suppose $J_{(l)_v} \cap f([-2, 2]^n) = \emptyset$.

For $i \in \{1, \dots, n\}$ we define U_i as the union of all $J_{(l)_{w_1, \dots, w_n}}$ such that $(w_1, \dots, w_n) \in \mathbb{N}_l^n$ and $w_i < v_i$ and we define V_i as the union of all $J_{(l)_{w_1, \dots, w_n}}$ such that $(w_1, \dots, w_n) \in \mathbb{N}_l^n$ and $w_i > v_i$. As before we conclude that

$$f([-2, 2]^n \cap \mathbb{H}^n) \subseteq (U_1 \cup \dots \cup U_{n-1}) \cup (V_1 \cup \dots \cup V_{n-1})$$

and

$$U_i \cap V_i = \emptyset, \quad U_i \cap f(\tilde{B}_i) = \emptyset, \quad V_i \cap f(\tilde{A}_i) = \emptyset$$

for each $i \in \{1, \dots, n-1\}$. We also have $U_n \cap V_n = \emptyset$. Since $v_n \leq u$, property (6) implies $U_n \cap f(\tilde{B}_n) = \emptyset$. On the other hand, by (5.10) we have $f(\tilde{A}_n) \subseteq T'$ and this, together with property (2), implies $f(\tilde{A}_n) \subseteq U_n$. Therefore $f(\tilde{A}_n) \cap V_n = \emptyset$.

We have the following conclusion: for each $i \in \{1, \dots, n\}$

$$f^{-1}(U_i) \cap \tilde{B}_i = \emptyset, \quad f^{-1}(V_i) \cap \tilde{A}_i = \emptyset, \quad f^{-1}(U_i) \cap f^{-1}(V_i) = \emptyset$$

and

$$[-2, 2]^n \cap \mathbb{H}^n \subseteq f^{-1}(U_1) \cup \dots \cup f^{-1}(U_n) \cup f^{-1}(V_1) \cup \dots \cup f^{-1}(V_n).$$

To get a contradiction with Theorem 5.1 we define the function $\gamma : \mathbb{R}^n \rightarrow \mathbb{H}^n$ by

$$\gamma(z_1, \dots, z_n) = \begin{cases} (z_1, \dots, z_{n-1}, \frac{z_n+2}{2}), & \text{if } z_n \geq -2 \\ (z_1, \dots, z_{n-1}, 0), & \text{if } z_n \leq -2. \end{cases}$$

This is clearly a continuous function and we have $\gamma([-2, 2]^n) = [-2, 2]^n \cap \mathbb{H}^n$. It follows

$$[-2, 2]^n \subseteq \gamma^{-1}(f^{-1}(U_1)) \cup \dots \cup \gamma^{-1}(f^{-1}(U_n)) \cup \gamma^{-1}(f^{-1}(V_1)) \cup \dots \cup \gamma^{-1}(f^{-1}(V_n)).$$

For each $i \in \{1, \dots, n\}$ clearly $\gamma^{-1}(f^{-1}(U_i)) \cap \gamma^{-1}(f^{-1}(V_i)) = \emptyset$,

$$\gamma^{-1}(f^{-1}(U_i)) \cap \gamma^{-1}(\tilde{B}_i) = \emptyset \text{ and } \gamma^{-1}(f^{-1}(V_i)) \cap \gamma^{-1}(\tilde{A}_i) = \emptyset.$$

It is immediate from the definition of γ that $A_i^n \subseteq \gamma^{-1}(\tilde{A}_i)$ and $B_i^n \subseteq \gamma^{-1}(\tilde{B}_i)$ for each $i \in \{1, \dots, n\}$, therefore $\gamma^{-1}(f^{-1}(U_i)) \cap B_i^n = \emptyset$ and $\gamma^{-1}(f^{-1}(V_i)) \cap A_i^n = \emptyset$. This altogether contradicts Theorem 5.1. So we finally conclude that (5.11) holds.

That (5.12) holds, it follows from properties (7) and (8). We note as before that $\Gamma \neq \emptyset$ and therefore $\tilde{e}(k) \leq \tilde{h}(k)$ for each $k \in \mathbb{N}$.

Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be some computable function such that

$$\Lambda_{g(k)} \approx_{2^{-k}} \bigcup_{\substack{\tilde{e}(k) \leq v_1, \dots, v_{n-1} \leq \tilde{h}(k), \\ 0 \leq v_n \leq \tilde{u}(k)}} J_{(\tilde{l}(k))_{v_1, \dots, v_n}}.$$

By (5.11) and (5.12)

$$S' \cap J_{x'} \prec_{2^{-k}} \Lambda_{g(k)} \text{ and } \Lambda_{g(k)} \prec_{2 \cdot 2^{-k}} S$$

for each $k \in \mathbb{N}$. Since $S' \cap J_{x'}$ is a neighborhood of x in S , the claim of the theorem follows. \square

6. SEMI-COMPUTABLE COMPACT MANIFOLDS WITH BOUNDARY

Let $n \in \mathbb{N}$, $n \geq 1$. A nonempty topological space X is said to be an n -**manifold** if each point of X has a neighborhood which is homeomorphic to \mathbb{R}^n . The following are some examples of n -manifolds: Euclidean space \mathbb{R}^n , the unit sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

in \mathbb{R}^{n+1} , the real projective space $P^n = \{\{x, -x\} \mid x \in S^n\}$ (with topology on P^n defined by $U \subseteq P^n$ is open if the union of the family U is open in S^n , see [3]). If X is an n -manifold and Y is an m -manifold, then $X \times Y$ is an $n \cdot m$ -manifold. Therefore, the space $S^1 \times \dots \times S^1$ (n -times) is an n -dimensional manifold. The **torus** is the space $S^1 \times S^1$, hence it is a 2-manifold.

A nonempty topological space X is said to be an n -**manifold with boundary** if each point of X has a neighborhood which is homeomorphic to an open subset of \mathbb{H}^n . If X is an n -manifold with boundary, then the subset of X consisting of all points $x \in X$ with the property that there exists a neighborhood N of x , an open subset U of \mathbb{H}^n and a homeomorphism $f : N \rightarrow U$ such that $f(x) \in \text{Bd } \mathbb{H}^n$ is called the **boundary** of X . We denote the boundary of X by ∂X . It can be shown (see [10]) that if $x \in \partial X$, then every homeomorphism between a neighborhood of x in X and an open subset of \mathbb{H}^n maps x to a point in $\text{Bd } \mathbb{H}^n$. It is interesting to remark here that if a point in some topological space has neighborhoods N_1 and N_2 which are homeomorphic to open subsets of \mathbb{H}^n and \mathbb{H}^m

respectively, then $n = m$; this is a consequence of Invariance of domain theorem, see [10]. In particular, the number n , called the dimension of X , is uniquely determined by X .

If U is an open set in \mathbb{H}^n and $z \in U$ is a point such that $z \in \mathbb{H}^n \setminus (\text{Bd } \mathbb{H}^n)$, then there exists an open ball in \mathbb{R}^n centered at x which is contained in U . Furthermore, each open ball in \mathbb{R}^n is homeomorphic to \mathbb{R}^n . Therefore, if X is an n -manifold with boundary and $x \in X \setminus \partial X$, then x has an open neighborhood in X homeomorphic to \mathbb{R}^n . On the other hand, each open ball in \mathbb{H}^n centered at a point in $\text{Bd } \mathbb{H}^n$ is homeomorphic to \mathbb{H}^n . It follows from this that each point in ∂X has an open neighborhood in X homeomorphic to \mathbb{H}^n .

If X is a manifold, then clearly X is a manifold with boundary and $\partial X = \emptyset$. In fact, if X is a manifold with boundary, then X is a manifold if and only if $\partial X = \emptyset$.

If X is a manifold with boundary and Y is homeomorphic to X , then clearly Y is also a manifold with boundary and $\partial Y = f(\partial X)$, where $f : X \rightarrow Y$ is a homeomorphism. Furthermore, if X and Y are manifolds with boundary, then $X \times Y$ is a manifold with boundary and the point (x, y) is in its boundary if and only if $x \in \partial X$ or $y \in \partial Y$.

It is not hard to check (see [10]) that the closed unit ball B^n in \mathbb{R}^n is a manifold with boundary and $\partial B^n = S^{n-1}$. For example $[0, 1]$ is a manifold with boundary and $\partial[0, 1] = \{0, 1\}$. Therefore, since S^1 is a manifold, $S^1 \times [0, 1]$ is a manifold with boundary and $\partial(S^1 \times [0, 1]) = S^1 \times \{0, 1\}$. (Note that the topological boundary of $S^1 \times [0, 1]$ in \mathbb{R}^3 differs from $\partial(S^1 \times [0, 1])$.)

Theorem 6.1. *Let (X, d, α) be a computable metric space and let $S \subseteq X$. Suppose S , as a subspace of (X, d) , is a manifold with boundary and suppose S and ∂S are semi-computable compact sets in (X, d, α) . Then S is a computable compact set in (X, d, α) .*

Proof. Let $n \geq 1$ be such that S is an n -manifold with boundary. If $x \in S \setminus \partial S$, then x has a neighborhood homeomorphic to \mathbb{R}^n and by Theorem 5.6 there exists a neighborhood of x in S which is computable up to S . On the other hand, if $x \in \partial S$, then there exists an open neighborhood U of x in S and a homeomorphism $f : \mathbb{H}^n \rightarrow U$. It is clear that $f(\text{Bd } \mathbb{H}^n) \subseteq U \cap \partial S$ and since for each $z \in \partial S$ every homeomorphism between a neighborhood of z in S and an open subset of \mathbb{H}^n maps z to a point in $\text{Bd } \mathbb{H}^n$, we have $f(\text{Bd } \mathbb{H}^n) = U \cap \partial S$. It follows from Theorem 5.7 that x has a neighborhood in S which is computable up to S .

Hence each $x \in S$ has a neighborhood N_x in S which is computable up to S . Let $\mathcal{U} = \{\text{Int}_S N_x \mid x \in S\}$. Then \mathcal{U} is an open cover of S . Since S is compact, there exists a finite subcover of \mathcal{U} for S , hence there exist $x_1, \dots, x_k \in S$ such that $S = N_{x_1} \cup \dots \cup N_{x_k}$. By Proposition 4.2 S is a computable compact set. \square

Corollary 6.2. *Let (X, d, α) be a computable metric space and let S be a semi-computable compact set in this space which is a manifold with boundary. Then the following implication holds:*

$$\partial S \text{ computable compact set} \Rightarrow S \text{ computable compact set.} \quad \square$$

Theorem 6.3. *Let (X, d, α) be a computable metric space. Let S be a semi-computable compact set in this space such that S , as a subspace of (X, d) , is a manifold. Then S is computable compact in (X, d, α) . \square*

In general, if X is an n -manifold with boundary and $\partial X \neq \emptyset$, then ∂X is an $(n - 1)$ -manifold. Therefore if S is a manifold with boundary in (X, d, α) , then ∂S is a manifold (boundaryless) and we have, by Theorem 6.3, that ∂S is semi-computable compact if and only if ∂S is computable compact. This means that Corollary 6.2 is an equivalent formulation of Theorem 6.1.

Theorem 6.4. *Let (X, d, α) be a computable metric space which is locally computable. Let S be a co-computably enumerable closed set in this space such that S , as a subspace of (X, d) , is a compact manifold with boundary. Then the following implication holds:*

$$\partial S \text{ computable closed set} \Rightarrow S \text{ computable closed set.} \quad \square$$

6.1. Application: regular level sets. Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function of class C^1 such that $y \in \mathbb{R}^m$ is a regular value of f , which means that the differential $D(f)(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of f in x is surjective for each $x \in f^{-1}\{y\}$. Suppose $f^{-1}\{y\} \neq \emptyset$. Then it is known from differential topology (see [5]) that $f^{-1}\{y\}$ is an $(n - m)$ -manifold.

Suppose now additionally that f is computable and $f^{-1}\{y\}$ is a bounded set. In general, if $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a computable function, then $g^{-1}\{0\}$ is a co-c.e. closed set in \mathbb{R}^n . Therefore $f^{-1}\{y\}$ is a co-c.e. closed set and, by Theorem 6.4, we conclude the following.

Corollary 6.5. *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a computable function of class C^1 and $y \in \mathbb{R}^m$ a regular value of f such that $f^{-1}\{y\}$ is bounded, then the set $f^{-1}\{y\}$ is computable.* \square

Example 6.6. (i) Let S be the set of all $(x, y, z) \in \mathbb{R}^3$ such that

$$x^2(1 + e^x) + y^2(1 + e^y) + z^2(1 + e^z) = 1.$$

We have $S = f^{-1}\{1\}$, where $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined in the obvious way. Since $\nabla f(x) = (\partial_1 f(x), \partial_2 f(x), \partial_3 f(x)) = 0$ only for the point $x = 0$ which is not in $f^{-1}\{1\}$, we have that 1 is a regular value for f . Hence S is computable.

(ii) Let S be the set of all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

and

$$\sin x_1 + \sin x_2 + \sin x_3 + \sin x_4 = 1.$$

The intersection of two computable sets need not be computable in general and therefore we cannot get that S is computable only by proving that the sets defined by above equations are computable. We have $S = f^{-1}\{(1, 1)\}$, where $f : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is defined in the obvious way. Let f_1, f_2 be the component functions of f . It is easy to check that the vectors $\nabla f_1(x)$ and $\nabla f_2(x)$ are linearly independent for each $x \in S$. By Corollary 6.5 S is computable.

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