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LOCATING $\mathfrak{A}x$, WHERE $\mathfrak{A}$ IS A SUBSPACE OF $\mathcal{B}(H)$

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Abstract. Given a linear space of operators on a Hilbert space, any vector in the latter determines a subspace of its images under all operators. We discuss, within a Bishop-style constructive framework, conditions under which the projection of the original Hilbert space onto the closure of the image space exists. We derive a general result that leads directly to both the open mapping theorem and our main theorem on the existence of the projection.

1. Introduction

Let H be a real or complex Hilbert space, $\mathcal{B}(H)$ the space of bounded operators on H, and $\mathfrak{A}$ a linear subspace of $\mathcal{B}(H)$. For each $x \in H$ write

$$\mathfrak{A}x \equiv \{Ax : A \in \mathfrak{A}\},$$

and, *if it exists*, denote the projection of H onto the closure $\overline{\mathfrak{A}x}$ of $\mathfrak{A}x$ by $[\mathfrak{A}x]$. Projections of this type play a very big part in the classical theory of operator algebras, in which context $\mathfrak{A}$ is normally a subalgebra of $\mathcal{B}(H)$; see, for example, [10, 11, 13, 15]. However, in the constructive[1] setting—the one of this paper—we cannot even guarantee that $[\mathfrak{A}x]$ exists. Our aim is to give sufficient conditions on $\mathfrak{A}$ and x under which $[\mathfrak{A}x]$ exists, or, equivalently, the set $\mathfrak{A}x$ is located, in the sense that

$$\rho(v, \mathfrak{A}x) \equiv \inf \{\|v - Ax\| : A \in \mathfrak{A}\}$$

exists for each $v \in H$.

We require some background on operator topologies. Specifically, in addition to the standard uniform topology on $\mathcal{B}(H)$, we need

▷ the ***strong operator topology:*** the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \rightsquigarrow Tx$ is continuous for all $x \in H$;
▷ the ***weak operator topology:*** the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \rightsquigarrow \langle Tx, y\rangle$ is continuous for all $x, y \in H$.

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[1] Our *constructive setting* is that of Bishop [2, 3, 6], in which the mathematics is developed with intuitionistic, not classical, logic, in a suitable set- or type-theoretic framework [1, 12] and with dependent choice permitted.

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These topologies are induced, respectively, by the seminorms of the form $T \rightsquigarrow \|Tx\|$ with $x \in H$, and $T \rightsquigarrow |\langle Tx, y\rangle|$ with $x, y \in H$. The unit ball[2]

$$\mathcal{B}_1(H) \equiv \{T \in \mathcal{B}(H) : \|T\| \leqslant 1\}$$

of $\mathcal{B}(H)$ is classically weak-operator compact, but constructively the most we can say is that it is weak-operator totally bounded (see [4]). The evidence so far suggests that in order to make progress when dealing constructively with a subspace or subalgebra $\mathfrak{A}$ of $\mathcal{B}(H)$, it makes sense to add the weak-operator total boundedness of

$$\mathfrak{A}_1 \equiv \mathfrak{A} \cap \mathcal{B}_1(H)$$

to whatever other hypothesis we are making; in particular, it is known that $\mathfrak{A}_1$ is located in the strong operator topology—and hence $\mathfrak{A}_1 x$ is located for each $x \in H$—if and only if it is weak-operator totally bounded [7, 14].

Recall that the ***metric complement*** of a subset S of a metric space X is the set $-S$ of those elements of X that are bounded away from X. When Y is a subspace of X, $y \in Y$, and $S \subset Y$, we define

$$\rho_Y(y, -S) \equiv \inf\{\rho(y, z) : z \in Y \cap -S\}$$

if that infimum exists.

We now state our main result.

Theorem 1.1. *Let $\mathfrak{A}$ be a uniformly closed subspace of $\mathcal{B}(H)$ such that $\mathfrak{A}_1$ is weak-operator totally bounded, and let x be a point of H such that $\mathfrak{A}x$ is closed and $\rho_{\mathfrak{A}x}(0, -\mathfrak{A}_1 x)$ exists. Then the projection $[\mathfrak{A}x]$ exists.*

Before proving this theorem, we discuss, in Section 2, some general results about the locatedness of sets like $\mathfrak{A}x$, and we derive, in Section 3, a generalisation of the open mapping theorem that leads to the proof of Theorem 1.1. Finally, we show, by means of a Brouwerian example, that the existence of $\rho_{\mathfrak{A}x}(0, -\mathfrak{A}_1 x)$ cannot be dropped from the hypotheses of our main theorem.

2. Some general locatedness results for $\mathfrak{A}x$

We now prove an elementary, but helpful, result on locatedness in a Hilbert space.

Proposition 2.1. *Let $(S_n)_{n\geqslant 1}$ be a sequence of located, convex subsets of a Hilbert space H such that $S_1 \subset S_2 \subset \cdots$, let $S_\infty = \bigcup_{n\geqslant 1} S_n$, and let $x \in H$. For each n, let $x_n \in S_n$ satisfy $\|x - x_n\| < \rho(x, S_n) + 2^{-n}$. Then*

$$\rho(x, S_\infty) = \inf_{n\geqslant 1} \rho(x, S_n) = \lim_{n\to\infty} \rho(x, S_n), \tag{2.1}$$

in the sense that if any of these three numbers exists, then all three do and they are equal. Moreover, $\rho(x, S_\infty)$ exists if and only if $(x_n)_{n\geqslant 1}$ converges to a limit $x_\infty \in H$; in that case, $\rho(x, S_\infty) = \|x - x_\infty\|$, and $\|x - y\| > \|x - x_\infty\|$ for all $y \in S_\infty$ with $y \neq x_\infty$.

[2] Note that it is not constructively provable that every element T of $\mathcal{B}(H)$ is normed, in the sense that the usual operator norm of T exists. Nevertheless, when we write '$\|T\| \leqslant 1$', we are using a shorthand for '$\|Tx\| \leqslant \|x\|$ for each $x \in H$'. Likewise, '$\|T\| < 1$' means that there exists $c < 1$ such that $\|Tx\| \leqslant c\|x\|$ for each $x \in H$; and '$\|T\| > 1$' means that there exists $x \in H$ such that $\|Tx\| > \|x\|$.

Proof. Suppose that $\rho(x, S_\infty)$ exists. Then $\rho(x, S_\infty) \leqslant \rho(x, S_n)$ for each n. On the other hand, given $\varepsilon > 0$ we can find $z \in S_\infty$ such that $\|x - z\| < \rho(x, S_\infty) + \varepsilon$. Pick N such that $z \in S_N$. Then for all $n \geqslant N$,

$$\rho(x, S_\infty) \leqslant \rho(x, S_n) \leqslant \rho(x, S_N) \leqslant \|x - z\| < \rho(x, S_\infty) + \varepsilon.$$

The desired conclusion (2.1) now follows.

Next, observe that (by the parallelogram law in H) if $m \geqslant n$, then

$$\begin{aligned} \|x_m - x_n\|^2 &\leqslant \|(x - x_m) - (x - x_n)\|^2 \\ &= 2\|x - x_m\|^2 + 2\|x - x_n\|^2 - 4\left\|x - \frac{1}{2}(x_m + x_n)\right\|^2 \\ &\leqslant 2\left(\rho(x, S_m) + 2^{-m}\right)^2 + 2\left(\rho(x, S_n) + 2^{-n}\right)^2 - 4\rho(x, S_m)^2, \end{aligned}$$

since $\frac{1}{2}(x_m + x_n) \in S_m$. Thus

$$\begin{aligned} \|x_m - x_n\|^2 &\leqslant 2\left(\left(\rho(x, S_m) + 2^{-m}\right)^2 - \rho(x, S_m)^2\right) \\ &\quad + 2\left(\left(\rho(x, S_n) + 2^{-n}\right)^2 - \rho(x, S_m)^2\right). \end{aligned} \tag{2.2}$$

If $\rho(x, S_\infty)$ exists, then, by the first part of the proof, $\rho(x, S_n) \to \rho(x, S_\infty)$ as $n \to \infty$. It follows from this and (2.2) that $\|x_m - x_n\|^2 \to 0$ as $m, n \to \infty$; whence $(x_n)_{n \geqslant 1}$ is a Cauchy sequence in H and therefore converges to a limit $x_\infty \in \overline{S_\infty}$. Then

$$\begin{aligned} \rho(x, S_\infty) &= \rho\left(x, \overline{S_\infty}\right) \leqslant \|x - x_\infty\| \\ &= \lim_{n \to \infty} \|x - x_n\| \\ &\leqslant \lim_{n \to \infty} \left(\rho(x, S_n) + 2^{-n}\right) = \rho(x, S_\infty). \end{aligned}$$

Thus $\rho(x, S_\infty) = \|x - x_\infty\|$.

Conversely, suppose that $x_\infty = \lim_{n \to \infty} x_n$ exists. Let $0 < \alpha < \beta$ and $\varepsilon = \frac{1}{3}(\beta - \alpha)$. Pick N such that $2^{-N} < \varepsilon$ and $\|x_\infty - x_n\| < \varepsilon$ for all $n \geqslant N$. Either $\|x - x_\infty\| > \alpha + 2\varepsilon$ or $\|x - x_\infty\| < \beta$. In the first case, for all $n \geqslant N$,

$$\begin{aligned} \rho(x, S_n) &> \|x - x_n\| - 2^{-n} \\ &\geqslant \|x - x_\infty\| - \|x_\infty - x_n\| - \varepsilon \\ &> (\alpha + 2\varepsilon) - \varepsilon - \varepsilon = \alpha. \end{aligned}$$

In the other case, there exists $\nu > N$ such that $\|x - x_\nu\| < \beta$; we then have

$$\rho(x, S_\nu) \leqslant \|x - x_\nu\| < \beta.$$

It follows from this and the constructive least-upper-bound principle ([6], Theorem 2.1.18) that

$$\inf\{\rho(x, S_n) : n \geqslant 1\}$$

exists; whence, by (2.1), $d \equiv \rho(x, S_\infty)$ exists.

Finally, suppose that x_∞ exists, and consider any $y \in S_\infty$ with $y \neq x_\infty$. We have

$$\begin{aligned} 0 &< \|y - x_\infty\|^2 = \|y - x - (x_\infty - x)\|^2 \\ &= 2\|y - x\|^2 + 2\|x_\infty - x\|^2 - 4\left\|\frac{y + x_\infty}{2} - x\right\|^2 \\ &= 2\left(\|y - x\|^2 - d^2\right) + 2\left(\|x_\infty - x\|^2 - d^2\right) = 2\left(\|y - x\|^2 - d^2\right), \end{aligned}$$

so $\|x - y\| > d$. □

For each positive integer n we write

$$\mathfrak{A}_n \equiv n\mathfrak{A}_1 = \{nA : A \in \mathfrak{A}_1\}.$$

If $\mathfrak{A}_1$ is weak-operator totally bounded and hence strong-operator located, then $\mathfrak{A}_n$ has those two properties as well.

Our interest in Proposition 2.1 stems from this:

Corollary 2.2. *Let $\mathfrak{A}$ be a linear subspace of $\mathcal{B}(H)$ with $\mathfrak{A}_1$ weak-operator totally bounded, and let $x, y \in H$. For each n, let $y_n \in \mathfrak{A}_n$ satisfy $\|y - y_n\| < \rho(x, \mathfrak{A}_n x) + 2^{-n}$. Then*

$$\rho(y, \mathfrak{A}x) = \inf_{n \geqslant 1} \rho(y, \mathfrak{A}_n x) = \lim_{n\to\infty} \rho(y, \mathfrak{A}_n x).$$

Moreover, $\rho(y, \mathfrak{A}x)$ exists if and only if $(y_n)_{n\geqslant 1}$ converges to a limit $y_\infty \in H$; in which case, $\rho(y, \mathfrak{A}x) = \|y - y_\infty\|$, and $\|y - Ax\| > \|y - y_\infty\|$ for each $A \in \mathfrak{A}$ such that $Ax \neq y_\infty$.

One case of this corollary arises when the sequence $(\rho(y, \mathfrak{A}_n x))_{n\geqslant 1}$ stabilises:

Proposition 2.3. *Let $\mathfrak{A}$ be a linear subspace of $\mathcal{B}(H)$ such that $\mathfrak{A}_1$ is weak-operator totally bounded. Let $x, y \in H$, and suppose that for some positive integer N, $\rho(y, \mathfrak{A}_N x) = \rho(y, \mathfrak{A}_{N+1} x)$. Then $\rho(y, \mathfrak{A}x)$ exists and equals $\rho(y, \mathfrak{A}_N x)$.*

Proof. By Theorem 4.3.1 of [6], there exists a unique $z \in \overline{\mathfrak{A}_N x}$ such that $\rho(y, \mathfrak{A}_N x) = \|y - z\|$. We prove that $y - z$ is orthogonal to $\mathfrak{A}x$. Let $A \in \mathfrak{A}$, and consider $\lambda \in \mathbf{C}$ so small that $\lambda A \in \mathfrak{A}_1$. Since,

$$z - \lambda Ax \in \overline{\mathfrak{A}_{N+1} x},$$

we have

$$\begin{aligned} \langle y - z - \lambda Ax, y - z - \lambda Ax \rangle &\geqslant \rho(y, \mathfrak{A}_{N+1} x)^2 \\ &= \rho(y, \mathfrak{A}_N x)^2 = \langle y - z, y - z \rangle. \end{aligned}$$

This yields

$$|\lambda|^2 \|Ax\|^2 + 2\,\mathrm{Re}\,(\lambda \langle y - z, Ax \rangle) \geqslant 0.$$

Suppose that $\mathrm{Re}\,\langle y - z, Ax \rangle \neq 0$. Then by taking a sufficiently small real λ with

$$\lambda\,\mathrm{Re}\,\langle y - z, Ax \rangle < 0,$$

we obtain a contradiction. Hence $\mathrm{Re}\,\langle y - z, Ax \rangle = 0$. Likewise, $\mathrm{Im}\,\langle y - z, Ax \rangle = 0$. Thus $\langle y - z, Ax \rangle = 0$. Since $A \in \mathfrak{A}$ is arbitrary, we conclude that $y - z$ is orthogonal to $\mathfrak{A}x$ and hence to $\overline{\mathfrak{A}x}$. It is well known that this implies that z is the unique closest point to y in the closed linear subspace $\overline{\mathfrak{A}x}$. Since $\mathfrak{A}x$ is dense in $\overline{\mathfrak{A}x}$, it readily follows that $\rho(y, \mathfrak{A}x) = \rho\left(y, \overline{\mathfrak{A}x}\right) = \|y - z\|$. □

The final result in this section will be used in the proof of our main theorem.

Proposition 2.4. *Let $\mathfrak{A}$ be a linear subspace of $\mathcal{B}(H)$ with weak-operator totally bounded unit ball, and let $x \in H$. Suppose that there exists $r > 0$ such that*

$$\mathfrak{A}_1 x \supset B_{\mathfrak{A}x}(0, r) \equiv \mathfrak{A}x \cap B(0, r).$$

Then $\mathfrak{A}x$ is located in H; in fact, for each $y \in H$, there exists a positive integer N such that $\rho(y, \mathfrak{A}x) = \rho(y, \mathfrak{A}_N x)$.

Proof. Fixing $y \in H$, compute a positive integer $N > 2\|y\|/r$. Let $A \in \mathfrak{A}$, and suppose that

$$\|y - Ax\| < \rho(y, \mathfrak{A}_N x).$$

We have either $\|Ax\| < Nr$ or $\|Ax\| > 2\|y\|$. In the first case, $N^{-1}Ax \in B_{\mathfrak{A}x}(0, r)$, so there exists $B \in \mathfrak{A}_1$ with $N^{-1}Ax = Bx$ and therefore $Ax = NBx$. But $NB \in \mathfrak{A}_N$, so

$$\|y - Ax\| = \|y - NBx\| \geqslant \rho(y, \mathfrak{A}_N x),$$

a contradiction. In the case $\|Ax\| \geqslant Nr > 2\|y\|$, we have

$$\|y - Ax\| \geqslant \|Ax\| - \|y\| > \|y\| \geqslant \rho(y, \mathfrak{A}_N x),$$

another contradiction. We conclude that $\|y - Ax\| \geqslant \rho(y, \mathfrak{A}_N x)$ for each $A \in \mathfrak{A}$. On the other hand, given $\varepsilon > 0$, we can find $A \in \mathfrak{A}_N$ such that $\|y - Ax\| < \rho(y, \mathfrak{A}_N x) + \varepsilon$. It now follows that $\rho(y, \mathfrak{A}x)$ exists and equals $\rho(y, \mathfrak{A}_N x)$. □

3. Generalising the open mapping theorem

The key to our main result on the existence of projections of the form $[\mathfrak{A}x]$ is a generalisation of the open mapping theorem from functional analysis ([6], Theorem 6.6.4). Before giving that generalisation, we note a proposition and a lemma.

Proposition 3.1. *If C is a balanced, convex subset of a normed space X, then $V \equiv \bigcup_{n \geqslant 1} nC$ is a linear subspace of X.*

Proof. Let $x \in V$ and $\alpha \in \mathbf{C}$. Pick a positive integer n and an element c of C such that $x = nc$. If $\alpha \neq 0$, then since C is balanced, $|\alpha|^{-1} \alpha c \in C$, so

$$\alpha x = \alpha nc = |\alpha| n |\alpha|^{-1} \alpha c \in |\alpha| nC \subset (1 + |\alpha|) nC.$$

In the general case, we can apply what we have just proved to show that

$$(1 + \alpha) x \in (1 + |1 + \alpha|) nC \subset (2 + |\alpha|) nC.$$

Now, since C is balanced,

$$-x = n(-c) \in nC \subset (2 + |\alpha|)nC.$$

Hence, by the convexity of $(2 + |\alpha|)nC$,

$$\alpha x = 2\frac{(1 + \alpha)x - x}{2} \in 2(2 + |\alpha|)nC.$$

Taking N as any integer $> 2(2 + |\alpha|)n$, we now see that $\alpha x \in NC \subset V$. In view of the foregoing and the fact that $(nC)_{n \geqslant 1}$ is an ascending sequence of sets, if x' also belongs to V

we can take N large enough to ensure that αx and x' both belong to NC. Picking $c, c' \in C$ such that $\alpha x = Nc$ and $x' = Nc'$, we obtain

$$\alpha x + x' = 2N\left(\frac{c+c'}{2}\right) \in 2NC,$$

so $\alpha x + x' \in V$. □

We call a bounded subset C of a Banach space X ***superconvex*** if for each sequence $(x_n)_{n\geqslant 1}$ in C and each sequence $(\lambda_n)_{n\geqslant 1}$ of nonnegative numbers such that $\sum_{n=1}^{\infty}\lambda_n$ converges to 1 and the series $\sum_{n=1}^{\infty}\lambda_n x_n$ converges, we have $\sum_{n=1}^{\infty}\lambda_n x_n \in C$. In that case, C is clearly convex.

Lemma 3.2. *Let C be a located, bounded, balanced, and superconvex subset of a Banach space X, such that $X = \bigcup_{n\geqslant 1} nC$. Let $y \in X$ and $r > \|y\|$. Then there exists $\xi \in 2C$ such that if $y \neq \xi$, then $\rho(z, C) > 0$ for some z with $\|z\| < r$.*

Proof. Either $\rho(y, C) > 0$ and we take $z = y$, or else, as we suppose, $\rho(y, C) < r/2$. Choosing $x_1 \in 2C$ such that $\left\|y - \frac{1}{2}x_1\right\| < r/2$ and therefore $\|2y - x_1\| < r$, set $\lambda_1 = 0$. Then either $\rho(2y - x_1, C) > 0$ or $\rho(2y - x_1, C) < r/2$. In the first case, set $\lambda_k = 1$ and $x_k = 0$ for all $k \geqslant 2$. In the second case, pick $x_2 \in 2C$ such that $\left\|2y - x_1 - \frac{1}{2}x_2\right\| < r/2$ and therefore $\left\|2^2 y - 2x_1 - x_2\right\| < r$, and set $\lambda_2 = 0$. Carrying on in this way, we construct a sequence $(x_n)_{n\geqslant 1}$ in $2C$, and an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ with the following properties.

- If $\lambda_n = 0$, then
$$\rho\left(2^{n-1}y - \sum_{i=1}^{n} 2^{n-i-1}x_i, C\right) < \frac{r}{2}$$
and
$$\left\|2^n y - \sum_{i=1}^{n} 2^{n-i}x_i\right\| < r.$$
- If $\lambda_n = 1 - \lambda_{n-1}$, then
$$\rho\left(2^{n-1}y - \sum_{i=1}^{n} 2^{n-i-1}x_i, C\right) > 0$$
and $x_k = 0$ for all $k \geqslant n$.

Compute $\alpha > 0$ such that $\|x\| < \alpha$ for all $x \in 2C$. Then the series $\sum_{i=1}^{\infty} 2^{-i}x_i$ converges, by comparison with $|\alpha|\sum_{i=1}^{\infty} 2^{-i}$, to a sum ξ in the Banach space X. Since $\sum_{i=1}^{\infty} 2^{-i} = 1$ and C is superconvex, we see that

$$\sum_{i=1}^{\infty} 2^{-i}x_i = 2\sum_{i=1}^{\infty} 2^{-i}\left(\frac{1}{2}x_i\right) \in 2C.$$

If $y \neq \xi$, then there exists N such that

$$\left\|y - \sum_{i=1}^{N} 2^{-i}x_i\right\| > 2^{-N}r$$

and therefore
$$\left\| 2^N y - \sum_{i=1}^{N} 2^{N-i} x_i \right\| > r.$$
It follows that we cannot have $\lambda_N = 0$, so $\lambda_N = 1$ and therefore there exists $\nu \leqslant N$ such that $\lambda_\nu = 1 - \lambda_{\nu-1}$. Setting
$$z \equiv 2^{\nu-1} y - \sum_{i=1}^{\nu-1} 2^{\nu-i-1} x_i,$$
we see that $\rho(z, C) > 0$ and $\|z\| < r$, as required. □

We now prove our generalisation of the open mapping theorem.

Theorem 3.3. *Let X be a Banach space,and C a located, bounded, balanced, and superconvex subset of X such that $\rho(0, -C)$ exists and $X = \bigcup_{n \geqslant 1} nC$. Then there exists $r > 0$ such that $B(0, r) \subset C$.*

Proof. Consider the identity
$$X = \bigcup_{n \geqslant 1} \overline{nC}.$$
By Theorem 6.6.1 of [6] (see also [8]), there exists N such that the interior of $\overline{NC}$ is inhabited. Thus there exist $y_0 \in NC$ and $R > 0$ such that $B(y_0, R) \subset \overline{NC}$. Writing $y_1 = N^{-1} y_0$ and $r = (2N)^{-1} R$, we obtain $B(y_1, 2r) \subset \overline{C}$.It follows from Lemma 6.6.3 of [6] that $B(0, 2r) \subset \overline{C}$. Now consider any $y \in B(0, 2r)$. By Lemma 3.2, there exists $\xi \in 2C$ such that if $y \neq \xi$, then there exists $z \in B(0, 2r)$ with $\rho(z, C) > 0$. Since $B(0, 2r) \subset \overline{C}$, this is absurd. Hence $y = \xi \in 2C$. It follows that $B(0, 2r) \subset 2C$ and hence that $B(0, r) \subset C$. □

Note that in Lemma 3.2 and Theorem 3.3 we can replace the superconvexity of C by these two properties: C is convex, and for each sequence $(x_n)_{n \geqslant 1}$ in C, if $\sum_{n=1}^{\infty} 2^{-n} x_n$ converges in H, then its sum belongs to C.

We now derive two corollaries of Theorem 3.3.

Corollary 3.4 (**The open mapping theorem** ([6], Theorem 6.6.4)[3])**.** *Let X, Y be Banach spaces, and T a sequentially continuous linear mapping of X onto Y such that $T\left(\overline{B(0,1)}\right)$ is located and $\rho\left(0, -T\left(\overline{B(0,1)}\right)\right)$ exists. Then there exists $r > 0$ such that $B(0, r) \subset T\left(\overline{B(0,1)}\right)$.*

Proof. In view of Theorem 3.3, it will suffice to prove that $C \equiv T\left(\overline{B(0,1)}\right)$ is superconvex. But if $(x_n)_{n \geqslant 1}$ is a sequence in $\overline{B(0,1)}$ and $(\lambda_n)_{n \geqslant 1}$ is a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$, then $\|\lambda_n x_n\| \leqslant \lambda_n$ for each n, so $\sum_{n=1}^{\infty} \lambda_n x_n$ converges in X; moreover,
$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| \leqslant \sum_{n=1}^{\infty} \lambda_n = 1,$$

[3]This is but one version of the open mapping theorem; for another, see [5].

so, by the sequential continuity of T,

$$T\left(\sum_{n=1}^{\infty}\lambda_n x_n\right)\in C.$$

Thus C is superconvex. $\square$

Theorem 3.3 also leads to the ***proof of Theorem 1.1:***

Proof. Taking $C\equiv\mathfrak{A}_1 x$, we know that C is located (since $\mathfrak{A}_1$ is weak-operator totally bounded and hence, by [7, 14], strong-operator located), as well as bounded and balanced. To prove that C is superconvex, consider a sequence $(A_n)_{n\geqslant 1}$ in $\mathfrak{A}_1$, and a sequence $(\lambda_n)_{n\geqslant 1}$ of nonnegative numbers such that $\sum_{n=1}^{\infty}\lambda_n$ converges to 1. For $k\geqslant j$ we have

$$\left\|\sum_{n=j}^{k}\lambda_n A_n\right\|\leqslant\sum_{n=j}^{k}\lambda_n,$$

so $\sum_{n=1}^{\infty}\lambda_n A_n$ converges uniformly to an element A of $\mathcal{B}_1(H)$. Since $\mathfrak{A}$ is uniformly closed, $A\in\mathfrak{A}_1$, so $\sum_{n=1}^{\infty}\lambda_n A_n x=Ax\in\mathfrak{A}_1 x$. Thus C is superconvex. We can now apply Theorem 3.3, to produce $r>0$ such that $B_{\mathfrak{A}x}(0,r)\subset C$. The locatedness of $\mathfrak{A}x$, and the consequent existence of the projection $[\mathfrak{A}x]$, now follow from Proposition 2.4. $\square$

We now discuss further the requirement, in Theorem 1.1, that $\rho_{\mathfrak{A}x}(0,-\mathfrak{A}_1 x)$ exist, where $\mathfrak{A}_1$ is weak-operator totally bounded. We begin by giving conditions under which that requirement is satisfied.

If $\mathfrak{A}x$ has positive, finite dimension—in which case it is both closed and located in H—then $\mathfrak{A}x-\mathfrak{A}_1 x$ is inhabited, so Proposition (1.5) of [9] can be applied to show that $\mathfrak{A}x-\mathfrak{A}_1 x$ is located in $\mathfrak{A}x$. In particular, $\rho_{\mathfrak{A}x}(0,-\mathfrak{A}_1 x)$ exists. On the other hand, if P is a projection in $\mathcal{B}(H)$ and

$$\mathfrak{A}\equiv\{PTP:T\in\mathcal{B}(H)\},$$

then $\mathfrak{A}$ can be identified with $\mathcal{B}(P(H))$, so $\mathfrak{A}_1$ is weak-operator totally bounded. Moreover, if $x\neq 0$, then $\mathfrak{A}x=P(H)$ and so is both closed and located, $\mathfrak{A}_1 x=\overline{B}(0,\|Px\|)\cap P(H)$, and $\rho_{\mathfrak{A}x}(0,-\mathfrak{A}_1 x)=\|Px\|$.

We end with a Brouwerian example showing that we cannot drop the existence of $\rho_{\mathfrak{A}x}(0,-\mathfrak{A}_1 x)$ from the hypotheses of Theorem 1.1. Consider the case where $H=\mathbf{R}\times\mathbf{R}$, and let $\mathfrak{A}$ be the linear subspace (actually an algebra) of $\mathcal{B}(H)$ comprising all matrices of the form

$$T_{a,b}\equiv\begin{pmatrix}a&0\\0&b\end{pmatrix}$$

with $a,b\in\mathbf{R}$. It is easy to show that $\mathfrak{A}$ is uniformly closed: if $(a_n),(b_n)$ are sequences in $\mathbf{R}$ such that $(T_{a_n,b_n})_{n\geqslant 1}$ converges uniformly to an element $T\equiv\begin{pmatrix}a_\infty&p\\q&b_\infty\end{pmatrix}$, then

$$a_n=T_{a_n,b_n}\begin{pmatrix}1\\0\end{pmatrix}\to T\begin{pmatrix}1\\0\end{pmatrix}=a_\infty,$$

Likewise, $b_n\to b_\infty$, $p=0$, and $q=0$. Hence $T=T_{a_\infty,b_\infty}\in\mathfrak{A}$.

Now, if (x,y) is in the unit ball of H, then

$$\begin{aligned}\left\|T_{a,b}\begin{pmatrix}x\\y\end{pmatrix}\right\|^2 &= \left\|\begin{pmatrix}ax\\by\end{pmatrix}\right\|^2 = a^2x^2+b^2y^2\\ &= a^2\left(x^2+y^2\right)+\left(b^2-a^2\right)y^2\\ &= a^2+\left(b^2-a^2\right)y^2.\end{aligned}$$

We see from this that if $a^2 \geqslant b^2$, then $\|T_{a,b}\|^2 \leqslant a^2$; moreover, $T_{a,b}(1,0) = a$, so $\|T_{a,b}\|^2 = a^2$. If $a^2 < b^2$, then a similar argument shows that $\|T_{a,b}\|^2 = b^2$. It now follows that $\|T_{a,b}\|$ exists and equals $\max\{|a|,|b|\}$. Also, since, relative to the uniform topology on $\mathcal{B}(H)$, $\mathfrak{A}_1$ is homeomorphic to the totally bounded subset

$$\{(a,b) : \max\{|a|,|b|\} \leqslant 1\}$$

of $\mathbf{R}^2$, it is uniformly, and hence weak-operator, totally bounded.

Consider the vector $\xi \equiv (1,c)$, where $c \in \mathbf{R}$. If $c = 0$, then $\mathfrak{A}\xi = \mathbf{R}\times\{0\}$, the projection of H on $\mathfrak{A}\xi$ is just the projection on the x-axis, and $\rho((0,1),\mathfrak{A}\xi) = 1$. If $c \neq 0$, then

$$\mathfrak{A}\xi = \{(a,cb) : a,b \in \mathbf{R}\} = \mathbf{R}\times\mathbf{R},$$

the projection of H on $\mathfrak{A}\xi$ is just the identity projection I, and $\rho((0,1),\mathfrak{A}\xi) = 0$. Suppose, then, that the projection P of H on $\mathfrak{A}\xi$ exists. Then either $\rho((0,1),\mathfrak{A}\xi) > 0$ or $\rho((0,1),\mathfrak{A}\xi) < 1$. In the first case, $c = 0$; in the second, $c \neq 0$. Thus if $[\mathfrak{A}x]$ exists for each $x \in H$, then we can prove that

$$\forall_{x\in\mathbf{R}}\,(x = 0 \vee x \neq 0),$$

a statement constructively equivalent to the essentially nonconstructive omniscience principle **LPO**:

> For each binary sequence $(a_n)_{n\geqslant 1}$, either $a_n = 0$ for all n or else there exists n such that $a_n = 1$.

It follows from this and our Theorem 1.1 that if $\rho_{\mathfrak{A}x}(0,-\mathfrak{A}_1x)$ exists for each $x \in H$, then we can derive **LPO**.

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