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TERMINATING TABLEAUX FOR GRADED HYBRID LOGIC WITH GLOBAL MODALITIES AND ROLE HIERARCHIES

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Abstract. We present a terminating tableau calculus for graded hybrid logic with global modalities, reflexivity, transitivity and role hierarchies. Termination of the system is achieved through pattern-based blocking. Previous approaches to related logics all rely on chain-based blocking. Besides being conceptually simple and suitable for efficient implementation, the pattern-based approach gives us a NExpTime complexity bound for the decision procedure.

1. Introduction

Graded modal logic [12] is a powerful generalization of basic modal logic. Most prominently, graded modalities are used in description logics, rich modal languages tailored for knowledge representation that have a wide range of practical applications [3]. Graded modal logic allows to constrain the number of accessible states satisfying a certain property. So, the modal formula $\Diamond_n p$ is true in a state x if x has at least $n+1$ successors satisfying p. Analogously to ordinary modal logic, graded modal logic can be extended by nominals [1]. The resulting language, graded hybrid logic, can be extended further by adding global modalities [13], which allow to specify properties that are to hold in all states.

Role hierarchies were first studied by Horrocks [16] in the context of description logics. Using inclusion assertions of the form $r \sqsubseteq r'$, one can specify that the role (relation) r is contained in the role r'. Role hierarchies are of particular interest when considered together with transitivity assertions for roles [30, 4]. The description logic $\mathcal{SHOQ}$ [18] combines the expressive means provided by nominals, graded modalities, role hierarchies and transitive roles.

We present a terminating tableau calculus for graded multimodal logic extended by nominals, global modalities, reflexive and transitive roles, and role hierarchies. The modal language under consideration in the present work is equivalent to $\mathcal{SHOQ}$ extended by reflexive roles and a universal role, both extensions also being known from $\mathcal{SROIQ}$ [17].

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The most important difference of our approach to existing calculi for $\mathcal{SHOQ}$ and stronger logics [18, 19, 17] is the technique used to achieve termination of the tableau construction. The established tableau algorithms all rely on modifications of Kripke's chain-based blocking technique [27]. Chain-based blocking assumes a precedence order on the nominals (also known as nodes or prefixes) of a tableau branch, and prevents processing of nominals that are subsumed by preceding nominals. In the simplest case, the precedence order is chosen to be the ancestor relation among nominals (ancestor blocking). In general, however, it may be any order that contains the ancestor relation (anywhere blocking [2, 28]). Ancestor blocking gives an exponential bound on the length of ancestor chains, resulting in a double exponential bound on the size of tableau branches. Depending on the choice of the precedence order, anywhere blocking can lower this bound to a single exponential. However, the size bound on tableau branches does not seem to translate easily to a complexity bound for the decision procedures in [18, 19, 17] ([18, 19] show a 2-NExpTime bound, while [17] leaves complexity open). We feel that the main difficulty in obtaining better complexity bounds is the algorithms being non-cumulative.

A tableau system is called cumulative if its rules never update or delete formulas. In contrast to most systems in the literature, calculi devised for description logics are often not cumulative. By giving up cumulativity, it is possible to obtain a more direct correspondence between tableau branches and the candidate models they represent. So, for instance, a non-cumulative calculus may merge several nominals into one if the nominals are found to be semantically equivalent. In this way, one can achieve that every state of a candidate model is represented by exactly one nominal. This close correspondence is intuitive and may simplify model existence arguments. At the same time, non-cumulative rules are typically more complex than their cumulative counterparts, which may complicate the presentation of a calculus. More importantly, cumulative systems are usually more amenable to termination and complexity analysis. The problem with non-cumulative systems is that rules that can update or delete formulas may potentially undo earlier changes made to a tableau branch. For instance, consider two tableau branches Γ and Δ, where Δ is obtained from Γ by some sequence of tableau rule applications. In a non-cumulative calculus, it is conceivable that by applying some rule to Δ, we may obtain Γ again. Clearly, such a calculus is non-terminating even if the size of tableau branches can be bounded. Often, termination of non-cumulative calculi can only be achieved if rule application follows some fixed strategy [5, 19, 17]. And even then, size bounds on tableau branches do not immediately yield time complexity bounds. To construct a branch of size n, a non-cumulative system may need significantly more than n rule applications. Cumulative calculi, on the other hand, are guaranteed to enlarge the branch by at least one formula in every step. Therefore, a size bound on tableau branches can immediately be interpreted as an upper bound on the non-deterministic time complexity of the decision procedure.

Unlike [18, 19, 17], our calculus is cumulative. Cumulativity of the calculus in the presence of nominals is achieved following [24] by representing equality constraints via an equivalence relation on nominals. Termination of our system is achieved through pattern-based blocking [23, 24]. Pattern-based blocking is conceptually simpler than chain-based techniques in that it does not need an order on the nominals, and seems promising as it comes to efficient implementation [15]. Pattern-based blocking provides an exponential bound on the size of tableau branches and on the number of tableau rule applications for a single branch. Thus it limits the complexity of the associated decision procedure to

NExpTime. To deal with graded modalities, we extend the blocking conditions in [23, 24], preserving the exponential size bound on tableau branches.

It is worth noting that, despite of the close interplay between pattern-based blocking and abstract representation of state equality in the present work, the two techniques should be seen as independent and applicable in isolation from each other. In fact, pattern-based blocking was introduced in [23] for a non-cumulative system where equality was treated by means of a substitution operation on branches. Also, in previous work [24], we show how abstract treatment of equality can be combined with chain-based blocking to obtain cumulative, terminating tableau calculi for hybrid logic with converse modalities and the difference modality.

We begin by presenting a calculus for graded hybrid logic with global modalities. We argue that the blocking conditions used in [23, 24] are insufficient in the presence of graded modalities. We extend pattern-based blocking to account for the increased expressive power and argue the completeness and termination of the resulting calculus. In the second part of the paper, we extend our calculus further by allowing reflexivity, transitivity and inclusion assertions. It turns out that in the presence of inclusion assertions, the blocking condition used for the basic calculus needs to be extended once again.

2. Graded Hybrid Logic with Global Modalities and Role Inclusion

Following [22, 24], we represent modal logic in simple type theory (see, e.g., [10, 9]). This way we can make use of a rich syntactic and semantic framework and modal logic does not appear as an isolated formal system. We start with two base types B and S. The interpretation of B is fixed and consists of two truth values. The interpretation of S is a nonempty set whose elements are called *worlds* or *states*. Given two types σ and τ, the *functional type* $\sigma\tau$ is interpreted as the set of all total functions from the interpretation of σ to the interpretation of τ. We write $\sigma_1\sigma_2\sigma_3$ for $\sigma_1(\sigma_2\sigma_3)$.

We assume a countable set of *names*, which we partition into a countable set of *variables* and a set of *constants*. We employ three kinds of variables: *Nominal variables* x, y, z of type S, *propositional variables* p, q of type SB, and *role variables* r of type SSB. Nominal variables are called *nominals* for short, and role variables are called *roles*. We assume there are infinitely many nominals. We use the logical constants

$$\bot, \top : \mathrm{B} \qquad \neg : \mathrm{BB} \qquad \vee, \wedge, \rightarrow : \mathrm{BBB} \qquad \doteq : \mathrm{SSB} \qquad \exists, \forall : (\mathrm{SB})\mathrm{B}$$

Terms are defined as usual. We write st for applications, $\lambda x.s$ for abstractions, and $s_1s_2s_3$ for $(s_1s_2)s_3$. We also use infix notation, e.g., $s \wedge t$ for $(\wedge)st$.

Terms of type B are called *formulas*. We employ some common notational conventions: $\exists x.s$ for $\exists(\lambda x.s)$, $\forall x.s$ for $\forall(\lambda x.s)$, and $x \dot{\neq} y$ for $\neg(x \doteq y)$. Given a set X of nominals, we use the following abbreviation:

$$DX \;:=\; \bigwedge_{\substack{x,y \in X \\ x \neq y}} x \dot{\neq} y$$

We use the following constants:

$$\begin{aligned} &\sqsubseteq : (\mathrm{SSB})(\mathrm{SSB})\mathrm{B} \qquad && r_1 \sqsubseteq r_2 \;=\; \forall xy.r_1xy \rightarrow r_2xy \\ &R : (\mathrm{SSB})\mathrm{B} && Rr \;=\; \forall x.rxx \\ &T : (\mathrm{SSB})\mathrm{B} && Tr \;=\; \forall xyz.rxy \wedge ryz \rightarrow rxz \end{aligned}$$

To the right of each constant is an equation defining its semantics. We call formulas of the form $r \sqsubseteq r'$ *(role) inclusion assertions*. Formulas Rr and Tr are called *reflexivity* and *transitivity assertions*, respectively.

We write $\exists^n X.s$ for $\exists x_1 \dots x_n.s$ if $|X| = n$ and $X = \{x_1, \dots, x_n\}$. The *modal constants* are then defined as follows:

$$
\begin{array}{rlrcl}
\dot\neg : & \mathrm{(SB)SB} & \dot\neg px & = & \neg(px) \\
\dot\wedge : & \mathrm{(SB)(SB)SB} & (p \mathbin{\dot\wedge} q)x & = & px \wedge qx \\
\dot\vee : & \mathrm{(SB)(SB)SB} & (p \mathbin{\dot\vee} q)x & = & px \vee qx \\
\langle \_ \rangle_n : & \mathrm{(SSB)(SB)SB} & \langle r \rangle_n px & = & \exists^{n+1} Y.\, DY \wedge (\bigwedge_{y \in Y} rxy \wedge py) \\
[\_]_n : & \mathrm{(SSB)(SB)SB} & [r]_n px & = & \forall^{n+1} Y.\, (\bigwedge_{y \in Y} rxy) \wedge DY \to \bigvee_{y \in Y} py \\
E_n : & \mathrm{(SB)SB} & E_n px & = & \exists^{n+1} Y.\, DY \wedge \bigwedge_{y \in Y} py \\
A_n : & \mathrm{(SB)SB} & A_n px & = & \forall^{n+1} Y.\, DY \to \bigvee_{y \in Y} py \\
\dot{\_} : & \mathrm{SSB} & \dot x y & = & x \doteq y
\end{array}
$$

where $n \geq 0$ in all equations

The semantics of boxes and diamonds is defined following [11, 31, 29]. Intuitively, it can be described as follows:

$E_n p$: There are at least $n+1$ states satisfying p.
$A_n p$: All states but possibly n exceptions satisfy p.
$\langle r \rangle_n p$: There are at least $n+1$ r-successors satisfying p.
$[r]_n p$: All r-successors but possibly n exceptions satisfy p.

In accordance with the usual modal intuition, "formulas" of modal logic are seen as predicates of type SB denoting sets of states. They can be represented as *modal expressions* according to the following grammar:

$$t \ ::= \ p \mid \dot x \mid \dot\neg t \mid t \mathbin{\dot\wedge} t \mid t \mathbin{\dot\vee} t \mid \langle r \rangle_n t \mid [r]_n t \mid E_n t \mid A_n t$$

As with the propositional connectives, we use infix notation for $\dot\wedge$ and $\dot\vee$. Unlike with the propositional connectives, we assume the application of modal operators to have a higher precedence than regular functional application. So, for instance, we write $\dot\neg \langle r \rangle_2 \dot y \mathbin{\dot\vee} p\ x$ for $((\dot\neg(\langle r \rangle_2(\dot y))) \mathbin{\dot\vee} p)x$.

An *interpretation* is a function $\mathcal{I}$ mapping B to the set $\{0, 1\}$, S to a non-empty set, a functional type $\sigma\tau$ to the set of all total functions from $\mathcal{I}\sigma$ to $\mathcal{I}\tau$, and every name $x : \sigma$ to an element of $\mathcal{I}\sigma$ (i.e., $\mathcal{I}x \in \mathcal{I}\sigma$) such that the logical constants get their usual meaning:

$$
\begin{array}{rclrcl}
\mathcal{I}\bot = 0 & \text{and} & \mathcal{I}\top = 1 & (\mathcal{I}\neg)a = 1 & \iff & a = 0 \\
(\mathcal{I}\wedge)ab = 1 & \iff & a = 1 \text{ and } b = 1 & (\mathcal{I}\vee)ab = 1 & \iff & a = 1 \text{ or } b = 1 \\
(\mathcal{I}\to)ab = 1 & \iff & a = 0 \text{ or } b = 1 & (\mathcal{I}\doteq)ab = 1 & \iff & a = b \\
(\mathcal{I}\exists)f = 1 & \iff & fa = 1 \text{ for some } a \in \mathcal{I}\mathrm{S} & (\mathcal{I}\forall)f = 1 & \iff & fa = 1 \text{ for all } a \in \mathcal{I}\mathrm{S}
\end{array}
$$

If $\mathcal{I}$ is an interpretation, $x : \sigma$ is a variable, and $a \in \mathcal{I}\sigma$, then $\mathcal{I}^x_a$ denotes the interpretation that agrees everywhere with $\mathcal{I}$ but possibly on x where it yields a. Every interpretation $\mathcal{I}$

can be extended to a function $\hat{\mathcal{I}}$ that maps every term $s:\sigma$ to an element of $\mathcal{I}\sigma$ such that:

$$\begin{aligned}\hat{\mathcal{I}}x &= \mathcal{I}x\\ \hat{\mathcal{I}}(st) &= (\hat{\mathcal{I}}s)(\hat{\mathcal{I}}t)\\ \hat{\mathcal{I}}(\lambda x.s) &= \{(a, \widehat{\mathcal{I}^x_a}s)\,|\,a\in\mathcal{I}\sigma\} \qquad \text{if } x:\sigma\end{aligned}$$

Since $\hat{\mathcal{I}}$ is uniquely determined by $\mathcal{I}$, in the following we write $\mathcal{I}s$ for $\hat{\mathcal{I}}s$ for convenience. A *modal interpretation* is an interpretation that, in addition, satisfies the above equations defining the constants $\sqsubseteq$, R, T, $\dot\neg$, $\dot\wedge$, $\dot\vee$, $\langle\_\rangle_n$, $[\_]_n$, E, A, $\dot{\_}$. If $\mathcal{I}s = 1$, we say that $\mathcal{I}$ *satisfies* s, or that $\mathcal{I}$ is a *model* of s. A modal interpretation $\mathcal{I}$ satisfies a set Γ of formulas ($\mathcal{I}$ is a *model* of Γ) if $\mathcal{I}$ satisfies every formula in Γ. A formula (a set of formulas) is called *satisfiable* if it has a model.

3. Graded Hybrid Logic with Global Modalities

We begin with a tableau calculus for the restricted language without inclusion, reflexivity or transitivity assertions.

3.1. **Branches.** For the sake of simplicity, we define our tableau calculus on negation normal expressions, i.e., terms of the form:

$$t \ ::= \ p \mid \dot\neg p \mid \dot x \mid \dot\neg\dot x \mid t\,\dot\wedge\,t \mid t\,\dot\vee\,t \mid \langle r\rangle_n t \mid [r]_n t \mid E_n t \mid A_n t$$

A *branch* Γ is a finite set of formulas s of the form

$$s \ ::= \ tx \mid rxy \mid x\dot=y \mid x\dot{\neq}y \mid \bot$$

where t is a negation-normal modal expression of the above form. Formulas of the form rxy are called *accessibility formulas* or *edges*. We use the formula $\bot$ to explicitly mark unsatisfiable branches. We call a branch Γ *closed* if $\bot\in\Gamma$. Otherwise, Γ is called *open*. The branch consisting of the initial formula (or formulas) to be tested for satisfiability is called the *initial branch*.

Let Γ be a branch. With $\sim_\Gamma$ we denote the least equivalence relation $\sim$ on nominals such that $x\sim y$ for every equation $x\dot=y\in\Gamma$. Let $R(x,y)$ denote a term of the form $x\dot=y$, $x\dot{\neq}y$, or rxy. We define the *equational closure* $\tilde\Gamma$ of a branch Γ as

$$\begin{aligned}\tilde\Gamma \ := \ &\Gamma\cup\{tx\,|\,t \text{ modal expression } \wedge\ \exists x':\ x'\sim_\Gamma x \ \wedge\ tx'\in\Gamma\}\\ &\cup\{R(x,y)\,|\,\exists x',y':\ x'\sim_\Gamma x\ \wedge\ y'\sim_\Gamma y\ \wedge\ R(x',y')\in\Gamma\}\end{aligned}$$

Note that for all nominals x and y, $x\sim_\Gamma y$ holds if and only if $x\dot=y\in\tilde\Gamma$. Since $\tilde\Gamma$ only contains nominals, modal expressions and roles that already occur on Γ, $\tilde\Gamma$ clearly is finite if Γ is finite. Reasoning with respect to $\tilde\Gamma$ can be implemented efficiently using disjoint-set forests, as demonstrated in [14, 15].

3.2. **Evidence.** The proof of model existence for our calculus proceeds in two stages. Applied to a satisfiable initial branch, the rules of the calculus (defined in Sect. 3.3) construct a *quasi-evident* branch (defined in Sect. 3.4). We show that every quasi-evident branch can be extended to an *evident* branch. For evident branches, we show model existence. Intuitively, we call a branch evident if it contains a complete syntactic description of a model of all of its formulas.

We write $D_\Gamma X$ as an abbreviation for $\forall x, y \in X\colon\ x \neq y \implies x \dot{\neq} y \in \tilde{\Gamma} \vee y \dot{\neq} x \in \tilde{\Gamma}$. A branch Γ is called *evident* if it satisfies all of the following *evidence conditions*:

$$
\begin{aligned}
(t_1 \mathbin{\dot\wedge} t_2)x \in \Gamma \ &\Rightarrow\ t_1x \in \tilde{\Gamma} \wedge t_2x \in \tilde{\Gamma}\\
(t_1 \mathbin{\dot\vee} t_2)x \in \Gamma \ &\Rightarrow\ t_1x \in \tilde{\Gamma} \vee t_2x \in \tilde{\Gamma}\\
\langle r\rangle_n tx \in \Gamma \ &\Rightarrow\ \exists^{n+1}Y\colon\ D_\Gamma Y \wedge \{rxy, ty \mid y \in Y\} \subseteq \tilde{\Gamma}\\
[r]_n tx \in \Gamma \ &\Rightarrow\ |\{y \mid rxy \in \tilde{\Gamma},\ ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n\\
E_n tx \in \Gamma \ &\Rightarrow\ \exists^{n+1}Y\colon\ D_\Gamma Y \wedge \{ty \mid y \in Y\} \subseteq \tilde{\Gamma}\\
A_n tx \in \Gamma \ &\Rightarrow\ |\{y \mid ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n\\
\dot{x}y \in \Gamma \ &\Rightarrow\ x \sim_\Gamma y\\
\dot\neg\dot{x}y \in \Gamma \ &\Rightarrow\ x \not\sim_\Gamma y\\
x \dot{\neq} y \in \Gamma \ &\Rightarrow\ x \not\sim_\Gamma y\\
\neg px \in \Gamma \ &\Rightarrow\ px \notin \tilde{\Gamma}
\end{aligned}
$$

A formula s is called *evident on* Γ if Γ satisfies the right-hand side of the evidence condition corresponding to s. For instance, $(t_1 \mathbin{\dot\wedge} t_2)x$ is evident on Γ if and only if $\{t_1x, t_2x\} \subseteq \tilde{\Gamma}$.

Given a term t, we write $\mathcal{N}t$ for the set of nominals that occur in t. The notation is extended to sets of terms in the natural way: $\mathcal{N}\Gamma := \bigcup\{\mathcal{N}t \mid t \in \Gamma\}$.

Theorem 3.1 (Model Existence)**.** *Every evident branch has a finite model.*

Proof. Let Γ be an evident branch and let $x_0 \in \mathcal{N}\Gamma$. Let ρ be a function from finite sets of nominals to nominals such that $\rho X \in X$ whenever X is nonempty. We define the interpretation $\mathcal{I}$ such that:

$$
\begin{aligned}
\mathcal{I}\mathrm{S} \ &:=\ \{\rho\{y \mid y \sim_\Gamma x\} \mid x \in \mathcal{N}\Gamma\}\\
\mathcal{I}x \ &:=\ \text{if } x \in \mathcal{N}\Gamma \text{ then } \rho\{y \in \mathcal{N}\Gamma \mid y \sim_\Gamma x\} \text{ else } \mathcal{I}x_0\\
\mathcal{I}p \ &:=\ \{x \in \mathcal{I}\mathrm{S} \mid px \in \tilde{\Gamma}\}\\
\mathcal{I}r \ &:=\ \{(x,y) \in (\mathcal{I}\mathrm{S})^2 \mid rxy \in \tilde{\Gamma}\}
\end{aligned}
$$

Intuitively, we construct $\mathcal{I}$ by interpreting S as the quotient of the nominals on Γ by $\sim_\Gamma$, where each equivalence class is represented by a fixed element of the class selected by ρ. Nominals on Γ are mapped to their corresponding equivalence classes. All other nominals are mapped to some arbitrary state. Propositional variables and roles are interpreted as the smallest sets that are consistent with the respective assertions on Γ. Since Γ is finite by definition, so is $\mathcal{I}$. Note that in the last two lines of the definition, we interpret the set notation as a convenient description for the respective characteristic functions.

We now show that, for all $s \in \Gamma$, $\mathcal{I}$ satisfies s by induction on s. Let $s \in \Gamma$. We proceed by case analysis.

- $s = px$. Since $\mathcal{I}x \sim_\Gamma x$, we have $p(\mathcal{I}x) \in \tilde{\Gamma}$. The claim follows.

- $s = \dot{\neg} px$. It suffices to show that $\mathcal{I}(px) = 0$. By the evidence condition for s, $px \notin \tilde{\Gamma}$. Hence $p(\mathcal{I}x) \notin \tilde{\Gamma}$. The claim follows.
- $s = rxy$. Then $r(\mathcal{I}x)(\mathcal{I}y) \in \tilde{\Gamma}$, and hence $(\mathcal{I}x, \mathcal{I}y) \in \mathcal{I}r$.
- $s = x \dot{=} y$. It suffices to show that $\mathcal{I}x = \mathcal{I}y$, which is the case as $x \sim_\Gamma y$ by the definition of $\sim_\Gamma$.
- $s = x \dot{\neq} y$. By the evidence condition for s, $x \not\sim_\Gamma y$. Hence $\mathcal{I}x \not\sim_\Gamma \mathcal{I}y$. The claim follows.
- $s = \langle r \rangle_n tx$. By the evidence condition for s, there is a set Y of cardinality $n+1$ such that $D_\Gamma Y$ and for all $y \in Y$, $\{rxy, ty\} \subseteq \tilde{\Gamma}$. By the inductive hypothesis for the disequations required by $D_\Gamma Y$, we have $|Y/_{\sim_\Gamma}| = |\{\mathcal{I}y \mid y \in Y\}| = n + 1$. By the inductive hypothesis for the formulas rxy and ty (for all $y \in Y$), we have $(\mathcal{I}x, \mathcal{I}y) \in \mathcal{I}r$, and $\mathcal{I}$ satisfies ty. The claim follows.
- $s = [r]_n tx$. By the evidence condition for s, $|\{y \mid rxy \in \tilde{\Gamma},\ ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n$. Since $\mathcal{I}x \sim_\Gamma x$ whenever $x \in \mathcal{N}\Gamma$, we have for all $x, y \in \mathcal{N}\Gamma$: $(\mathcal{I}x, \mathcal{I}y) \in \mathcal{I}r \;\Leftrightarrow\; r(\mathcal{I}x)(\mathcal{I}y) \in \tilde{\Gamma} \;\Leftrightarrow\; rxy \in \tilde{\Gamma}$. Hence $|\{y \mid rxy \in \tilde{\Gamma},\ ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| = |\{\mathcal{I}y \mid (\mathcal{I}x, \mathcal{I}y) \in \mathcal{I}r,\ ty \notin \tilde{\Gamma}\}| \leq n$. Moreover, by the inductive hypothesis, $\mathcal{I}$ satisfies ty whenever $ty \in \tilde{\Gamma}$. The claim follows.

The cases $s = (t_1 \dot{\vee} t_2)x$, $s = (t_1 \dot{\wedge} t_2)x$ are straightforward. The cases $s = \dot{x}y$ and $s = \dot{\neg}\dot{x}y$ proceed analogously to $s = x \dot{=} y$ and, respectively, $s = x \dot{\neq} y$, and the cases $s = E_n tx$ and $s = A_n tx$ are analogous but simpler than $s = \langle r \rangle_n tx$ and, respectively, $s = [r]_n tx$. $\square$

3.3. **Tableau Rules.** The tableau rules of our basic calculus $\mathcal{T}$ are defined in Fig. 1. In the rules, we write $\exists x \in X : \Gamma(x)$ for $\Gamma(x_1) \mid \ldots \mid \Gamma(x_n)$, where $X = \{x_1, \ldots, x_n\}$ and $\Gamma(x)$ is a set of formulas parameterized by x. In case $X = \emptyset$, the notation translates to $\perp$. Dually, we write $\forall x \in X : \Gamma(x)$ for $\Gamma(x_1), \ldots, \Gamma(x_n)$ ($X = \{x_1, \ldots, x_n\}$). If $X = \emptyset$, the notation stands for the empty set of formulas.

The side condition of $\mathcal{R}_\Diamond$ uses the notion of quasi-evidence that we will introduce in Sect. 3.4. For now, we assume the rule is formulated with the restriction "$\langle r \rangle_n tx$ not evident on Γ".

Note that for $n = 0$, the rules $\mathcal{R}_\Diamond$ and $\mathcal{R}_\Box$ instantiate, modulo obvious simplifications, to their respective non-graded counterparts:

$$\frac{\langle r \rangle_0 tx}{rxy,\ ty}\ y \text{ fresh},\ \langle r \rangle_0 tx \text{ not quasi-evident on } \Gamma \qquad\qquad \frac{[r]_0 tx}{ty}\ rxy \in \tilde{\Gamma}$$

A branch Δ is called a *proper extension* of a branch Γ if $\Delta \supseteq \Gamma$ and $\tilde{\Delta} \supsetneq \tilde{\Gamma}$. Note that if Δ is a proper extension of Γ, in particular it holds $\Delta \supsetneq \Gamma$. The converse does not hold: Let $\Gamma := \{\dot{x}y,\ x \dot{=} z,\ z \dot{=} y\}$ and $\Delta := \Gamma \cup \{x \dot{=} y\}$. Then $\Delta \supsetneq \Gamma$ but Δ is not a proper extension of Γ. We implicitly restrict the applicability of the tableau rules so that a rule $\mathcal{R}$ is only applicable to a formula $s \in \Gamma$ if all of the alternative branches $\Delta_1, \ldots, \Delta_n$ resulting from this application are proper extensions of Γ. Moreover, we require that for every i, j with $1 \leq i < j \leq n$, $\tilde{\Delta}_i \neq \tilde{\Delta}_j$. Whenever a rule produces several alternative branches whose equational closure is equal, by the following proposition it suffices to consider only one of them to preserve soundness.

Proposition 3.2. *Let $\mathcal{I}$ be a modal interpretation and Γ, Δ be branches such that $\tilde{\Gamma} = \tilde{\Delta}$. Then $\mathcal{I}$ satisfies Γ if and only if $\mathcal{I}$ satisfies Δ.* $\square$

$$\mathcal{R}_{\dot\wedge}\ \frac{(s \mathbin{\dot\wedge} t)x}{sx,\ tx} \qquad\qquad \mathcal{R}_{\dot\vee}\ \frac{(s \mathbin{\dot\vee} t)x}{sx \mid tx}$$

$$\mathcal{R}_{\Diamond}\ \frac{\langle r\rangle_n tx}{\forall y \in Y\colon rxy,\ ty,\ \forall z \in Y, y \neq z\colon y \dot{\neq} z}\ Y \text{ fresh},\ |Y| = n+1,\ \langle r\rangle_n tx \text{ not quasi-evident on } \Gamma$$

$$\mathcal{R}_{\Box}\ \frac{[r]_n tx}{\exists y, z \in Y,\ y \neq z\colon y \doteq z \mid \exists y \in Y\colon ty}\ Y \subseteq \{y \mid rxy \in \tilde{\Gamma}\},\ |Y| = |Y/{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_{E}\ \frac{E_n tx}{\forall y \in Y\colon ty,\ \forall z \in Y,\ y \neq z\colon y \dot{\neq} z}\ Y \text{ fresh},\ |Y| = n+1,\ E_n tx \text{ not evident on } \Gamma$$

$$\mathcal{R}_{A}\ \frac{A_n tx}{\exists y, z \in Y,\ y \neq z\colon y \doteq z \mid \exists y \in Y\colon ty}\ Y \subseteq \mathcal{N}\Gamma,\ \ |Y| = |Y/{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_{N}\ \frac{\dot{x}y}{x \doteq y} \qquad \mathcal{R}_{\bar{N}}\ \frac{\dot\neg \dot{x}y}{x \dot{\neq} y} \qquad \mathcal{R}^{\bot}_{\dot\neg}\ \frac{\dot\neg px}{\bot}\ px \in \tilde{\Gamma} \qquad \mathcal{R}^{\bot}_{\dot\neq}\ \frac{x \dot{\neq} y}{\bot}\ x \sim_\Gamma y$$

Γ is the branch to which a rule is applied.
"Y fresh" stands for $Y \cap \mathcal{N}\Gamma = \emptyset$.

Figure 1: Tableau rules for $\mathcal{T}$

Proposition 3.3 (Soundness)**.** *Let $\Delta_1, \dots, \Delta_n$ be the branches obtained from a branch Γ by a rule of $\mathcal{T}$. Then Γ is satisfiable if and only if there is some $i \in \{1, \dots, n\}$ such that Δ_i is satisfiable.* □

Example 3.4. Consider the unsatisfiable formula $(\langle r\rangle_1 p \mathbin{\dot\wedge} [r]_1 \dot\neg p)x$. Applied to the formula, our tableau rules produce three closed branches as shown in Fig. 2. All the rule applications except $\mathcal{R}_\Box$ produce exactly one extension. The rule $\mathcal{R}_\Box$ applies to the formula $[r]_1 \dot\neg px$ and the set $Y = \{y, z\}$ producing three extensions. The leftmost branch is closed with $\mathcal{R}^{\bot}_{\dot\neq}$ applied to $y \dot{\neq} z$, the other two branches are closed with $\mathcal{R}^{\bot}_{\dot\neg}$ applied to the respective two formulas introduced by the application of $\mathcal{R}_\Box$. Note that without the restriction that the equational closures of alternative extensions must be different the application of $\mathcal{R}_\Box$ would introduce an additional fourth extension, namely by the equation $z \doteq y$.

3.4. **Control.** The restrictions on the applicability of the tableau rules given by the evidence conditions are not sufficient for termination. Consider $\Gamma_0 := \{A_0 \langle r\rangle_0 px\}$. An application of $\mathcal{R}_A$ to Γ_0 yields $\Gamma_1 := \Gamma_0 \cup \{\langle r\rangle_0 px\}$, which can be extended by $\mathcal{R}_\Diamond$ to $\Gamma_2 := \Gamma_1 \cup \{rxy, py\}$. Now $\mathcal{R}_A$ is applicable again and yields $\Gamma_3 := \Gamma_2 \cup \{\langle r\rangle_0 py\}$, which in turn can be extended by $\mathcal{R}_\Diamond$, and so ad infinitum.

To obtain a terminating calculus, the rule $\mathcal{R}_\Diamond$ needs to be restricted further. We do so by weakening the notion of evidence for diamond formulas. The weaker notion, called

$$
\begin{array}{c}
(\langle r\rangle_1 p \mathbin{\dot\wedge} [r]_1 \dot\neg p)x \\
\langle r\rangle_1 px,\ [r]_1 \dot\neg px \qquad \mathcal{R}_{\dot\wedge} \\
rxy,\ py,\ rxz,\ pz,\ y \not\doteq z \qquad \mathcal{R}_{\Diamond} \\
\hline
\begin{array}{cc|cc|cc}
y \doteq z & \mathcal{R}_{\Box} & \dot\neg py & \mathcal{R}_{\Box} & \dot\neg pz & \mathcal{R}_{\Box} \\
\bot & \mathcal{R}^{\bot}_{\not\doteq} & \bot & \mathcal{R}^{\bot}_{\dot\neg} & \bot & \mathcal{R}^{\bot}_{\dot\neg}
\end{array}
\end{array}
$$

Figure 2: Tableau derivation for $(\langle r\rangle_1 p \mathbin{\dot\wedge} [r]_1 \dot\neg p)x$

quasi-evidence, is then used in the side condition of $\mathcal{R}_{\Diamond}$ in place of evidence. As we have mentioned before, an evident branch contains a complete description of a model of all of its formulas. A quasi-evident branch will contain only a partial description of such a model. In particular, quasi-evidence will not require that for every diamond $\langle r\rangle_n tx$, we have $n+1$ outgoing edges rxy. However, we require that the partial description given by a quasi-evident branch can always be completed to a full model of the branch by adding edges. So, in particular, every quasi-evident branch will be satisfiable. In the above example, Γ_3 will turn out to be quasi-evident and hence terminal. And indeed, Γ_3 is clearly satisfiable and can be completed to an evident branch by adding the edge ryy.

While quasi-evidence was introduced in the context of pattern-based blocking, it can also be made sense of in the context of chain-based blocking. Unlike with pattern-based blocking, calculi using chain-based blocking usually terminate with branches that are not quasi-evident, which is due to the presence of "blocked" parts, i.e., parts of the branch that have at some point been identified as irrelevant for the model construction and so have been excluded from further processing. The parts that are not blocked form a kernel from which a model can be constructed. And in many cases, this kernel is precisely what we call a quasi-evident branch. A concrete example relating chain-based blocking and quasi-evidence is given in [24].

Our task is now to define a notion of quasi-evidence that is weak enough to guarantee termination of our calculus but strong enough to preserve completeness in the presence of graded modalities. The notions of quasi-evidence used in previous work on pattern-based blocking [23, 24] turn out to be too weak. For instance, intuitively adapting the notion in [23] would give us the following candidate definition:

A formula $\langle r\rangle_m sx$ is quasi-evident on Γ if there are nominals $y, z_1, \ldots, z_{m+1}$ such that $\{ryz_1, sz_1, \ldots, ryz_{m+1}, sz_{m+1}\} \subseteq \tilde\Gamma$ and $\{[r]_n ty \mid [r]_n tx \in \tilde\Gamma\} \subseteq \tilde\Gamma$. (We also say: $\langle r\rangle_m sx$ is quasi-evident if the corresponding *pattern* $\{\langle r\rangle_m s\} \cup \{[r]_n t \mid [r]_n tx \in \tilde\Gamma\}$ is *expanded*).

With this definition of quasi-evidence, no rule of our calculus would apply to the following branch:

$$\Gamma := \{ryz,\ qz,\ [r]_1(p \mathbin{\dot\wedge} \dot\neg p)y,\ \langle r\rangle_0 qx,\ [r]_1(p \mathbin{\dot\wedge} \dot\neg p)x,\ rxu,\ \dot\neg qu\}$$

As Γ is clearly unsatisfiable, the notion of quasi-evidence needs to be adapted.

Given a branch Γ and a role r, an *r-pattern* is a set of expressions of the form μs, where $\mu \in \{\langle r\rangle_n, [r]_n \mid n \in \mathbb{N}\}$. We write $P^r_\Gamma x$ for the largest r-pattern P such that $P \subseteq \{t \mid tx \in \tilde\Gamma\}$. We call $P^r_\Gamma x$ the r-pattern of x on Γ. An r-pattern P is *expanded on* Γ if there are nominals x, y such that $rxy \in \tilde\Gamma$ and $P \subseteq P^r_\Gamma x$. In this case, we say that the nominal x *expands* P *on* Γ.

A diamond formula $\langle r\rangle_n sx \in \Gamma$ is *quasi-evident on* Γ if it is either evident on Γ or x has no *r-successor* on Γ (i.e., there is no y such that $rxy \in \tilde{\Gamma}$) and $P^r_\Gamma x$ is expanded on Γ. The rule $\mathcal{R}_\Diamond$ can only be applied to diamond formulas that are not quasi-evident.

Note that whenever $\langle r\rangle_n sx \in \Gamma$ is quasi-evident but not evident on Γ, there is a nominal y that expands $P^r_\Gamma x$ on Γ.

We call a branch Γ quasi-evident if it satisfies all of the evidence conditions but the one for diamond formulas, which we replace by:

$$\langle r\rangle_n tx \in \Gamma \;\Rightarrow\; \langle r\rangle_n tx \text{ is quasi-evident on } \Gamma$$

Example 3.5. Figure 3 shows a tableau derivation resulting in a quasi-evident branch. Let us write Γ_n for the branch obtained in line n of the derivation. Note that $P^r_{\Gamma_3} x = \{\langle r\rangle_0 p,\ \langle r\rangle_0 q\}$ is expanded on Γ_3. The notion of expandedness is such that, once expanded, a pattern remains expanded on all extensions of the branch. In particular, if $P^r_{\Gamma_i} x$ is expanded on Γ_i, then $P^r_{\Gamma_i} x$ (not, however, $P^r_{\Gamma_j} x$) will be expanded on Γ_j for all $j \geq i$. Note that the pattern of a nominal may change over time, i.e., $P^r_{\Gamma_i} x$ and $P^r_{\Gamma_j} x$ may be different if $i \neq j$. So, in the example, $P^r_{\Gamma_1} x = \emptyset \subsetneq P^r_{\Gamma_3} x$. In general, we have $P^r_{\Gamma_i} x \subseteq P^r_{\Gamma_j} x$ whenever $i \leq j$. However, if $x \sim_{\Gamma_i} y$ and x expands $P^r_{\Gamma_i} y$ on Γ_i, then x will expand $P^r_{\Gamma_j} y$ on Γ_j for all $j \geq i$.

Since $P^r_{\Gamma_5} x = P^r_{\Gamma_5} y$, $P^r_{\Gamma_5} y$ is expanded on Γ_5, and hence both $\langle r\rangle_0 py$ and $\langle r\rangle_0 qy$ are quasi-evident on Γ_5. The pattern $P^{r'}_{\Gamma_5} y = \{\langle r'\rangle_0 q\}$ is not expanded on Γ_5, so $\mathcal{R}_\Diamond$ is applicable to $\langle r'\rangle_0 qy$. On the branch Γ_6 resulting from this application, the pattern becomes expanded, and so does $P^{r'}_{\Gamma_6} x$. The only diamond formula that is not quasi-evident on Γ_6 is $\langle r\rangle_0 qx$ (since it is not evident and x has a successor on Γ_6). After applying $\mathcal{R}_\Diamond$ to $\langle r\rangle_0 qx$, Γ_7 contains only quasi-evident diamond formulas. To make the branch evident, it remains to propagate the universal constraint $\langle r\rangle_0 p \,\dot\wedge \langle r\rangle_0 q \,\dot\wedge \langle r'\rangle_0 q$ to z and u (steps 8-11). Since this introduces no new patterns (we have $P^r_{\Gamma_{11}} z = P^r_{\Gamma_{11}} u = P^r_{\Gamma_{11}} y = P^r_{\Gamma_{11}} x$ and $P^{r'}_{\Gamma_{11}} z = P^{r'}_{\Gamma_{11}} u = P^{r'}_{\Gamma_{11}} x = P^{r'}_{\Gamma_{11}} y$), Γ_{11} is quasi-evident.

Lemma 3.6. *Let Γ be a quasi-evident branch and let $\langle r\rangle_n sx \in \Gamma$ be not evident on Γ. Let y be a nominal that expands $P^r_\Gamma x$ on Γ and let $\Delta := \Gamma \cup \{rxz \mid ryz \in \tilde{\Gamma}\}$. Then:*

(1) $\forall z:\ rxz \in \tilde{\Delta} \iff ryz \in \tilde{\Gamma}$,
(2) $\forall m, t:\ \langle r\rangle_m t \in P^r_\Gamma x \implies \langle r\rangle_m tx$ *evident on* Δ,
(3) $\langle r\rangle_n sx$ *evident on* Δ,
(4) $\forall r', m, t, z:\ \langle r'\rangle_m tz$ *evident on* $\Gamma \implies \langle r'\rangle_m tz$ *evident on* Δ,
(5) Δ *quasi-evident.*

Proof. We begin with (1). Let z be a nominal. By construction, it holds $ryz \in \tilde{\Gamma} \Rightarrow rxz \in \Delta$. The converse implication holds by the fact that $\langle r\rangle_n sx$ is quasi-evident but not evident on Γ, meaning that x has no r-successor on Γ. It remains to show: $rxz \in \Delta \Leftrightarrow rxz \in \tilde{\Delta}$. The direction from left to right is obvious. For the other direction, assume $rxz \in \tilde{\Delta}$. Then there are x', z' such that $x' \sim_\Gamma x$, $z' \sim_\Gamma z$, and $rx'z' \in \Delta$. Since x has no r-successor on Γ, neither does x'. Hence, by the definition of Δ, we must have $x' = x$, and so $rxz' \in \Delta$. But then $ryz' \in \tilde{\Gamma}$, and consequently $ryz \in \tilde{\Gamma}$. The claim follows by the definition of Δ.

Now to (2). Let $\langle r\rangle_m t \in P^r_\Gamma x$. Since $P^r_\Gamma y \supseteq P^r_\Gamma x$, in particular it holds $\langle r\rangle_m ty \in \tilde{\Gamma}$, i.e., there is some $y' \sim_\Gamma y$ such that $\langle r\rangle_m ty' \in \Gamma$. By (1), it suffices to show that $\langle r\rangle_m ty$ is evident on Γ. This is the case since $\langle r\rangle_m ty'$ is quasi-evident on Γ (as Γ is quasi-evident) and y' has an r-successor on Γ (as y has one on Γ).

0.	$A_0(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q \mathbin{\dot\wedge} \langle r'\rangle_0 q)x$	
1.	$(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q \mathbin{\dot\wedge} \langle r'\rangle_0 q)x$	$\mathcal{R}_A$
2.	$(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q)x,\ \langle r\rangle_0 px,\ \langle r\rangle_0 qx,\ \langle r'\rangle_0 qx$	$2\times\mathcal{R}_{\dot\wedge}$
3.	$rxy,\ py$	$\mathcal{R}_\Diamond$
4.	$(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q \mathbin{\dot\wedge} \langle r'\rangle_0 q)y$	$\mathcal{R}_A$
5.	$(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q)y,\ \langle r\rangle_0 py,\ \langle r\rangle_0 qy,\ \langle r'\rangle_0 qy$	$2\times\mathcal{R}_{\dot\wedge}$
6.	$r'yz,\ qz$	$\mathcal{R}_\Diamond$
7.	$rxu,\ qu$	$\mathcal{R}_\Diamond$
8.	$(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q \mathbin{\dot\wedge} \langle r'\rangle_0 q)z$	$\mathcal{R}_A$
9.	$(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q)z,\ \langle r\rangle_0 pz,\ \langle r\rangle_0 qz,\ \langle r'\rangle_0 qz$	$2\times\mathcal{R}_{\dot\wedge}$
10.	$(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q \mathbin{\dot\wedge} \langle r'\rangle_0 q)u$	$\mathcal{R}_A$
11.	$(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q)u,\ \langle r\rangle_0 pu,\ \langle r\rangle_0 qu,\ \langle r'\rangle_0 qu$	$2\times\mathcal{R}_{\dot\wedge}$

Figure 3: Tableau derivation for $A_0(\langle r\rangle_0 p \mathbin{\dot\wedge} \langle r\rangle_0 q \mathbin{\dot\wedge} \langle r'\rangle_0 q)x$

Claim (3) immediately follows from (2).

Claim (4) is obvious as the evidence of diamonds on a branch cannot be destroyed by adding edges.

Now to (5). The only conditions that might in principle be violated on Δ are the quasi-evidence condition for diamonds of the form $\langle r\rangle_m tz \in \Delta$ where $z \sim_\Delta x$, and the evidence condition for boxes $[r]_m tz \in \Delta$ where $z \sim_\Delta x$.

For diamonds of the above form, the quasi-evidence condition holds by (2).

If $[r]_m tz \in \Delta$ and $z \sim_\Delta x$, it holds $[r]_m ty \in \tilde\Gamma$ since $P^r_\Gamma y \supseteq P^r_\Gamma x = P^r_\Delta x$. Hence by (1) it suffices to show that $[r]_m ty$ is evident on Γ, which is the case since Γ is quasi-evident. □

Theorem 3.7 (Evidence Completion)**.** *For every quasi-evident branch Γ there is an evident branch Δ such that $\Gamma \subseteq \Delta$.*

Proof. For every branch Γ we define:

$$\varphi\Gamma := |\{\langle r\rangle_n sx \mid \langle r\rangle_n sx \in \Gamma \wedge \langle t\rangle_n sx \text{ not evident on } \Gamma\}|$$

Let Γ be quasi-evident. We proceed by induction on $\varphi\Gamma$. If $\varphi\Gamma = 0$, then Γ is evident and we are done. Otherwise, there is a diamond $\langle r\rangle_n sx \in \Gamma$ that is not evident on Γ. Let y be a nominal that expands $P^r_\Gamma x$ on Γ, and let $\Gamma' := \Gamma \cup \{rxz \mid ryz \in \tilde\Gamma\}$. By Lemma 3.6(3-5), Γ' is quasi-evident and $\varphi\Gamma' < \varphi\Gamma$. So, by the inductive hypothesis, there is some evident branch Δ such that $\Gamma \subseteq \Gamma' \subseteq \Delta$. □

A branch is called *maximal* if it cannot be extended by any tableau rule.

Theorem 3.8 (Quasi-evidence)**.** *Every open and maximal branch in $\mathcal{T}$ is quasi-evident.*

Proof. Let Γ be an open and maximal branch. Note that we have no evidence or quasi-evidence conditions for formulas of the form px, rxy or $x \dot= y$. We show that every $s \in \Gamma$

that is not of the form px, rxy or $x\dot{=}y$ is (quasi-)evident on Γ by case analysis on the shape of s.

- $s = \dot{\neg}px$. The claim, $px \notin \tilde{\Gamma}$, follows by $\mathcal{R}^{\perp}_{\dot{\neg}}$ (and the assumption that Γ is open and maximal).
- $s = x\dot{\neq}y$. The claim, $x \not\sim_{\Gamma} y$, follows by $\mathcal{R}^{\perp}_{\dot{\neq}}$ (and the assumption that Γ is open and maximal).
- $s = \dot{x}y$. By $\mathcal{R}_N$, $x\dot{=}y \in \tilde{\Gamma}$ and hence $x \sim_{\Gamma} y$.
- $s = \dot{\neg}\dot{x}y$. By $\mathcal{R}_{\bar{N}}$, $x\dot{\neq}y \in \tilde{\Gamma}$. Then there are some x' and y' such that $x' \sim_{\Gamma} x$, $y' \sim_{\Gamma} y$, and $x'\dot{\neq}y' \in \Gamma$. By $\mathcal{R}^{\perp}_{\dot{\neq}}$, we have $x' \not\sim_{\Gamma} y'$ (cf. $s = x\dot{\neq}y$). The claim follows by the transitivity of $\sim_{\Gamma}$.
- $s = [r]_n tx$. To show: $|\{y \mid rxy \in \tilde{\Gamma},\ ty \notin \tilde{\Gamma}\}/_{\sim_{\Gamma}}| \leq n$. This is clearly the case if $|\{y \mid rxy \in \tilde{\Gamma}\}| \leq n$. Otherwise, it suffices to show that for every $Y \subseteq \{y \mid rxy \in \tilde{\Gamma}\}$ such that $|Y| = n+1$, it either holds $|Y/_{\sim_{\Gamma}}| < |Y|$ or $ty \in \tilde{\Gamma}$ for some $y \in Y$. This follows by $\mathcal{R}_{\Box}$ since $y\dot{=}z \in \Gamma$ implies $y \sim_{\Gamma} z$ for all $y, z \in Y$.

The cases $s = (t_1 \dot{\vee} t_2)x$, $s = (t_1 \dot{\wedge} t_2)x$, and $s = \langle r\rangle_n tx$ are immediate by, respectively, $\mathcal{R}_{\dot{\vee}}$, $\mathcal{R}_{\dot{\wedge}}$, and $\mathcal{R}_{\Diamond}$. The cases $s = E_n tx$ and $s = A_n tx$ are proved analogously to $s = \langle r\rangle_n tx$ and, respectively, $s = [r]_n tx$. ☐

3.5. **Termination.** We will now show that every tableau derivation is finite. As usual, the main difficulty is bounding the number of applications of generative rules, in particular of $\mathcal{R}_{\Diamond}$. The present proof is notably more complex than the proofs in [23, 24] since now, an application of $\mathcal{R}_{\Diamond}$ does not necessarily expand a new pattern. Hence, we need to combine the pattern-counting argument from [23, 24] with a bound on the number of non-expanding applications of $\mathcal{R}_{\Diamond}$.

Since the rules $\mathcal{R}_{\dot{\vee}}$, $\mathcal{R}_{\Box}$, and $\mathcal{R}_A$ are all finitely branching, by König's lemma it suffices to show that the construction of every individual branch terminates. Since tableau rule application always produces proper extensions of branches, it then suffices to show that the size (i.e., cardinality) of an individual branch is bounded.

First, we show that the size of a branch Γ is bounded by a function in the number of nominals on Γ. Then, we show that this number itself is bounded from above, completing the termination proof.

We write $\Gamma \xrightarrow{\mathcal{R}} \Delta$ to denote that the branch Δ is obtained from Γ by the rule $\mathcal{R}$. We write $\Gamma \rightarrow \Delta$ if Δ is obtained from Γ by a single rule application. We write $\mathcal{S}\Gamma$ for the set of all modal expressions occurring on Γ, possibly as subterms of other expressions, and $\mathrm{Rel}\,\Gamma$ for the set of all roles that occur on Γ.

Crucial for the termination argument is the fact the tableau rules cannot introduce any modal expressions that do not already occur on the initial branch.

Proposition 3.9. *If Γ, Δ are branches such that Δ is obtained from Γ by any rule of $\mathcal{T}$, then $\mathcal{S}\Delta = \mathcal{S}\Gamma$.* ☐

For every pair of nominals x, y a branch Γ may contain an equation $x\dot{=}y$ or a disequation $x\dot{\neq}y$. For every pair x, y and every role r, Γ may contain an edge rxy. Moreover, for every expression $s \in \mathcal{S}\Gamma$, Γ may contain a formula sx. Hence, the size of Γ is bounded by $(2+|\mathrm{Rel}\,\Gamma|)\cdot|\mathcal{N}\Gamma|^2 + |\mathcal{S}\Gamma|\cdot|\mathcal{N}\Gamma|$. By Proposition 3.9, we know that $|\mathcal{S}\Gamma|$ and $|\mathrm{Rel}\,\Gamma|$ depend only on the initial branch. Clearly, $|\mathcal{S}\Gamma|$ and $|\mathrm{Rel}\,\Gamma|$ are bounded from above by the size of the input, i.e., the sum of the sizes of the initial formulas.

By the above, it suffices to show that $|\mathcal{N}\Gamma|$ is exponentially bounded in the size of the input. We do so by giving a bound on the number of applications of $\mathcal{R}_\Diamond$ and $\mathcal{R}_E$ that can occur in the derivation of a branch, which suffices since $\mathcal{R}_\Diamond$ and $\mathcal{R}_E$ are the only two rules that can introduce new nominals.

We begin by showing that $\mathcal{R}_E$ can be applied at most as many times as there are distinct modal expressions of the form $E_n s$ on the initial branch. For this purpose, we define a function ψ_E such that $\psi_E\Gamma := \{E_n s \in \mathcal{S}\Gamma \mid \exists x \in \mathcal{N}\Gamma :\ E_n sx \text{ not evident on } \Gamma\}$. Since $|\psi_E\Gamma|$ is bounded from below by 0, it suffices to show that the number decreases with every application of $\mathcal{R}_E$ (and is non-increasing otherwise, which is obvious).

Lemma 3.10. *Let s be of the form $\langle r\rangle_n tx$ or $E_n tx$. If s is evident on Γ and $\Gamma \subseteq \Delta$, then s is evident on Δ.* □

Proposition 3.11. $\Gamma \overset{\mathcal{R}_E}{\to} \Delta \implies |\psi_E\Gamma| > |\psi_E\Delta|$

Proof. Let $\Gamma \overset{\mathcal{R}_E}{\to} \Delta$. By Lemma 3.10, $\psi_E\Gamma \supseteq \psi_E\Delta$. Hence it suffices to show that $\psi_E\Gamma - \psi_E\Delta$ is non-empty. Let Δ be obtained from Γ by applying $\mathcal{R}_E$ to $s = E_n tx$. Then, by $\mathcal{R}_E$, $E_n t \in \psi_E\Gamma$. On the other hand, s is evident on Δ, and it is easy to see that the evidence of s implies the evidence of $E_n ty$ for every $y \in \mathcal{N}\Delta$. Hence $E_n t \notin \psi_E\Delta$. □

Now we show that $\mathcal{R}_\Diamond$ can be applied at most finitely often in a derivation. Since there are only finitely many roles, it suffices to show that $\mathcal{R}_\Diamond$ can be applied at most finitely often for each role. Observe that since $\mathcal{R}_\Diamond$ is only applicable to diamond formulas that are not quasi-evident, it holds:

Proposition 3.12. *If $\mathcal{R}_\Diamond$ is applicable to a formula $\langle r\rangle_n sx \in \Gamma$, then either*
(1) *x has an r-successor on Γ, or*
(2) *$P^r_\Gamma x$ is not expanded on Γ.* □

Let Γ and Δ be branches such that Δ is obtained from Γ by applying $\mathcal{R}_\Diamond$ to a formula $\langle r\rangle_n sx \in \Gamma$ such that $P^r_\Gamma x$ is not expanded on Γ. It is easy to see that $P^r_\Delta x$ must be expanded on Δ. Let us call such an application of $\mathcal{R}_\Diamond$ *pattern-expanding.*

Let $\mathrm{Pat}^{\,r}\Gamma := \mathcal{P}(\{\langle r\rangle_n s \in \mathcal{S}\Gamma\} \cup \{[r]_n s \in \mathcal{S}\Gamma\})$. In other words, $\mathrm{Pat}^{\,r}\Gamma$ contains all the possible sets of r-diamonds and r-boxes from $\mathcal{S}\Gamma$. Since $\Gamma \to \Delta$ implies $\tilde{\Gamma} \subseteq \tilde{\Delta}$, it holds:

Lemma 3.13. *Let $\Gamma \to \Delta$ and $P \in \mathrm{Pat}^{\,r}\Gamma$. If P is expanded on Γ, then P is expanded on Δ.* □

So, for each role r the derivation of a branch has at most $|\mathrm{Pat}^{\,r}\Gamma_0|$ pattern-expanding applications of $\mathcal{R}_\Diamond$, where Γ_0 is the initial branch. Clearly, $|\mathrm{Pat}^{\,r}\Gamma_0|$ is exponentially bounded in the size of the input.

Hence, it remains to show that a derivation can contain only finitely many applications of $\mathcal{R}_\Diamond$ assuming that none of the applications is pattern-expanding. We say a nominal x has a *successor* on Γ if x has an r-successor on Γ for any role r. A set of nominals X has a successor on Γ if there is some $x \in X$ that has a successor on Γ. We define

$$\psi^X_\Diamond\Gamma := |\{\langle r\rangle_n s \in \mathcal{S}\Gamma \mid \exists x \in X \colon \langle r\rangle_n sx \text{ not evident on } \Gamma\}|$$

and

$$\psi_\Diamond\Gamma := \sum_{\substack{X\in\mathcal{N}\Gamma/\sim_\Gamma \\ X \text{ has a successor on } \Gamma}} \psi_\Diamond^X\Gamma \ .$$

Lemma 3.14. *Let $X, Y \in \mathcal{N}\Gamma/\sim_\Gamma$, $x \in X$, $y \in Y$, and let $\Delta := \Gamma \cup \{x\dot{=}y\}$. Then $\psi_\Diamond^X\Gamma \geq \psi_\Diamond^{X\cup Y}\Delta$.* □

Proposition 3.15. *Let $\Gamma \to \Delta$ such that Δ is obtained from Γ by some rule application other than a pattern-expanding application of $\mathcal{R}_\Diamond$.*

(1) *If Δ is obtained from Γ by $\mathcal{R}_\Diamond$, then $\psi_\Diamond\Gamma > \psi_\Diamond\Delta$.*
(2) *Otherwise, $\psi_\Diamond\Gamma \geq \psi_\Diamond\Delta$.*

Proof.

(1) Clearly, nominals introduced by $\mathcal{R}_\Diamond$ are fresh and hence cannot have any successors on Δ. Hence $\psi_\Diamond\Gamma \geq \psi_\Diamond\Delta$. Therefore, it suffices to find a set $X \in \mathcal{N}\Gamma/\sim_\Gamma$ that has a successor on Γ, a nominal $x \in X$ and a formula $\langle r\rangle_n sx \in \Gamma$ that is not evident on Γ but is evident on Δ.

    Assume Δ is obtained from Γ by $\mathcal{R}_\Diamond$ applied to a formula $\langle r\rangle_n sx \in \Gamma$. Clearly, $\langle r\rangle_n sx$ is not evident on Γ but is evident on Δ. Since the rule application is not pattern-expanding, x has an r-successor on Γ. Hence there is some $X \in \mathcal{N}\Gamma/\sim_\Gamma$ such that $x \in X$ and X has a successor on Γ. The claim follows.
(2) Since cumulativity of tableau construction preserves the evidence of diamond formulas (Lemma 3.10), the only interesting rules are those modifying $\mathcal{N}\Gamma/\sim_\Gamma$. Nominals introduced by $\mathcal{R}_E$ are fresh and hence do not have any successors on Δ. Therefore, the only remaining cases are $\mathcal{R}_N$, $\mathcal{R}_\Box$ and $\mathcal{R}_A$. Clearly, none of the three rules can increase the cardinality of $\{X \in \mathcal{N}\Gamma/\sim_\Gamma \mid X \text{ has a successor on } \Gamma\}$. The claim follows by Lemma 3.14. □

This completes the termination proof. Since the cardinalities of the sets $\mathrm{Pat}^r\Gamma$ are exponentially bounded in the size n_0 of the input, $|\psi_E\Gamma|$ is polynomial in n_0, and $\psi_\Diamond\Gamma$ polynomial in $|\Gamma|$ and n_0, $|\mathcal{N}\Gamma|$ is exponentially bounded in n_0. Since $|\Gamma|$ is polynomial in $|\mathcal{N}\Gamma|$, we conclude that $|\Gamma|$ is at most exponential in n_0. By cumulativity, the construction of Γ terminates in at most exponentially many steps in n_0. This suffices to give us a NExpTime complexity bound for the decision procedure based on the calculus.

4. Adding Reflexivity, Transitivity and Role Inclusion

We now extend $\mathcal{T}$ to deal with reflexivity, transitivity and inclusion assertions. As in related work on description logic [16, 20, 18, 19, 17], we restrict our modal expressions to contain no graded boxes for roles that have transitive subroles.

We define $\subseteq^*_\Gamma$ as the smallest reflexive and transitive relation such that $r \subseteq^*_\Gamma r'$ whenever $r \sqsubseteq r' \in \Gamma$. A role r is called *simple* on a branch Γ (or just simple if Γ is clear from the context) if there is no r' such that $r' \subseteq^*_\Gamma r$ and $Tr' \in \Gamma$. Observe that all subroles of a simple role are in turn simple. Also, since our tableau rules will not introduce new inclusion assertions, a role r will be simple on a given branch Γ if and only if r is simple on the initial branch from which Γ is obtained.

Our branches may now contain inclusion, reflexivity and transitivity assertions:

$$s \ ::= \ tx \mid rxy \mid x\dot{=}y \mid x\dot{\neq}y \mid \perp \mid r \sqsubseteq r' \mid Rr \mid Tr$$

The modal expressions t in formulas of the form tx are restricted to contain no boxes $[r]_n s$ with $n > 0$ unless r is simple.

Following the ideas in [16, 18, 19, 17], we introduce the *induced transition relation* $\trianglerighteq^r_\Gamma$ to reason about accessibility in the presence of inclusion axioms. Intuitively, $x \trianglerighteq^r_\Gamma y$ means that in every model of Γ, y is accessible from x via r.

4.1. **Extending Evidence.** To account for the new types of formulas, we extend the evidence conditions as follows:

$$\begin{aligned} r \sqsubseteq r' \in \Gamma \;&\Rightarrow\; \forall x, y \in \mathcal{N}\Gamma :\; rxy \in \tilde{\Gamma} \Rightarrow r'xy \in \tilde{\Gamma} \\ Rr \in \Gamma \;&\Rightarrow\; \forall x \in \mathcal{N}\Gamma :\; rxx \in \tilde{\Gamma} \\ Tr \in \Gamma \;&\Rightarrow\; \forall x, y, z \in \mathcal{N}\Gamma :\; rxy \in \tilde{\Gamma} \wedge ryz \in \tilde{\Gamma} \Rightarrow rxz \in \tilde{\Gamma} \end{aligned}$$

It is easy to see that if Γ satisfies the extended evidence conditions, the interpretation $\mathcal{I}$ constructed in the proof of Theorem 3.1 will satisfy the new formulas. Hence, Theorem 3.1 adapts to the extended system.

Theorem 4.1 (Model Existence)**.** *Every evident branch has a finite model.* □

4.2. **Pre-evidence.** To account for the new evidence conditions, one could imagine the following rules.

$$\frac{r \sqsubseteq r',\ rxy}{r'xy} \qquad\qquad \frac{Rr}{rxx}\ x \in \mathcal{N}\Gamma \qquad\qquad \frac{Tr,\ rxy,\ ryz}{rxz}$$

In the presence of blocking, however, the rules are problematic. In particular, the rule for reflexivity renders the notion of quasi-evidence that we use for $\mathcal{T}$ ineffective to ensure termination. Once we add a reflexive edge rxx to a branch Γ, x will have an r-successor on Γ, meaning quasi-evidence will coincide with evidence for all r-diamonds on x. Similarly, the rule for transitivity is known to be incomplete in the presence of blocking [24].

We solve the problem by defining a weaker notion of evidence, called *pre-evidence*. To satisfy the pre-evidence conditions, we do not have to explicitly add reflexive or transitive edges during tableau construction. We will extend our tableau rules and the notion of quasi-evidence such that every open and maximal branch in the extended calculus can be completed to a pre-evident branch, which in turn can be made evident by adding the implicit edges.

We define the relation $\triangleright^r_\Gamma$ as the least relation such that:

$$\begin{aligned} rxy \in \tilde{\Gamma} \quad &\Rightarrow \quad x \triangleright^r_\Gamma y \\ r' \sqsubseteq r \in \Gamma,\ x \triangleright^{r'}_\Gamma y \quad &\Rightarrow \quad x \triangleright^r_\Gamma y \end{aligned}$$

The relation $\triangleright^r_\Gamma$ does not account for reflexivity. To do so, we extend it as follows:

$$\trianglerighteq^r_\Gamma \;:=\; \begin{cases} \triangleright^r_\Gamma \cup \{(x, y) \mid x, y \in \mathcal{N}\Gamma \wedge x \sim_\Gamma y\} & \text{if } \exists r' :\ r' \sqsubseteq^*_\Gamma r \wedge Rr' \in \Gamma \\ \triangleright^r_\Gamma & \text{otherwise} \end{cases}$$

The *pre-evidence conditions* are obtained from the evidence conditions by omitting the conditions for inclusion and reflexivity assertions and replacing the conditions for diamonds, boxes and transitivity assertions as follows:

$$\langle r\rangle_n tx \in \Gamma \;\Rightarrow\; \exists^{n+1} Y\colon\; D_\Gamma Y \;\wedge\; \forall y \in Y:\; x \trianglerighteq^r_\Gamma y \;\wedge\; ty \in \tilde{\Gamma}$$
$$[r]_n tx \in \Gamma \;\Rightarrow\; |\{y \,|\, x \trianglerighteq^r_\Gamma y,\; ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n$$
$$Tr \in \Gamma \;\Rightarrow\; \forall r', t, x, y:\; [r']_0 tx \in \tilde{\Gamma} \;\wedge\; r \subseteq^*_\Gamma r' \;\wedge\; x \triangleright^r_\Gamma y \;\Rightarrow\; [r]_0 ty \in \tilde{\Gamma}$$

Note that we do not need pre-evidence conditions for inclusion or reflexivity assertions as their semantics is taken care of by the way we define the relation $x \trianglerighteq^r_\Gamma y$. Pre-evidence of individual formulas is defined analogously to the corresponding notion of evidence.

We now show that every pre-evident branch can be extended to an evident branch. Let the *evidence closure* $\hat{\Gamma}$ of a branch Γ be defined as the least superset of Γ such that:

$$x \trianglerighteq^r_\Gamma y \;\Rightarrow\; rxy \in \hat{\Gamma}$$
$$Tr \in \Gamma \;\wedge\; rxy \in \hat{\Gamma} \;\wedge\; ryz \in \hat{\Gamma} \;\Rightarrow\; rxz \in \hat{\Gamma}$$
$$r \sqsubseteq r' \in \Gamma \;\wedge\; rxy \in \hat{\Gamma} \;\Rightarrow\; r'xy \in \hat{\Gamma}$$

Note that, by construction, we have $rxy \in \hat{\hat{\Gamma}} \iff rxy \in \hat{\Gamma}$.

Lemma 4.2. *Let Γ be a branch and r be simple on Γ. Then $x \trianglerighteq^r_\Gamma y \iff rxy \in \hat{\Gamma}$*

Proof. Let r be simple on Γ. The direction from left to right is immediate. The other direction can be shown by induction on the construction of $\hat{\Gamma}$ from Γ. ☐

Lemma 4.3. *Let Γ be a branch and let $rxy \in \hat{\Gamma}$. Then either $x \trianglerighteq^r_\Gamma y$, or there is some r' such that $\{r' \sqsubseteq r,\, Tr'\} \subseteq \Gamma$ and*

$$\exists\, n{\geq}2\, \exists x_1, \ldots, x_n\colon\; x_1 = x \wedge x_n = y \wedge \forall 1{\leq}i{<}n:\; x_i \triangleright^{r'}_\Gamma x_{i+1}\;\; .$$

Proof. By induction on the construction of $\hat{\Gamma}$. ☐

Theorem 4.4 (Evidence Completion)**.** *Γ pre-evident $\Longrightarrow$ $\hat{\Gamma}$ evident*

Proof. It is easy to see that $\hat{\Gamma}$ satisfies the evidence conditions for inclusion, reflexivity and transitivity assertions. The only remaining evidence conditions that may be affected by adding edges to Γ are the ones for diamonds and boxes. The rest of the evidence conditions is already satisfied by Γ and hence also holds on $\hat{\Gamma}$.

The evidence condition for diamonds holds on $\hat{\Gamma}$ since the corresponding pre-evidence condition holds on Γ and $x \trianglerighteq^r_\Gamma y$ implies $rxy \in \hat{\Gamma}$ for all nominals x, y and roles r.

It remains to show the evidence condition for boxes. Let $[r]_n sx \in \Gamma$ and $|\{y \,|\, x \trianglerighteq^r_\Gamma y,$ $sy \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n$. It suffices to show: $|\{y \,|\, rxy \in \hat{\Gamma},\; sy \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n$. We distinguish two cases. If r is simple, the claim follows by Lemma 4.2. Otherwise, we must have $n = 0$. Hence, it suffices to show that we have $sy \in \tilde{\Gamma}$ for every edge $rxy \in \hat{\Gamma}$. Let $rxy \in \hat{\Gamma}$. Then, by Lemma 4.3, two cases are possible. Either $x \trianglerighteq^r_\Gamma y$, in which case the claim follows by the pre-evidence condition for boxes, or there is a transitive subrole r' of r such that there are nominals $x_1, \ldots, x_m$ ($m \geq 2$) such that $x_1 = x$, $x_m = y$ and $x_i \triangleright^{r'}_\Gamma x_{i+1}$ for all $1 \leq i < n$. In this case, by induction on m one can show that the pre-evidence condition for transitivity assertions applied to r' and $[r]_n sx \in \Gamma$ implies either $[r]_n sx_{m-1} \in \Gamma$ (true by assumption for $m = 2$) or $[r']_n sx_{m-1} \in \tilde{\Gamma}$ (if $m > 2$). Either way, the claim follows by the pre-evidence condition for boxes. ☐

$$\mathcal{R}_\Box \; \frac{[r]_n tx}{\exists y, z \in Y,\; y \neq z\colon y \dot{=} z \mid \exists y \in Y : ty} \; Y \subseteq \{y \mid x \trianglerighteq^r_\Gamma y\},\; |Y| = |Y/_{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_T \; \frac{Tr,\; [r']_0 tx}{[r]_0 ty} \; r \sqsubseteq^*_\Gamma r',\; x \triangleright^r_\Gamma y$$

Figure 4: New rules for $\mathcal{T}_\sqsubseteq$

4.3. **Tableau Rules.** The tableau rules for the extended calculus $\mathcal{T}_\sqsubseteq$ in Fig. 4 replace the original rule $\mathcal{R}_\Box$ from Fig. 1 and add a new rule $\mathcal{R}_T$, which is necessary to achieve the pre-evidence condition for transitivity assertions. While the formulation of $\mathcal{R}_\Diamond$ remains unchanged, the rule will now have to use an adapted notion of quasi-evidence, which will be introduced in Sect. 4.4. For now, we assume $\mathcal{R}_\Diamond$ is formulated with the restriction "$\langle r\rangle_n tx$ not pre-evident on Γ" instead. Again, it is not hard to verify that the extended rules are sound.

4.4. **Control.** As it turns out, in the presence of role inclusion we have to modify the definition of patterns. It no longer suffices to consider patterns separately for each role. This is due to the fact that now, different roles may be constrained by inclusion assertions. Consider, for instance, the unsatisfiable branch

$$\Gamma := \{r \sqsubseteq r',\; \langle r\rangle_0 px,\; \langle r'\rangle_0 \dot{\neg} px,\; [r']_1(p \dot{\wedge} \dot{\neg} p)x,\; r'xy,\; \dot{\neg} py,\; \langle r\rangle_0 pz,\; rzu,\; pu\}$$

According to our previous notion of quasi-evidence, $\langle r\rangle_0 px$ is quasi-evident on Γ as x has no r-successor (even if we extend the set of successors to $\{y \mid x \triangleright^r_\Gamma y\}$) and $P^r_\Gamma x$ is expanded. Since the other two diamonds on Γ are evident, Γ is quasi-evident, witnessing the incompleteness of our previous definition of patterns.

Hence, we redefine the notion of a pattern as follows. Given a branch Γ, a *pattern* is a set of terms of the form μs, where $\mu \in \{\langle r\rangle_n, [r]_n \mid r \in \operatorname{Rel}\Gamma,\; n \in \mathbb{N}\}$. We write $P_\Gamma x$ for the largest pattern P such that $P \subseteq \{t \mid tx \in \tilde{\Gamma}\}$. We call $P_\Gamma x$ the pattern of x on Γ. A pattern P is *expanded on* Γ if there are nominals x, y and a role r such that $x \triangleright^r_\Gamma y$ and $P \subseteq P_\Gamma x$. In this case, we say that x *expands* P *on* Γ. Note that here we use the relation $\triangleright^r_\Gamma$ rather than $\trianglerighteq^r_\Gamma$. Otherwise, we would get the same problems with termination as outlined in Sect. 4.2.

A diamond formula $\langle r\rangle_n sx$ is *quasi-evident on* Γ if it is either pre-evident on Γ or x has no *successor* on Γ (i.e., there is no y and r such that $x \triangleright^r_\Gamma y$) and $P_\Gamma x$ is expanded on Γ. As before, we restrict the rule $\mathcal{R}_\Diamond$ such that it can only be applied to diamond formulas that are not quasi-evident, and call a branch Γ quasi-evident if it satisfies all of the pre-evidence conditions but the one for diamond formulas, which we again replace by

$$\langle r\rangle_n tx \in \Gamma \;\Rightarrow\; \langle r\rangle_n tx \text{ is quasi-evident on } \Gamma$$

but now with the adapted notion of quasi-evidence.

Example 4.5. Figure 5 shows a tableau derivation in $\mathcal{T}_\sqsubseteq$ resulting in a quasi-evident branch. As in Example 3.5, we write Γ_n for the branch up to line n. We observe:

- Since r is reflexive and $r \sqsubseteq r' \in \Gamma_0$, r' is also reflexive. Consequently, we have $x \trianglerighteq^{r'}_{\Gamma_0} x$, which explains why $\mathcal{R}_\Box$ applies to $[r']_0\langle r\rangle_0 px \in \Gamma_0$.

0.	$r \sqsubseteq r',\ Rr,\ Tr',\ [r']_0\langle r\rangle_0 px,\ \langle r'\rangle_0 qx$	
1.	$\langle r\rangle_0 px$	$\mathcal{R}_\Box$
2.	$rxy,\ py$	$\mathcal{R}_\Diamond$
3.	$r'xz,\ qz$	$\mathcal{R}_\Diamond$
4.	$[r']_0\langle r\rangle_0 pz$	$\mathcal{R}_T$
5.	$\langle r\rangle_0 pz$	$\mathcal{R}_\Box$
6.	$rzu,\ pu$	$\mathcal{R}_\Diamond$

Figure 5: Tableau derivation for $\{r \sqsubseteq r',\ Rr,\ Tr',\ [r']_0\langle r\rangle_0 px,\ \langle r'\rangle_0 qx\}$

- The rule $\mathcal{R}_T$ propagates $[r']_0\langle r\rangle_0 p$ to z but not to y since r is not (necessarily) transitive.
- In $\mathcal{T}$, $\langle r\rangle_0 pz \in \Gamma_5$ would be quasi-evident since $P^r_{\Gamma_5}x = P^r_{\Gamma_5}z$. In $\mathcal{T}_\sqsubseteq$, however, $\mathcal{R}_\Diamond$ applies to $\langle r\rangle_0 pz$ since $P_{\Gamma_5}x = \{\langle r\rangle_0 p,\ \langle r'\rangle_0 q,\ [r']_0\langle r\rangle_0 p\} \neq \{\langle r\rangle_0 p,\ [r']_0\langle r\rangle_0 p\} = P_{\Gamma_5}z$.

Lemma 4.6. *Let Γ, Δ be branches such that $\{r \sqsubseteq r' \mid r \sqsubseteq r' \in \Gamma\} = \{r \sqsubseteq r' \mid r \sqsubseteq r' \in \Delta\}$. Let x, y, u, v be nominals such that $\{r \mid rxy \in \tilde{\Gamma}\} = \{r \mid ruv \in \tilde{\Delta}\}$. Then, for all r, $x \rhd^r_\Gamma y \ \Leftrightarrow\ u \rhd^r_\Delta v$.*

Proof. Let Γ, Δ, x, y, u and v be as required. Let r be a role. We show $x \rhd^r_\Gamma y \ \Rightarrow\ u \rhd^r_\Delta v$ by induction on the derivation of $x \rhd^r_\Gamma y$. The other direction follows analogously by induction on the derivation of $u \rhd^r_\Delta v$. Assume $x \rhd^r_\Gamma y$. We distinguish two cases:

- $rxy \in \tilde{\Gamma}$. Then, by assumption, $ruv \in \tilde{\Delta}$, and so $u \rhd^r_\Delta v$.
- There is some r' such that $r' \sqsubseteq r \in \Gamma$ and $x \rhd^{r'}_\Gamma y$. By the inductive hypothesis, we have $u \rhd^{r'}_\Delta v$. Moreover, by assumption, $r' \sqsubseteq r \in \Delta$. Hence, $u \rhd^r_\Delta v$. □

Lemma 4.7. *Let Γ be a quasi-evident branch and let $\langle r\rangle_n sx$ be not pre-evident on Γ. Let y expand $P_\Gamma x$ on Γ and let $\Delta := \Gamma \cup \{r'xz \mid r'yz \in \tilde{\Gamma}\}$. Then:*

(1) $\forall r', z:\ x \rhd^{r'}_\Delta z \iff y \rhd^{r'}_\Gamma z$ *and* $x \unrhd^{r'}_\Delta z \iff y \unrhd^{r'}_\Gamma z$,
(2) $\forall r', m, t:\ \langle r'\rangle_m t \in P_\Gamma x \implies \langle r'\rangle_m tx$ *pre-evident on* Δ,
(3) $\langle r\rangle_n sx$ *pre-evident on* Δ,
(4) $\forall r', m, t, z:\ \langle r'\rangle_m tz$ *pre-evident on* $\Gamma \implies \langle r'\rangle_m tz$ *pre-evident on* Δ,
(5) Δ *quasi-evident.*

Proof. We begin with (1). Let r' be a role and z a nominal. We will only show the first equivalence since the other claim easily follows. Since $\langle r\rangle_n sx$ is quasi-evident but not evident on Γ, x has no successor on Γ. Hence, by construction, $\{r' \mid r'xz \in \tilde{\Delta}\} = \{r' \mid r'yz \in \tilde{\Gamma}\}$. The claim follows by Lemma 4.6.

Claims (2–4) are shown analogously to the corresponding claims of Lemma 3.6.

Now to (5). The only conditions that might in principle be violated in Δ are the quasi-evidence condition for diamonds of the form $\langle r'\rangle_m tz \in \Delta$ where $z \sim_\Delta x$, the evidence condition for boxes $[r']_m tz \in \Delta$ where $z \sim_\Delta x$, and the evidence condition for transitivity assertions $Tr' \in \Delta$.

For diamonds of the above form, the quasi-evidence condition holds by (2).

For transitivity assertions, it suffices to show that for every r_1, r_2 such that $Tr_1 \in \Gamma$, $r_1 \subseteq^*_\Gamma r_2$, and $[r_2]_0 tx \in \tilde{\Gamma}$, and for all z such that $x \rhd^{r_1}_\Delta z$, it holds $[r_1]_0 tz \in \tilde{\Gamma}$. Since

$P_\Gamma y \supseteq P_\Gamma x$, we have $[r_2]_0 ty \in \tilde{\Gamma}$. The claim now follows by (1) and the quasi-evidence condition for $Tr_1 \in \Gamma$.

The claim for boxes follows analogously (we exploit $P_\Gamma y \supseteq P_\Gamma x$ and (1)). $\square$

Theorem 4.8 (Pre-evidence Completion)**.** *For every quasi-evident branch Γ there is a pre-evident branch Δ such that $\Gamma \subseteq \Delta$.*

Proof. Proceeds analogously to the proof of Theorem 3.7 with Lemma 4.7 in place of Lemma 3.6. $\square$

Theorem 4.9 (Quasi-evidence)**.** *Every open and maximal branch in $\mathcal{T}_\sqsubseteq$ is quasi-evident.*

Proof. Proceeds analogously to the proof of Theorem 3.8. The additional case for transitivity assertions is straightforward. $\square$

4.5. **Termination.** The termination proof for $\mathcal{T}_\sqsubseteq$ proceeds analogously to the proof for $\mathcal{T}$. Let us sketch what needs to be adapted. Because of the rule $\mathcal{R}_T$, the set $\mathcal{S}\Gamma$ of modal expressions occurring on Γ needs to be extended as follows: $\mathcal{S}'\Gamma := \mathcal{S}\Gamma \cup \{[r]_0 s \mid r \subseteq^*_\Gamma r' \wedge [r']_0 s \in \mathcal{S}\Gamma\}$. With the extended definition of $\mathcal{S}$, Proposition 3.9 holds for $\mathcal{T}_\sqsubseteq$. Lemma 3.10 is modified as follows:

Lemma 4.10. *Let s be of the form $\langle r\rangle_n tx$ or $E_n tx$. If s is (pre-)evident on Γ and $\Gamma \subseteq \Delta$, then s is (pre-)evident on Δ.* $\square$

Proposition 3.11 is unaffected by the extensions to the calculus. Proposition 3.12 is adapted as follows:

Proposition 4.11. *If $\mathcal{R}_\Diamond$ is applicable to a formula $\langle r\rangle_n sx \in \Gamma$, then either*
(1) *x has a successor on Γ, or*
(2) *$P_\Gamma x$ is not expanded on Γ.* $\square$

Also, analogously to Lemma 3.13, the expandedness of our extended patterns is preserved by tableau rule application. Lemma 3.14 and Proposition 3.15 remain valid if we redefine

$$\psi^X_\Diamond \Gamma := |\{\langle r\rangle_n s \in \mathcal{S}'\Gamma \mid \exists x \in X\colon \langle r\rangle_n sx \text{ not pre-evident on } \Gamma\}|$$

and $\psi_\Diamond\Gamma$ accordingly, with the modified definition of a successor.

5. Conclusion

We have presented a terminating tableau calculus for graded hybrid logic with global modalities and role hierarchies. Following [8, 7, 24], our calculus is cumulative, representing state equality abstractly via an equivalence relation (declarative approach). The existing calculi for equivalent and stronger logics [18, 19, 17] work on possibly cyclic graph structures and treat equality by destructive graph transformation during tableau construction (procedural approach). The procedural approach encompasses algorithmic decisions that are not present in the more abstract declarative approach. From a declarative calculus we can always obtain a procedural system by refinement.

Exploiting an extended pattern-based blocking technique and the cumulativity of our calculus, we have proved a NExpTime complexity bound for the associated decision procedure. To ensure termination of pattern-based blocking in the presence of reflexivity, we differentiated between the induced transition relation $\trianglerighteq_\Gamma^r$ and its non-reflexive counterpart $\triangleright_\Gamma^r$. The implementation of pattern-based blocking for a hybrid language with global modalities [15] reveals its considerable practical potential. We consider it a promising project to implement the extended version of pattern-based blocking presented in this paper and compare its performance to that of established blocking techniques.

Following related work [16, 20, 18, 19, 17], we restrict the language decided by our calculus to contain no graded boxes on complex roles. As shown by Horrocks, Sattler and Tobies [20], this restriction is essential for decidability of logics extending $\mathcal{SHIN}$. In the absence of inverse roles ($\mathcal{I}$), however, the restriction of graded boxes to simple roles can be significantly relaxed [26]. In [25], we give a terminating tableau calculus for $\mathcal{SOQ}$ extended by graded boxes on transitive roles. The logic extends the decidable fragment of [26] by nominals but lacks inclusion assertions that are allowed (with some restrictions) in [26]. It remains an open problem to design an efficient tableau calculus for the full decidable fragment of [26]. Also, it is still open if the fragment of [26] remains decidable when extended by nominals.

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