

CC-CIRCUITS AND THE EXPRESSIVE POWER OF NILPOTENT ALGEBRAS

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ABSTRACT. We show that CC-circuits of bounded depth have the same expressive power as circuits over finite nilpotent algebras from congruence modular varieties. We use this result to phrase a new algebraic version of Barrington, Straubing and Thérien’s conjecture, which states that CC-circuits of bounded depth need exponential size to compute AND.

Furthermore we investigate the complexity of deciding identities and solving equations in a fixed nilpotent algebra. Under the assumption that the conjecture is true, we obtain quasipolynomial algorithms for both problems. On the other hand, if AND is computable by uniform CC-circuits of bounded depth and polynomial size, we can construct a nilpotent algebra in which checking identities is coNP-complete, and solving equations is NP-complete.

1. INTRODUCTION

Proving lower bounds on the size of Boolean circuits needed to compute explicit functions is a fundamental, but also notoriously hard problem in theoretical computer science. A rare exception with known sharp lower bounds is the parity function: In [FSS84] it was shown that parity cannot be computed by bounded depth circuits of polynomial size; [Yao85] then provided exponential lower bounds, which were further improved by Håstad [Hås86] to the almost optimal $2^{\Omega(n^{(d-1)^{-1}})}$, where d stands for the depth of the circuits.

These results led to the question how much computational power we gain, if we also allow gates that describe parity or other counting functions in the construction of bounded depth circuits. More precisely, by such ‘counting gates’ we mean MOD_m -gates (for some $m \in \mathbb{N}$) of unbounded fan-in that output 1, if the inputs sum up to 0 modulo m , and 0 otherwise (This use of MOD_m follows the notational convention used e.g. in [Vol99, MPT91]. We remark that other authors such as [CGPT06, Cau96] actually denote the *negation* of this function by MOD_m). The class of functions that can be expressed by polynomially growing such bounded depth circuits is denoted by $\text{AC}^0[m]$, and its union by $\text{ACC}^0 = \bigcup_{m>1} \text{AC}^0[m]$.

Key words and phrases: CC-circuits, Maltsev algebras, nilpotent algebras, Circuit Equivalence Problem, Circuit Satisfaction Problem.

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An important step towards a characterization of ACC^0 is to understand circuits that *only* consist of MOD_m -gates first. Such circuits are called $\text{CC}[m]$ -circuits. The functions that can be computed by bounded depth $\text{CC}[m]$ -circuits of polynomial size are denoted by $\text{CC}^0[m]$, and their union over all m by $\text{CC}^0 = \bigcup_{m>1} \text{CC}^0[m]$. Despite being studied extensively, many questions about $\text{CC}[m]$ -circuits are still wide open. For instance their relationship for different values of m is not well-understood, although this would be central for proving or disproving a conjecture of Smolensky [Smo87], which says that constant depth circuits having only AND, OR and MOD_m gates cannot compute MOD_q in sub-exponential size, when m is a prime power, and q is coprime to it.

Another big open question is whether bounded depth $\text{CC}[m]$ -circuits can efficiently compute AND or not. If it is not the case, it would signify that there is a fundamental difference between circuits that use logical gates versus circuits that use counting gates. Barrington, Straubing and Thérien first conjectured that AND has exponential lower bounds, mirroring Håstad's result:

Conjecture 1.1 (Barrington, Straubing and Thérien [BST90]). Let $m \in \mathbb{N}$ and let $(C_n)_{n \in \mathbb{N}}$ be a family of bounded depth $\text{CC}[m]$ -circuits that compute AND. Then C_n is of size $2^{\Omega(n^q)}$ for some $q > 0$.

For values of m that have more than one prime divisor one can construct $\text{CC}[m]$ -circuits $(C_n)_{n \in \mathbb{N}}$ of depth d and size $\mathcal{O}(2^{n^{(d-1)^{-1}}})$ that compute AND (see for instance [IKK20]). Thus one might be tempted to conjecture (in analogy to Håstad's result) that a value of q close to $(d-1)^{-1}$ gives us optimal lower bound. However, this is not the case: By the recent results in [CW21], for every $q > 0$, there is a choice of m , such that there are depth 3 $\text{CC}[m]$ -circuits of size $2^{\mathcal{O}(n^q)}$ computing AND. It was further shown in [IKK21] that for the correct choice of m there are even $\text{CC}[m]$ -circuits of depth 2 and size $2^{\mathcal{O}(n^q \log(n))}$ computing AND. Thus any optimal bound in Conjecture 1.1 needs to depend not only on the depth d but also on the modulus m .

Also weaker versions of Conjecture 1.1 can be found in the literature, e.g. [MPT91] conjectured that AND is not in CC^0 . Both the strong and the weak version of Conjecture 1.1 are open until today, with the best known general lower bound for AND being superlinear [CGPT06]. It was further shown in [HK10] that AND is in uniform *probabilistic* CC^0 (i.e. for every $c > 0$, AND_n can be computed up to error $< \frac{1}{n^c}$ by P-uniform, bounded depth CC -circuits with $\mathcal{O}(\log(n))$ random bits), which could be interpreted as evidence contrary to the conjecture.

However, for some special cases Conjecture 1.1 is confirmed: For prime powers p^k it is well known that $\text{CC}[p^k]$ -circuits of bounded depth *cannot* compute AND [BST90]. The conjecture is further confirmed for the special case of bounded depth circuits with MOD_m -gates at the input level, and MOD_{p^k} -gates everywhere else, where $m \in \mathbb{N}$ and p^k is a prime power (see [BST90], or [Cau96, Theorem 1]).

The first results about CC -circuits arose from a characterization of them in the language of groups/monoids: In [BST90] Barrington, Straubing and Thérien used the notion of NUDFA (non uniform deterministic finite automata, introduced in [Bar89]) to show that a language is in ACC^0 if and only if it is accepted by a NUDFA over a solvable monoid, and in CC^0 if and only if it is accepted by a NUDFA over a solvable group.¹ NUDFAs proved not

¹We remark that this characterization of CC^0 was not explicitly stated in [BST90], as the paper does not use the notion of CC -circuits. However the relevant literature often attributes it to [BST90], as it is a

only to be a fruitful tool in circuit complexity, but also led to new developments in algebra, regarding the study of equations in monoids and groups [BMM⁺00].

In this paper we give a new algebraic description of CC-circuits, using concepts from universal algebra, more specifically *commutator theory*. We show that, in some sense, CC-circuits of bounded depth can be represented by the circuits over a nilpotent algebra from a congruence modular variety and vice-versa. As a corollary we obtain the following theorem:

Theorem 1.2. *Conjecture 1.1 is true, if and only if for every finite nilpotent algebra \mathbf{A} from a congruence modular variety, and for every $0 \in A$ there is a $q > 0$, such that every sequence $(p_n)_{n \in \mathbb{N}}$ of nonconstant 0-absorbing circuits $p_n(x_1, \dots, x_n)$ over \mathbf{A} is of size $2^{\Omega(n^q)}$.*

Here, for a set A and an element $0 \in A$ we call an operation $f: A^n \rightarrow A$ 0-absorbing if $0 = f(0, a_2, \dots, a_n) = f(a_1, 0, a_3, \dots, a_n) = \dots = f(a_1, \dots, a_{n-1}, 0)$ holds for all values $a_1, \dots, a_n \in A$. Note that on the Boolean domain $A = \{0, 1\}$ the only n -ary 0-absorbing operations are the n -ary conjunction and the constant 0-function. Thus nonconstant 0-absorbing operations can be seen as a natural generalization of conjunctions to arbitrary domains. We remark that 0-absorbing operations are of independent interest in commutator theory, as they characterize properties of the so called higher commutator [AM10].

In the course of the proof of Theorem 1.2 we are actually going to prove a stronger, but more technical result in Theorem 3.6, which allows us to compute explicit bounds for AND from bounds on nonconstant 0-absorbing circuits in nilpotent algebras, and vice-versa. We are further going to discuss how known results about $\text{CC}[m]$ -circuits correspond to known results about nilpotent algebras: the fact that for primes p , $\text{CC}[p]$ -circuits of bounded depth cannot compute AND, corresponds for instance to the result that finite nilpotent algebras of prime power size have only nonconstant 0-absorbing polynomial operations up to some arity [BB87]. The fact that Conjecture 1.1 holds for MOD_p - MOD_q circuits [BST90] was essentially reproven in the language of nilpotent algebras in [IKK18].

At last we discuss the impact of Conjecture 1.1 on two computational problems, namely the circuit satisfaction problem $\text{CSAT}(\mathbf{A})$ and the circuit equivalence problem $\text{CEQV}(\mathbf{A})$ for fixed nilpotent algebras \mathbf{A} . Here $\text{CSAT}(\mathbf{A})$ models the decision problem, whether an equation over the algebra \mathbf{A} (encoded by two circuits) has a solution, while $\text{CEQV}(\mathbf{A})$ asks, whether two circuits are equivalent. In [IK18] Idziak and Krzaczkowski gave an almost complete complexity classification of both problems for algebras from congruence modular varieties, relating the complexity to the commutator theoretical properties of the given algebra. Essentially the only case left open are nilpotent, but not supernilpotent algebras (Problem 2 in [IK18]). We show that, under the assumption of Conjecture 1.1, there are quasipolynomial algorithms for both CSAT and CEQV of such algebras. On the other hand we show that, if AND is in P-uniform CC^0 , then there is a nilpotent algebra with NP-complete circuit satisfaction problem, and coNP -complete circuit equivalence problem.

We remark that, since this paper first appeared as a preprint, there were some new developments in the study of CSAT and CEQV for nilpotent algebras. Under the assumption of the Exponential Time Hypothesis (ETH), there are nilpotent algebras, for which both problems have proper quasipolynomial lower bounds [IKK20]. Together with the results in this paper, this implies that, under the ETH and Conjecture 1.1, there are some nilpotent algebras, in which CSAT and CEQV can be solved in quasipolynomial, but not polynomial time. In particular no P/NP-c dichotomy result holds for CSAT . This is an interesting

direct consequence of the presented results. One of the first explicit mentions of it can be found in [MPT91, Theorem 2.8]. Also Conjecture 1.1 was originally stated as a conjecture about NUDFAs in [BST90].

contrast to the problem of solving *systems* of equations in a given algebra \mathbf{A} , since for every \mathbf{A} , this problem is equivalent to a finite constraint satisfaction problem and hence either in P or NP -complete by the CSP dichotomy theorem (independently proven by Zhuk [Zhu17, Zhu20] and Bulatov [Bul17]).

The structure of this paper is as follows: In Section 2 we discuss some necessary background from universal algebra and give a characterization of nilpotent algebras. In Section 3 we prove Theorem 1.2 and discuss its implications on CC-circuits and nilpotent algebras. In Section 4 we derive the complexity results for CSAT and CEQV of nilpotent algebras.

2. BACKGROUND FROM UNIVERSAL ALGEBRA

2.1. Polynomials and circuits over algebras. An algebra \mathbf{A} is a pair $(A, (f_i^{\mathbf{A}})_{i \in I})$, where A is a set (the *universe* of \mathbf{A}), and every element of the family $(f_i^{\mathbf{A}})_{i \in I}$ is a finitary operation $f_i^{\mathbf{A}}: A^{k_i} \rightarrow A$ (the *basic operations* of \mathbf{A}). The *signature* of \mathbf{A} is the family $(k_i)_{i \in I}$, i.e. it describes which basic operations are of which arity. In this paper we are only going to consider *finite algebras*, i.e. algebras that have a finite universe and only finitely many basic operations. By $\text{ar}(\mathbf{A})$ we denote the maximal arity of the basic operations of \mathbf{A} .

A *term operation* of \mathbf{A} is an operation that can be obtained as a composition of basic operations of \mathbf{A} . A *polynomial operation* allows also the use of elements of A in its construction. For the ring of integers $(\mathbb{Z}, +, 0, -, \cdot)$ for instance, the polynomial operations are just the polynomial operations in the conventional sense (e.g. the operation $p: \mathbb{Z}^3 \rightarrow \mathbb{Z}$ defined by $p(x_1, x_2, x_3) = (2x_1 \cdot x_3 + 4) \cdot x_1 \cdot x_2 + 1$). The set of all polynomial operations of \mathbf{A} is denoted by $\text{Pol}(\mathbf{A})$. Let \mathbf{A} and \mathbf{B} be two algebras with the same universe; if $\text{Pol}(\mathbf{A}) = \text{Pol}(\mathbf{B})$ we say that \mathbf{A} and \mathbf{B} are *polynomially equivalent*, if $\text{Pol}(\mathbf{A}) \subseteq \text{Pol}(\mathbf{B})$, we say \mathbf{B} is a *polynomial extension* of \mathbf{A} .

Given a finite algebra, there are different ways of encoding its polynomial operations. The naive way is to just encode them by the string defining them. Such a string is usually referred to as a *polynomial* over \mathbf{A} . However, in an effort to compress the input, one can also consider *circuits* over \mathbf{A} , i.e. A -valued circuits with a unique output gate, whose gates are labelled by the basic operation of \mathbf{A} . We say that two circuits (respectively two polynomials) are *equivalent*, if they define the same operation.

By the following folklore result, the circuit approach does not only allow for a more concise, but also more stable representation of polynomial operations:

Lemma 2.1. *Let \mathbf{A} and \mathbf{B} be two finite algebras with the same universe and $\text{Pol}(\mathbf{A}) \subseteq \text{Pol}(\mathbf{B})$. Then we can rewrite every circuit $C(\bar{x})$ over \mathbf{A} to an equivalent circuit $C'(\bar{x})$ over \mathbf{B} in linear time.*

The circuit C' in Lemma 2.1 can be easily obtained, by substituting every gate in C by a circuit over \mathbf{B} defining the corresponding basic operation of \mathbf{A} . Note that it follows from Lemma 2.1 that the size of C' is linear in the size of C . An analogue statement is however not true for the encoding by polynomials; in fact, the results in [HS12] imply that under the assumption $\mathsf{P} \neq \mathsf{NP}$ it even fails for quite well-behaved algebras such as the alternating group A_4 . In this paper we are therefore only going to discuss the circuit encoding of polynomial operations.

For a circuit C over \mathbf{A} with input gates $\bar{x} = (x_1, \dots, x_n)$ we will write $C(\bar{x})$ both for the circuit, and the n -ary polynomial operation induced by it, but this should never cause any confusion. We call a circuit C constant/0-absorbing/etc., if the polynomial operation $C(\bar{x})$ defined by it is constant/0-absorbing/etc.

2.2. Nilpotent algebras from congruence modular varieties. Commutator theory is a field of universal algebra that generalizes concepts from the commutator theory of groups to arbitrary algebras. In particular there is the notion of a central series of congruences, which allows to define nilpotent algebras (as a generalization of nilpotent groups and nilpotent rings). Since we are only interested in nilpotent algebras from congruence modular varieties, we refrain from giving the original definition (via the so called *term congruence commutator*) and refer to [FM87] for background.

Algebras from *congruence modular varieties* form a quite broad class that contains many examples of interest in abstract algebra and computer science, such as: lattices, Boolean algebras, fields, rings, modules, groups, quasigroups, and all their polynomial extensions. Commutator theory works especially well in the congruence modular case.

In a congruence modular variety, an algebra is *Abelian* (or *1-nilpotent*) if and only if it is polynomially equivalent to a *module*. Here, a module \mathbf{A} is an algebra $\mathbf{A} = (A, +, 0, -, (r)_{r \in R})$, where $(A, +, 0, -)$ is the underlying Abelian group and every ring element $r \in R$ is considered as a unary operation $r(x) = r \cdot x$. Therefore all operations of such an Abelian algebra \mathbf{A} are affine operations $p(x_1, \dots, x_n) = \sum_{i=1}^n r_i x_i + c$ for some scalars $r_i \in R$ and a constant $c \in A$.

General nilpotent algebras from congruence modular varieties can, in some sense, be decomposed into Abelian algebras (this follows from Proposition 7.1. in [FM87]). We are going to treat this characterization as a definition of nilpotence:

Definition 2.2. An algebra $\mathbf{A} = (A, (f^{\mathbf{A}})_{f \in I})$ from a congruence modular variety is

- *1-nilpotent* (or *Abelian*) if and only if it is polynomially equivalent to a module;
- *n-nilpotent*, if there are algebras $\mathbf{L} = (L, (f^{\mathbf{L}})_{f \in I})$ and $\mathbf{U} = (U, (f^{\mathbf{U}})_{f \in I})$ in the same signature as \mathbf{A} such that
 - \mathbf{L} is Abelian and \mathbf{U} is $(n-1)$ -nilpotent,
 - $A = L \times U$, where L and U are the universes of \mathbf{L} and \mathbf{U} respectively,
 - Every basic operation $f^{\mathbf{A}}$ of \mathbf{A} is of the form

$$f^{\mathbf{A}}((l_1, u_1), \dots, (l_k, u_k)) = (f^{\mathbf{L}}(l_1, \dots, l_k) + \hat{f}(u_1, \dots, u_k), f^{\mathbf{U}}(u_1, \dots, u_k)),$$
 for a function $\hat{f}: U^k \rightarrow L$.

Here $+$ denotes the addition of the module that is polynomially equivalent to \mathbf{L} . We write $\mathbf{A} = \mathbf{L} \otimes \mathbf{U}$ for the decomposition of \mathbf{A} into \mathbf{L} and \mathbf{U} .

When talking about nilpotent algebras in this paper, we will from now on always implicitly assume that they are from a congruence modular variety.

Note that by Definition 2.2, n -nilpotent algebras can be regarded as an $(n-1)$ -nilpotent algebra \mathbf{U} ‘acting’ on the Abelian algebra \mathbf{L} by the operations $\hat{f}: U^k \rightarrow L$. The *degree of nilpotency* of \mathbf{A} is the smallest n , such that \mathbf{A} is n -nilpotent. Note that this degree is always bounded by $\lfloor \log_2 |A| \rfloor$.

By a recent result of Aichinger every nilpotent algebra has a nicely behaved extension by some Abelian group operations:

Theorem 2.3 (Corollary of Theorem 4.2. in [Aic19a]). *Let \mathbf{A} be a finite nilpotent algebra from a congruence modular variety and let $0 \in A$. Then there exists a nilpotent algebra \mathbf{B} with the same universe A , such that*

- $\text{Pol}(\mathbf{A}) \subseteq \text{Pol}(\mathbf{B})$,
- $\mathbf{B} = \mathbf{L}_1 \otimes (\mathbf{L}_2 \otimes (\cdots \otimes \mathbf{L}_n))$, such that
 - every \mathbf{L}_i is polynomially equivalent to a module with group $(L_i, +, 0, -) = \mathbb{Z}_{q_i}^{k_i}$ and a ring $R \leq \text{End}(\mathbb{Z}_{q_i}^{k_i})$, for a prime q_i and $k_i \in \mathbb{N}$,
 - \mathbf{B} contains basic operations $+$, 0 , $-$ such that $(A, +, 0, -) = \prod_{i=1}^n (L_i, +, 0, -)$.

We remark that Theorem 4.2. in [Aic19a] is phrased in a different way from our statement: it states that to every central series of congruences $0_A = \alpha_0 < \alpha_1 < \dots < \alpha_n = 1_A$ of \mathbf{A} and every constant $0 \in A$, we can associate an Abelian group $(A, +, 0, -)$, such that $(\alpha_i)_{i=0}^n$ is still a central series of \mathbf{B} , the extension of \mathbf{A} by $+$, 0 , $-$. By applying Proposition 7.1. from [FM87] iteratively to \mathbf{B} , we then obtain the algebras \mathbf{L}_i such that $\mathbf{B} = \mathbf{L}_1 \otimes (\mathbf{B}/\alpha_1) = \mathbf{L}_1 \otimes (\mathbf{L}_2 \otimes (\mathbf{B}/\alpha_2)) = \cdots = \mathbf{L}_1 \otimes (\mathbf{L}_2 \otimes (\cdots \otimes \mathbf{L}_n))$ such that $(A, +, 0, -) = \prod_{i=1}^n (L_i, +, 0, -)$. If we pick the central series to be of maximal length, it holds that $(L_i, +, 0, -) = \mathbb{Z}_{q_i}^{k_i}$ for some prime q_i by Lemma 4.1. in [Aic19a].

By Lemma 2.1 every circuit over \mathbf{A} can be rewritten in linear time to a circuit over its polynomial extension \mathbf{B} given by Theorem 2.3; working in the extension \mathbf{B} instead of \mathbf{A} will simplify our proof. In particular, we are going to use that every element $x \in A$ can be identified with a tuple in $(A, +, 0, -) = \prod_{i=1}^n (L_i, +, 0, -) = \prod_{j=1}^s \mathbb{Z}_{p_j}$ such that $|A| = \prod_{j=1}^s p_j$ is the prime decomposition of $|A|$ (possibly containing repetitions).

We want to stress here, that only the group $(A, +, 0, -)$ can be decomposed into a direct product of prime sized algebras, but not the entire algebra \mathbf{B} . Moreover, the Abelian algebras \mathbf{L}_i can in general not be further decomposed with respect to \otimes ; an example is the Abelian algebra $(\mathbb{Z}_2 \times \mathbb{Z}_2, +, 0, -, (r)_{r \in R})$, where $R = \text{End}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is the full endomorphism ring.

3. THE EQUIVALENCE OF CC-CIRCUITS AND CIRCUITS OVER NILPOTENT ALGEBRAS

Our proof of Theorem 1.2 is based on expressing circuits over a fixed nilpotent algebra \mathbf{A} as $\text{CC}[m]$ -circuits, and vice-versa. It is however a priori not clear if and how this is possible, as $\text{CC}[m]$ -circuits are Boolean valued, whereas the universe of \mathbf{A} can be arbitrary. Because of this, most of the time we are not going to work with $\text{CC}[m]$ -circuits themselves, but an m -valued analogue, which we call $\text{CC}^+[m]$ -circuits. We define $\text{CC}^+[m]$ -circuits and discuss some of their properties in the next subsection. This is then followed by the proof and discussion of Theorem 3.6, which in particular implies Theorem 1.2.

3.1. $\text{CC}^+[m]$ -circuits.

Definition 3.1. A $\text{CC}^+[m]$ -circuit C is a circuit over the set $\mathbb{Z}_m = \{0, 1, \dots, m-1\}$ containing $+$ -gates, constant gates, and MOD_m -gates of arbitrary fan in. We interpret $+$ as addition modulo m . As in the Boolean case, MOD_m -gates output 1, if their inputs sum up to 0 modulo m , and 0 otherwise.

Without loss of generality we can assume that in such a $\text{CC}^+[m]$ -circuit C there are at most m wires between two gates (otherwise we take their number modulo m). Furthermore, every wire from a $+$ -gate to some MOD_m -gate can be substituted by wires that go directly

from the inputs of the $+$ -gate to the MOD_m -gate. By doing this substitution for all such wires (and reducing their number in each step modulo m) we obtain an equivalent circuit, whose size is linear in the size of the original. Thus, from now on, we assume that every $\text{CC}^+[m]$ -circuit has at most one $+$ -gate, and this is the output gate.

We next discuss the relationship between nonconstant 0-absorbing $\text{CC}^+[m]$ - and $\text{CC}[m]$ -circuits. Note that an n -ary $\text{CC}[m]$ -circuit C is nonconstant and 0-absorbing if and only if $C(\bar{x}) = 1$ if $\bar{x} = (1, 1, \dots, 1)$ and $C(\bar{x}) = 0$ else. So it defines the n -ary AND operation.

Lemma 3.2.

- (1) For every nonconstant 0-absorbing $\text{CC}^+[m]$ -circuit $C(\bar{x})$ of depth d we can compute in linear time a $\text{CC}[m]$ -circuit $C'(\bar{x})$ of depth d that defines AND.
- (2) For every $\text{CC}[m]$ -circuit $C(\bar{x})$ of depth d defining AND we can compute in linear time a nonconstant 0-absorbing $\text{CC}^+[m]$ -circuit $C'(\bar{x})$ of depth $d + 1$.

Proof. For (1), let $C(x_1, \dots, x_n)$ be a $\text{CC}^+[m]$ -circuit of depth d that defines a nonconstant 0-absorbing function. Thus there is some tuple $\bar{a} \in (\mathbb{Z}_m \setminus \{0\})^n$ such that $C(\bar{a}) \neq 0$. Without loss of generality we can assume that $\bar{a} = (1, 1, \dots, 1)$, otherwise we duplicate every wire connected to the input gate x_i a_i -many times. Without loss of generality we can further assume that C has only constant gates labelled by 1 (otherwise we substitute the constant gate c by c -many copies of 1). If C contains no $+$ -gate at all, we set $C = C'$ and are done. If C has a $+$ -gate as an output gate that sums up g_1, \dots, g_k we construct C' by substituting this $+$ -gate by a MOD_m -gate that has g_1, \dots, g_k and $m - C(\bar{a})$ as input.

For (2) note that C itself might not be 0-absorbing when evaluated over \mathbb{Z}_m . However the circuit $C'(\bar{x}) = C(\text{MOD}_m(1 - x_1), \dots, \text{MOD}_m(1 - x_n))$ is. \square

As a consequence of Lemma 3.2 (1), lower bounds on the size of $\text{CC}[m]$ -circuit of depth d defining AND are also lower bounds on the size of nonconstant 0-absorbing $\text{CC}^+[m]$ -circuit of depth d . By Lemma 3.2 (2) also the reverse statement holds, up to decreasing the depth by 1.

We continue by discussing representations of arbitrary functions by $\text{CC}^+[m]$ -circuits. A useful concept for it is the *essential depth* of a $\text{CC}^+[m]$ -circuit C , which we define to be the depth of C , not counting its $+$ -gates.

Lemma 3.3.

- (1) If $m > 2$, then every function $f: \mathbb{Z}_m^n \rightarrow \mathbb{Z}_m$ can be represented by a $\text{CC}^+[m]$ -circuit of essential depth $\lceil \log_2(n) \rceil + 1$.
- (2) If m has two distinct prime factors, then every function $f: \mathbb{Z}_m^n \rightarrow \mathbb{Z}_m$ can be represented by a $\text{CC}^+[m]$ -circuit of essential depth 3.

Proof. In order to prove (1), let us define the series of circuits

$$C_1(x_1) = \text{MOD}_m(x_1), \text{ and}$$

$$C_{2^{k+1}}(x_1, \dots, x_{2^{k+1}}) = \text{MOD}_m(C_{2^k}(x_1, \dots, x_{2^k}), C_{2^k}(x_{2^k+1}, \dots, x_{2^{k+1}}), m - 2),$$

for every $k \in \mathbb{N}$. Note that $C_{2^{k+1}}$ outputs 1 if all inputs are equal to 0 and 0 otherwise. We can further construct circuits C_n with this behaviour for all arities $2^k < n \leq 2^{k+1}$ by identifying some of the variables of $C_{2^{k+1}}$. Now (1) follows from the fact that every function $f: \mathbb{Z}_m^n \rightarrow \mathbb{Z}_m$ can be obtained as the sum of translations of C_n by constant tuples, i.e. $f(\bar{x}) = \sum_{\bar{a} \in A^n} f(\bar{a}) \cdot C_n(\bar{x} - \bar{a})$ (where $f(\bar{a}) \cdot y$ is a shorthand for the sum of $f(\bar{a})$ -many copies of y).

For (2) recall that all Boolean operations, in particular the n -ary AND, can be written as $\text{CC}[m]$ -circuits of depth 2 (see [BST90]). Thus $C_n(\bar{x}) = \text{AND}(\text{MOD}_m(x_1), \dots, \text{MOD}_m(x_n))$ is a $\text{CC}^+[m]$ -circuit of depth 3 that describes the characteristic function of $(0, \dots, 0)$. As in the proof of (1), we can obtain all other functions as linear combinations $f(\bar{x}) = \sum_{\bar{a} \in A^n} f(\bar{a}) \cdot C_n(\bar{x} - \bar{a})$. \square

3.2. The main result. Recall that in Definition 2.2 we considered the operations of n -nilpotent algebras as ‘actions’ of an $(n - 1)$ -nilpotent algebra on an Abelian one. Also in a $\text{CC}^+[m]$ -circuit we can think of a MOD_m -gates as receiving inputs from gates of lower depth and having an output in the Abelian group \mathbb{Z}_m . This viewpoint allows us to construct a nilpotent algebra in which we can interpret all $\text{CC}^+[m]$ -circuits of fixed depth d :

Lemma 3.4. *For all $m, d \in \mathbb{N}$ there is a $(d+1)$ -nilpotent algebra $\mathbf{D} = (\mathbb{Z}_m^{d+1}, +, 0, -, h, *, (f_i)_{i=1}^d)$, such that for every $\text{CC}^+[m]$ -circuit C of essential depth d , there is a circuit C' over \mathbf{D} with $C'(x_1, \dots, x_n) = (C(\pi_{d+1}(x_1), \dots, \pi_{d+1}(x_n)), 0, \dots, 0)$ (where π_i denotes the projection of D to the i -th component of $(\mathbb{Z}_m)^{d+1}$). Further C' can be computed from C in linear time.*

Proof. As usual, $+, 0, -$ denote the Abelian group operation on \mathbb{Z}_m^{d+1} . Further we define the unary operation $h(x) = (0, \dots, 0, \pi_{d+1}(x))$, the binary operation $x_1 * x_2 = (\sum_{i=1}^{d+1} \pi_i(x_1) + \pi_i(x_2), 0, \dots, 0)$, and the unary operations $(f_i)_{i=1}^d$ by

$$f_i(x) = \begin{cases} (0, \dots, 0, 1, 0, \dots, 0) & \text{if } \pi_{i+1}(x) + \pi_{i+2}(x) + \dots + \pi_{d+1}(x) = 0, \\ (0, 0, \dots, 0) & \text{else;} \end{cases}$$

here 1 lies on the i -th coordinate. Note that $\mathbf{D} = \mathbf{L}_1 \otimes (\mathbf{L}_2 \otimes \dots (\mathbf{L}_d \otimes \mathbf{L}_{d+1}) \dots)$, such that every \mathbf{L}_i is polynomially equivalent to \mathbb{Z}_m . Since \mathbf{D} contains group operations, and groups have modular congruence lattices, \mathbf{D} generates a congruence modular variety. Thus, by Definition 2.2, \mathbf{D} is $d + 1$ -nilpotent. Let $C(y_1, \dots, y_n)$ now be a $\text{CC}^+[m]$ -circuit of essential depth d . Without loss of generality we can assume that $C(y_1, \dots, y_n)$ has at most one $+$ -gate at its output level.

By the depth r of a gate g of C we denote the distance of g from one of the input or constant gates. Further let C_g be the subcircuit of C , consisting of all vertices that have a directed path to g . We are going to show by induction over the depth $r = 0, 1, \dots, d$ that for every gate g of depth r there is a circuit C'_g over \mathbf{A} , such that $\pi_{d+1-r} C'_g(x_1, \dots, x_n) = C_g(\pi_{d+1}(x_1), \dots, \pi_{d+1}(x_n))$ and $\pi_j C'_g(x_1, \dots, x_n) = 0$, for all $j \neq d + 1 - r$. Note that for $r = d$, this is essentially the statement of the lemma.

The gates g of depth $r = 0$ are either input gates or constant gates. For every constant gate $C_g = c$ of C we set $C'_g = (0, 0, \dots, c)$ and every input gate $C_g = y_i$ of C we set $C'_g = h(x_i)$. Clearly C'_g then satisfies the our claim.

For an induction step $r - 1 \rightarrow r$, let g be a gate of depth r . Note that g must be a MOD_m -gate, and thus $C_g = \text{MOD}_m(C_{g_1} + C_{g_2} + \dots + C_{g_n})$, where all g_i are gates of depth $r - 1$ or lower. By induction hypothesis, for every g_i there exists a circuit C'_{g_i} over \mathbf{A} , whose projection to the component of \mathbb{Z}_m^{d+1} corresponding to the depth of g_i is $C_{g_i}(\pi_{d+1}(x_1), \dots, \pi_{d+1}(x_n))$, and whose projection to every other component is 0. We then define $C'_g = f_{d+1-r}(C'_{g_1} + C'_{g_2} + \dots + C'_{g_n})$. By the definition of the function f_{d+1-r} , this circuit satisfies $\pi_{d+1-r} C'_g(x_1, \dots, x_n) = C_g(\pi_{d+1}(x_1), \dots, \pi_{d+1}(x_n))$ and $\pi_j C'_g = 0$ for all $j \neq d + 1 - r$, which is what we wanted to prove.

If C has a MOD_m -gate g of depth d as output gate, then we simply set $C' = C'_g$ and are done. In the other case C' has a $+$ -gate (of depth $d + 1$) as output gate. Hence $C = C_{g_1} + C_{g_2} + \dots + C_{g_n}$, for gates g_1, \dots, g_n of depth at most d . Let $C' = C'_{g_1} * C'_{g_2} * \dots * C'_{g_n}$. By definition of $*$, this C' also satisfies the statement of the lemma. \square

The other direction, i.e. showing that circuits over a fixed nilpotent algebra \mathbf{A} can be expressed by bounded depth $\text{CC}^+[m]$ -circuits, requires some more work.

Recall that by Theorem 2.3 every nilpotent algebra \mathbf{A} can be extended by group operations $+, 0, -$ such that $(A, +, 0, -)$ is the direct product of prime order groups $\prod_{i=1}^s \mathbb{Z}_{p_i}$, with $|A| = \prod_{j=1}^s p_j$. So we can identify an element $a \in A$ with the tuple $(\pi_1(a), \dots, \pi_s(a)) \in \prod_{j=1}^s \mathbb{Z}_{p_j}$. Let m be the product of all *distinct* prime factors of $|A|$. Every factor \mathbb{Z}_{p_j} clearly embeds into \mathbb{Z}_m (by the map $x \mapsto (mp_j^{-1})x$). Therefore there is also a natural embedding $e: (A, +, 0, -) = \prod_{j=1}^s \mathbb{Z}_{p_j} \rightarrow (\mathbb{Z}_m)^s$.

Our goal is to interpret circuits over the algebra \mathbf{A} as $\text{CC}^+[m]$ -circuits using this embedding. So for a circuit C over \mathbf{A} , we want to find a $\text{CC}^+[m]$ -circuit C' such that $C'(e(x_1), \dots, e(x_k)) = eC(x_1, \dots, x_k)$. Note that for every input gate x_i of C , the $\text{CC}^+[m]$ -circuit C' needs to have s input gates for $e(x_i) = (\pi_1(x_i), \dots, \pi_s(x_i))$. For the same reason C' will also have s output gates.

Proposition 3.5. *Let \mathbf{A} be a finite nilpotent algebra, and \mathbf{B} be its extension by the Abelian group operations $+, 0, -$, given by Theorem 2.3.*

Let $m > 2$ be the product of prime divisors of $|A|$, and $e: (A, +, 0, -) \rightarrow (\mathbb{Z}_m)^s$ the natural group embedding. Then there is a $d \in \mathbb{N}$ such that for every circuit C over \mathbf{B} there is a $\text{CC}^+[m]$ -circuit C' of essential depth d with $C'(e(x_1), \dots, e(x_k)) = eC(x_1, \dots, x_k)$.

- (1) *In general $d \leq (\lceil \log_2(s \cdot \text{ar}(\mathbf{B})) \rceil + 1) \cdot (\log_2 |A| - 1)$,*
- (2) *If m is not a prime, then $d \leq 3(\log_2 |A| - 1)$.*

Furthermore C' can be computed from C in linear time.

Proof. Recall that, by Theorem 2.3,

$$\mathbf{B} = \mathbf{L}_1 \otimes (\mathbf{L}_2 \otimes \dots (\mathbf{L}_{n-1} \otimes \mathbf{L}_n) \dots),$$

such that every \mathbf{L}_i is polynomially equivalent to a module, whose group is $\mathbb{Z}_{p_i}^{k_i}$ for a prime p_i . With respect to this representation, every basic operation of \mathbf{B} is of the form

$$f = (f^{\mathbf{L}_1} + \hat{f}_1, f^{\mathbf{L}_2} + \hat{f}_2, \dots, f^{\mathbf{L}_{n-1}} + \hat{f}_{n-1}, f^{\mathbf{L}_n}),$$

where $f^{\mathbf{L}_i}$ is an operation of \mathbf{L}_i and \hat{f}_i only depends on the projection of A to $L_{i+1} \times \dots \times L_n$. Without loss of generality we can assume that each basic operation of \mathbf{B} is either ‘of affine type’, meaning that $f = (f^{\mathbf{L}_1}, f^{\mathbf{L}_2}, \dots, f^{\mathbf{L}_{n-1}}, f^{\mathbf{L}_n})$, or of ‘hat type’ meaning that $f = (0, \dots, 0, \hat{f}_j, 0, \dots, 0)$. If this is not the case we substitute f by the basic operations $(f^{\mathbf{L}_1}, \dots, f^{\mathbf{L}_n})$ and $(0, \dots, 0, \hat{f}_j, 0, \dots, 0)$ for every $j = 1, \dots, n - 1$. The resulting algebra is also n -nilpotent and a polynomial extension of \mathbf{B} . Note further that this extension has the same maximal arity as \mathbf{B} .

Every basic operation $f^{\mathbf{L}_i}: L_i^l \rightarrow L_i$ of \mathbf{L}_i is equal to an affine combination $f^{\mathbf{L}_i}(z_1, \dots, z_l) = \sum_{j=1}^l r_i \cdot z_j + c$, where the r_i are linear maps. Therefore we can consider $f^{\mathbf{L}_i}$ as an affine map from the vector space $\mathbb{Z}_{p_i}^{l \cdot k_i}$ to $\mathbb{Z}_{p_i}^{k_i}$. As such, it can be represented by a circuit over \mathbb{Z}_{p_i} with lk_i inputs and k_i outputs, consisting only of $+$ -gates. By duplicating every wire in this circuit mp^{-1} -many times, we obtain a corresponding $+$ -circuit over \mathbb{Z}_m . By doing this for every projections $f^{\mathbf{L}_i}$ of a basic operation $f = (f^{\mathbf{L}_1}, \dots, f^{\mathbf{L}_n})$ of ‘affine type’,

we can show that there is a $\text{CC}^+[m]$ circuits C_f only consisting of $+$ -gates, such that $C_f(e(x_1), \dots, e(x_l)) = ef(x_1, \dots, x_l)$.

Next, for every operation of ‘hat type’ $f: A^l \rightarrow A$, we take an arbitrary extension of it to a function $g: (\mathbb{Z}_m^s)^l \rightarrow \mathbb{Z}_m^s$. Since $l \leq \text{ar}(\mathbf{B})$, by Lemma 3.3 (1), this g can be represented by a $\text{CC}^+[m]$ -circuit C_f of essential depth $\lceil \log_2(s \cdot \text{ar}(\mathbf{B})) \rceil + 1$. If m is the product of two or more prime factors, then by Lemma 3.3 (2), C_f can even be written as a $\text{CC}^+[m]$ -circuits of essential depth 3.

Now let C be an arbitrary circuit of the algebra \mathbf{B} . For every basic operation of hat-type f , there is a $j \in \{1, \dots, n-1\}$ such that f_j only depends on $L_{j+1} \times \dots \times L_n$, and all other coordinates are 0. For any other operation g that is of hat-type with respect to a smaller index $p \leq j$ the identity $f(x_1, \dots, x_k + c \cdot g(\bar{y}), \dots, x_l) = f(x_1, \dots, x_k, \dots, x_l)$ holds, for all $k = 1, \dots, l$. Using these identities, the circuit C can be rewritten in linear time to a circuit, in which no directed path contains more than $n-1$ gates of hat type.

If we then substitute all gates f of C by the corresponding $\text{CC}^+[m]$ -circuits C_f as described above, we obtain a $\text{CC}^+[m]$ -circuit C' of essential depth $(\lceil \log_2(s \cdot \text{ar}(\mathbf{B})) \rceil + 1)(n-1)$. In the case, in which m has two or more prime factors, the same argument gives us a circuit of essential depth $3(n-1)$.

Since $n \leq \log_2 |A|$, this proves the proposition. \square

Note that the sk -ary $\text{CC}^+[m]$ -circuit $C'(y_1, \dots, y_{sk})$ in Proposition 3.5 does not need to be 0-absorbing, if C is 0-absorbing. However we can obtain a 0-absorbing circuit with the same range as eC by taking the circuit $C''(y_1, \dots, y_{sk})$ that computes $C'(c_1 y_1, c_2 y_2, \dots, c_s y_{sk})$, with $c_i = mp_i^{-1}$.

We are now ready to prove the main result.

Theorem 3.6. *For a fixed nilpotent algebra \mathbf{A} and $0 \in A$, let $f_{\mathbf{A}}(n)$ denote the minimal size of a nonconstant 0-absorbing n -ary circuit over \mathbf{A} . For two integers d, m , let $g_{m,d}(n)$ be the smallest size of an n -ary $\text{CC}[m]$ -circuit of depth d computing AND. Then:*

- (1) *Let \mathbf{A} be a finite nilpotent algebra, and m be the product of prime divisors of $|A|$. If m is not a prime, then $f_{\mathbf{A}}(n) \geq Kg_{m,d}(n)$ for $d = 3\lceil \log_2 |A| \rceil - 2$ and some $K > 0$.*
- (2) *Vice versa, for every $d > 1$ and every product of two or more distinct primes m , there is a $(d+2)$ -nilpotent algebra \mathbf{B} with $g_{m,d}(n) \geq K'f_{\mathbf{B}}(n)$ for some $K' > 0$.*

Proof. To see (1), let $(C_n)_{n \in \mathbb{N}}$ be a sequence of nonconstant 0-absorbing circuits over \mathbf{A} such that $|C_n| = f_{\mathbf{A}}(n)$. We can regard every C_n also as a circuit over the nilpotent extension \mathbf{B} of \mathbf{A} given by Theorem 2.3. This extension \mathbf{B} is nilpotent and contains a group operation $+$ such that $(A, +, 0, -) = \prod_{j=1}^s \mathbb{Z}_{p_j}$. By Proposition 3.5, for every n there is a sn -ary $\text{CC}^+[m]$ -circuit C'_n of essential depth at most $3\lceil \log_2 |A| \rceil - 3$ such that $C'_n(e(x_1), \dots, e(x_n)) = eC_n(x_1, \dots, x_n)$. Note that, by eliminating all $+$ -gates, except the output gates, we obtain a $\text{CC}^+[m]$ -circuit C''_n of depth at most $d = 3\lceil \log_2 |A| \rceil - 2$.

Since C_n is 0-absorbing, the circuit $C'_n(c_1 y_1, \dots, c_s y_{sn})$, for $c_i = (mp_i^{-1})$ is 0-absorbing and has the same range as eC_n . As C_n is not constant, there is an output gate of C'_n that induces a nonconstant 0-absorbing operation. By Lemma 3.2 (1) we can compute in linear time a $\text{CC}[m]$ -circuit of the same depth, that defines the sn -ary AND. Therefore there is a K , such that $f_{\mathbf{A}}(n) \geq Kg_{m,d}(sn) \geq Kg_{m,d}(n)$; the constant K results from the fact that all computations only required linear time.

For (2) note that by Lemma 3.2 (2), for every $\text{CC}[m]$ -circuit C of depth d defining AND, we can construct in linear time a nonconstant, 0-absorbing $\text{CC}^+[m]$ -circuit C' of depth $d + 1$. Let us define \mathbf{B} to be the $(d + 2)$ -nilpotent algebra \mathbf{D} given by Lemma 3.4 for depth $d + 1$. We then can compute a circuit C'' over \mathbf{B} such that $C''(x_1, \dots, x_n)$ evaluates to $(C'(\pi_{d+2}(x_1), \dots, \pi_{d+2}(x_n)), 0, \dots, 0)$. C'' is clearly nonconstant and 0-absorbing. This concludes the proof of (2). \square

Theorem 3.6 directly implies that Conjecture 1.1 is true, if and only if nonconstant 0-absorbing polynomial operations in nilpotent algebras require circuits of exponential size. So we obtain Theorem 1.2 as a direct corollary.

Moreover Theorem 3.6 implies that also the weak version of Conjecture 1.1, which states that AND is not in CC^0 , has an algebraic counterpart by the following equivalence:

Corollary 3.7. *AND is in CC^0 if and only if there is a nilpotent algebra \mathbf{A} (from a congruence modular variety) and a series of circuits $(C_n(\bar{x}))_{n \in \mathbb{N}}$ over \mathbf{A} that grows polynomially and defines nonconstant 0-absorbing polynomials of all arities.*

We further remark that, if m is an odd prime, Proposition 3.5 allows us to reprove results about nilpotent algebras: Since we know that bounded depth $\text{CC}[m]$ -circuits are not able to define AND, Proposition 3.5 implies that finite nilpotent algebras of prime power order only have nonconstant 0-absorbing polynomials up to some arity. This was independently already shown in [BB87]. In fact, finite nilpotent algebras that have nonconstant 0-absorbing polynomials only up to some fixed arity (so called *supernilpotent* algebras) are characterized by being direct products of nilpotent algebras of prime power size [Kea99]. Nilpotent groups and rings are examples of supernilpotent algebras.

In [IKK18] it was further shown that, if \mathbf{A} is 2-nilpotent and $\mathbf{A} = \mathbf{L} \otimes \mathbf{U}$, where \mathbf{L} and \mathbf{U} are two vector spaces of different characteristics p and q , there are nonconstant 0-absorbing polynomial operations that can be computed by circuits of exponential size $\mathcal{O}(c^n)$; however they cannot be computed by circuits of smaller size. In the spirit of Theorem 3.6, this can be also seen as the algebraic counterpart of the well-known fact that MOD_p - MOD_q circuits need at least size $\Omega(c^n)$ to compute AND [BST90].

4. CIRCUIT SATISFIABILITY AND EQUIVALENCE

In this section we discuss the complexity of the circuit satisfiability and the circuit equivalence problem for nilpotent algebras. The circuit satisfiability problem $\text{CSAT}(\mathbf{A})$ models the question, whether a single equation over the algebra \mathbf{A} has a solution; the circuit equivalence problem $\text{CEQV}(\mathbf{A})$ asks whether an equation holds for all assignments of variables. Both problems were introduced in [IK18] and are formally defined as follows:

CIRCUIT SATISFIABILITY $\text{CSAT}(\mathbf{A})$

INPUT: Two circuits C, C' over \mathbf{A} with input gates $\bar{x} = (x_1, \dots, x_n)$

QUESTION: Is there a tuple $\bar{a} \in A^n$ such that $C(\bar{a}) = C'(\bar{a})$?

CIRCUIT EQUIVALENCE $\text{CEQV}(\mathbf{A})$

INPUT: Two circuits C, C' over \mathbf{A} with input gates $\bar{x} = (x_1, \dots, x_n)$

QUESTION: Is $C(\bar{a}) = C'(\bar{a})$ for all $\bar{a} \in A^n$?

Circuits over finite algebras can be evaluated in polynomial time. Therefore $\text{CSAT}(\mathbf{A})$ is always in NP and $\text{CEQV}(\mathbf{A})$ in coNP. The major question then is, for which algebras CSAT and CEQV are tractable, and for which they are NP-complete, respectively coNP-complete. In particular this is still open for nilpotent algebras from congruence modular varieties. We first show that - under the assumption that Conjecture 1.1 is true - there are algorithms for both $\text{CEQV}(\mathbf{A})$ and $\text{CSAT}(\mathbf{A})$ that run in quasipolynomial time. This gives us a conditional answer to Problem 2 in [IK18].

Theorem 4.1. *Assume that Conjecture 1.1 is true. Then, for every finite nilpotent algebra \mathbf{A} from a congruence modular variety $\text{CSAT}(\mathbf{A})$ and $\text{CEQV}(\mathbf{A})$ can be solved in quasipolynomial time $\mathcal{O}(2^{(\log n)^t})$ (where $t > 0$ depends on \mathbf{A}).*

Proof. We start with the equivalence problem $\text{CEQV}(\mathbf{A})$. Without loss of generality we assume that \mathbf{A} contains a group operation $+$, such that $(A, +, 0, -) = \prod_{i=1}^s \mathbb{Z}_{p_i}$ (otherwise we reduce to such an algebra by Theorem 2.3 and Lemma 2.1). When solving $\text{CEQV}(\mathbf{A})$ it is sufficient to find an algorithm to check whether some input circuit is equivalent to the constant 0-circuit (as $C = C'$ if and only if $C - C' = 0$). Thus we are only considering inputs C and 0 to $\text{CEQV}(\mathbf{A})$.

By Proposition 3.5 we can identify $C(x_1, \dots, x_n)$ with a $\text{CC}^+[m]$ -circuit $C'(y_1, \dots, y_{sn})$ having a depth d and s output gates, where d and s only depend on \mathbf{A} . Then $C(x_1, \dots, x_n)$ is equivalent to 0 if and only if $C'(c_1 y_1, \dots, c_s y_{sn})$ is constant and equivalent to $(0, 0, \dots, 0)$ (see also the discussion after Proposition 3.5). Let us denote the output gates of this circuit by C_1, \dots, C_s .

In the case where one of the output gates C_i does not compute the constant 0 function, there is a tuple $\bar{a} = (a_1, \dots, a_{sn})$ such $C_i(\bar{a}) \neq 0$. Let us pick \bar{a} such that the number of coordinates j with $a_j = 0$ is maximal. For the sake of simplicity, let us assume that $a_j = 0$ iff $j > k$ for some k . Then, the circuit $C_i(x_1, \dots, x_k, 0, \dots, 0)$ defines a nonconstant 0-absorbing operation. Since we assume that Conjecture 1.1 is true, we have that the size $|C_i|$ of this circuit is bigger than 2^{k^q} for some $q > 0$. In other words $k \leq \log(|C_i|)^l$, with $l = q^{-1}$.

Thus in order to check, whether $C(\bar{x})$ is equivalent to 0, we only need to check, whether all C_i evaluate to 0 for all tuples that have at most $\log(|C_i|)^l$ -many non-0 elements. There are $\binom{|C_i|}{\log(|C_i|)^l} = \mathcal{O}(|C_i|^{\log(|C_i|)^l})$ many such tuples. Since $|C_i|$ is linear in the size of C we obtain an algorithm that runs in time $\mathcal{O}(|C|^{\log(|C|)^l}) = \mathcal{O}(2^{\log(|C|)^{l+1}})$. Thus the lemma holds for $t = l + 1$.

The satisfiability problem $\text{CSAT}(\mathbf{A})$ can similarly be reduced to checking whether the bounded depth $\text{CC}^+[m]$ -circuit $C'(c_1 y_1, \dots, c_s y_{sm})$ outputs the s -ary tuple $(0, 0, \dots, 0)$ for some input \bar{a} . Let $f: \mathbb{Z}_m^s \rightarrow \mathbb{Z}_m$ be the function that outputs 0 if and only if $x \neq (0, 0, \dots, 0)$, and 1 else. By Lemma 3.3 (2), f can be computed by a $\text{CC}^+[m]$ -circuit of essential depth 3. So clearly $C(\bar{x}) = 0$ has no solution, if and only if $f(C'(c_1 y_1, \dots, c_s y_{sm}))$ is constant and equal to 0. Thus we reduced $\text{CSAT}(\mathbf{A})$ to the equivalence problem for $\text{CC}^+[m]$ -circuit of a fixed depth, which has a quasipolynomial algorithm by the above. Note that the depth of this circuit is bigger than in the proof for CEQV, thus we might obtain a bigger value for the constant t . \square

We remark that already [BMM⁺00] used the same approach to show that the program satisfiability problem over solvable monoids has a quasipolynomial algorithm (supposed that Conjecture 1.1 holds). Moreover the algorithm described in Theorem 4.1 was already studied for the special case where $|A|$ is power of some prime p . As already mentioned, then

$\text{CC}[p]$ -circuits of bounded depth cannot compute AND, which corresponds to \mathbf{A} only having nonconstant 0-absorbing polynomials of arity smaller than some constant k . So, in this case we only need to evaluate the circuits at tuples with k -many non 0-entries, which gives us an algorithm that runs in polynomial time $\mathcal{O}(|C|^k)$. For CEQV this was observed in [AM10], algorithms for CSAT were constructed in [Kom18], [IK18] and [Aic19b].

Finite nilpotent algebras of prime power order and their direct products are provably the only finite nilpotent algebras, where we have a bound on the arity of nonconstant 0-absorbing polynomials. Thus the algorithm described in Theorem 4.1 cannot be refined to run in polynomial time for general nilpotent algebras. However we remark that there are examples of 2-nilpotent, not supernilpotent algebras for which we can obtain other polynomial-time algorithms: It was shown in [IKK18] that for nilpotent algebras $\mathbf{L} \otimes \mathbf{U}$, where both \mathbf{L} and \mathbf{U} are polynomially equivalent to finite vector space, CEQV and CSAT are in P. For CEQV this was generalized to all 2-nilpotent algebras in [KKK19].

At last we show that, under the assumption that there is an efficient way of *uniformly* computing AND by bounded depth $\text{CC}[m]$ -circuits, there is a nilpotent algebra with intractable CSAT and CEQV problem (note however that this is a stronger assumption than AND being in CC^0).

Theorem 4.2. *Assume that there is a family $(C_n)_{n \in \mathbb{N}}$ of $\text{CC}[m]$ -circuits of depth d , that defines AND, and is enumerable by a polynomial time Turing machine (i.e. it outputs $C_n(x_1, \dots, x_n)$ on input 1^n). Then there exists a finite nilpotent algebra \mathbf{A} such that $\text{CSAT}(\mathbf{A})$ is NP-complete and $\text{CEQV}(\mathbf{A})$ is coNP-complete.*

Proof. Since $(C_n)_{n \in \mathbb{N}}$ defines AND, m cannot be a prime. In particular $m > 2$. We are first going to reduce the graph-colouring problem with m -colors to the circuit satisfiability problem for $\text{CC}^+[m]$ -circuits of depth bounded by $d + 1$.

So let $G = (V, E)$ a graph. We are going to identify every vertex $v \in V$ with a variable x_v over \mathbb{Z}_m , representing the color of v . Note that the $\text{CC}^+[m]$ -circuit $1 - \text{MOD}_m(x_v - x_w)$ outputs 1 if $x_v \neq x_w$, and 0 else. Thus, if we define the circuit $H((x_v)_{v \in V}) = C_{|E|}((1 - \text{MOD}_m(x_v - x_w))_{(v,w) \in E})$, then H outputs 1 if the assignment $v \mapsto x_v$ is a proper coloring of the graph and 0 else. So G is yes-instance of the m -coloring problem if and only if $H((x_v)_{v \in V}) = 1$ has a solution.

By our assumption H can be computed in polynomial time from G , thus we reduced m -coloring to the satisfiability problem for $\text{CC}^+[m]$ -circuits of depth bounded by $d + 1$. Furthermore, note that there is no m -coloring of G if and only if $H((x_v)_{v \in V})$ is constant and equal to 0. Therefore the complement of the m -coloring problem, reduces to checking the equivalence of $\text{CC}^+[m]$ -circuits of depth $d + 1$.

By Lemma 3.4 there is a nilpotent algebra \mathbf{A} , such that we can encode every $\text{CC}^+[m]$ -circuit H of depth $d + 1$ by a circuit H' over \mathbf{A} . Thus m -coloring reduces to $\text{CSAT}(\mathbf{A})$ and its complement to $\text{CEQV}(\mathbf{A})$. \square

The proof of Theorem 4.2 uses a reduction of the coloring problems to CSAT with the help of 0-absorbing polynomials; we remark that this idea was used in several other results in the relevant literature [BL93, GR02, HS11].

If AND satisfies the criterion in Theorem 4.2, we also say that AND is in P-*uniform* CC^0 , for short. Note that, by Theorem 3.6, this is equivalent to a statement about nilpotent algebras:

Corollary 4.3. *AND is in P-uniform CC^0 if and only if there is a finite nilpotent algebra \mathbf{A} from a congruence modular variety, a family of nonconstant 0-absorbing circuits $(C_n(x_1, \dots, x_n))_{n \in \mathbb{N}}$ over \mathbf{A} , and a polynomial time Turing machine that outputs $C_n(x_1, \dots, x_n)$ on input 1^n .*

Proof. This follows directly from the proof of Theorem 3.6, since we showed how to compute an n -ary circuit computing AND from a circuit $(C_n(x_1, \dots, x_n))_{n \in \mathbb{N}}$ over \mathbf{A} in linear time; and vice-versa. \square

In Corollary 4.3 it is essential that the nilpotent algebra is finite. If we however also look at nilpotent algebras with infinite signature, it is quite easy to construct examples that have nonconstant 0-absorbing circuits, which are enumerable in polynomial time. Theorem 6.1 in [IKK18] gives an example of a 2-nilpotent algebra $\mathbf{A} = (A, (f_n)_{n \in \mathbb{N}})$, for which this is the case, and which has hard CSAT and CEQV problems. Interestingly this algebra \mathbf{A} is also polynomially equivalent to a finite nilpotent algebra \mathbf{A}' , which in turn however fails to efficiently enumerate the same polynomial operations and satisfies $CSAT(\mathbf{A}') \in P$ and $CEQV(\mathbf{A}') \in P$.

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REFERENCES

- [Aic19a] Erhard Aichinger. Bounding the free spectrum of nilpotent algebras of prime power order. *Israel Journal of Mathematics*, 230(2):919–947, 2019. doi:10.1007/s11856-019-1846-x.
- [Aic19b] Erhard Aichinger. Solving systems of equations in supernilpotent algebras. In *44th International Symposium on Mathematical Foundations of Computer Science (MFCS 2019)*, volume 138, pages 72:1–72:9, 2019. doi:10.4230/LIPIcs.MFCS.2019.72.
- [AM10] Erhard Aichinger and Nebojša Mudrinski. Some applications of higher commutators in Mal’cev algebras. *Algebra universalis*, 63(4):367–403, 2010. doi:10.1007/s00012-010-0084-1.
- [Bar89] David A Barrington. Bounded-width polynomial-size branching programs recognize exactly those languages in NC1. *Journal of Computer and System Sciences*, 38(1):150–164, 1989. doi:10.1016/0022-0000(89)90037-8.
- [BB87] Joel Berman and W.J. Blok. Free spectra of nilpotent varieties. *Algebra Universalis*, 24(3):279–282, 1987. doi:10.1007/BF01195266.
- [BL93] Stanley Burris and John Lawrence. The equivalence problem for finite rings. *Journal of Symbolic Computation*, 15(1):67–71, 1993. doi:10.1006/jscs.1993.1004.
- [BMM⁺00] David Mix Barrington, Pierre McKenzie, Cris Moore, Pascal Tesson, and Denis Thérien. Equation satisfiability and program satisfiability for finite monoids. In *Proceedings of the 25th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 172–181. Springer, 2000. doi:10.1007/3-540-44612-5_13.
- [BST90] David Mix Barrington, Howard Straubing, and Denis Thérien. Non-uniform automata over groups. *Information and Computation*, 89(2):109–132, 1990. doi:10.1016/0890-5401(90)90007-5.
- [Bul17] Andrei A Bulatov. A dichotomy theorem for nonuniform CSPs. In *Proceedings of the 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 319–330. IEEE, 2017. doi:10.1109/FOCS.2017.37.
- [Cau96] Hervé Caussinus. A note on a theorem of Barrington, Straubing and Thérien. *Information Processing Letters*, 58(1):31–33, 1996. doi:10.1016/0020-0190(96)00029-4.

- [CGPT06] Arkadev Chattopadhyay, Navin Goyal, Pavel Pudlák, and Denis Thérien. Lower bounds for circuits with MOD_m gates. In *Proceedings of the 47th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 709–718. IEEE, 2006. doi:10.1109/FOCS.2006.46.
- [CW21] Brynmor Chapman and Ryan Williams. Smaller ACC0 circuits for symmetric functions. arXiv preprint arXiv:2107.04706, 2021.
- [FM87] Ralph Freese and Ralph McKenzie. *Commutator theory for congruence modular varieties*, volume 125. CUP Archive, 1987.
- [FSS84] Merrick Furst, James B. Saxe, and Michael Sipser. Parity, circuits, and the polynomial-time hierarchy. *Mathematical systems theory*, 17(1):13–27, 1984. doi:10.1007/BF01744431.
- [GR02] Mikael Goldmann and Alexander Russell. The complexity of solving equations over finite groups. *Information and Computation*, 178(1):253–262, 2002.
- [Hås86] Johan Håstad. Almost optimal lower bounds for small depth circuits. In *Proceedings of the 18th annual ACM symposium on Theory of computing (STOC)*, pages 6–20, 1986. doi:10.1145/12130.12132.
- [HK10] Kristoffer Arnsfelt Hansen and Michal Koucký. A new characterization of ACC0 and probabilistic CC0. *Computational Complexity*, 19(2):211–234, 2010. doi:10.1007/s00037-010-0287-z.
- [HS11] Gábor Horváth and Csaba Szabó. The extended equivalence and equation solvability problems for groups. *Discrete Mathematics & Theoretical Computer Science*, 13(4):23–32, 2011.
- [HS12] Gábor Horváth and Csaba Szabó. Equivalence and equation solvability problems for the alternating group A_4 . *Journal of Pure and Applied Algebra*, 2012.
- [IK18] Paweł M. Idziak and Jacek Krzaczkowski. Satisfiability in multi-valued circuits. In *Proceedings of the 33rd Annual Symposium on Logic in Computer Science (LICS)*, pages 550–558. ACM, 2018.
- [IKK18] Paweł M. Idziak, Piotr Kawalek, and Jacek Krzaczkowski. Expressive power, satisfiability and equivalence of circuits over nilpotent algebras. In *Proceedings of the 43rd International Symposium on Mathematical Foundations of Computer Science (MFCS)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018. doi:10.4230/LIPIcs.MFCS.2018.17.
- [IKK20] Paweł M. Idziak, Piotr Kawalek, and Jacek Krzaczkowski. Intermediate problems in modular circuits satisfiability. In *Proceedings of the 35th Annual Symposium on Logic in Computer Science (LICS)*, pages 578–590, 2020. doi:10.1145/3373718.3394780.
- [IKK21] Paweł M Idziak, Piotr Kawalek, and Jacek Krzaczkowski. Complexity of modular circuits. arXiv preprint arXiv:2106.02947, 2021.
- [Kea99] Keith Kearnes. Congruence modular varieties with small free spectra. *Algebra Universalis*, 42(3):165–181, 1999. doi:10.1007/s000120050132.
- [KKK19] Michael Kompatscher, Jacek Krzaczkowski, and Piotr Kawalek. Circuit equivalence in 2-nilpotent algebras. aXiv preprint, arXiv:1909.12256, 2019.
- [Kom18] Michael Kompatscher. The equation solvability problem over supernilpotent algebras with Mal'cev term. *International Journal of Algebra and Computation*, 28(06):1005–1015, 2018. doi:10.1142/S0218196718500443.
- [MPT91] Pierre McKenzie, Pierre Péladeau, and Denis Thérien. NC1: The automata-theoretic viewpoint. *Computational Complexity*, 1(4):330–359, 1991. doi:10.1007/BF01212963.
- [Smo87] Roman Smolensky. Algebraic methods in the theory of lower bounds for boolean circuit complexity. In *Proceedings of the 19th annual ACM symposium on Theory of computing (STOC)*, pages 77–82. ACM, 1987. doi:10.1145/28395.28404.
- [Vol99] Heribert Vollmer. *Introduction to circuit complexity: a uniform approach*. Springer Science & Business Media, 1999.
- [Yao85] Andrew Chi-Chih Yao. Separating the polynomial-time hierarchy by oracles. In *Proceedings of the 26th Annual Symposium on Foundations of Computer Science (SFCS)*, pages 1–10. IEEE, 1985. doi:10.1109/SFCS.1985.49.
- [Zhu17] Dmitriy Zhuk. A proof of CSP dichotomy conjecture. In *Proceedings of the 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 331–342. IEEE, 2017. doi:10.1109/FOCS.2017.38.
- [Zhu20] Dmitriy Zhuk. A proof of the CSP dichotomy conjecture. *Journal of the ACM (JACM)*, 67(5):1–78, 2020. doi:10.1145/3402029.