
ON THE CORRESPONDENCE BETWEEN DISPLAY POSTULATES AND DEEP INFERENCE IN NESTED SEQUENT CALCULI FOR TENSE LOGICS*

RAJEEV GORÉ, LINDA POSTNIECE, AND ALWEN TIU

Logic and Computation Group, College of Engineering and Computer Science, The Australian
National University

e-mail address: {Rajeev.Gore,Alwen.Tiu}@anu.edu.au, linda.postniece@gmail.com

ABSTRACT. We consider two styles of proof calculi for a family of tense logics, presented in a formalism based on nested sequents. A nested sequent can be seen as a tree of traditional single-sided sequents. Our first style of calculi is what we call “shallow calculi”, where inference rules are only applied at the root node in a nested sequent. Our shallow calculi are extensions of Kashima’s calculus for tense logic and share an essential characteristic with display calculi, namely, the presence of structural rules called “display postulates”. Shallow calculi enjoy a simple cut elimination procedure, but are unsuitable for proof search due to the presence of display postulates and other structural rules. The second style of calculi uses deep-inference, whereby inference rules can be applied at any node in a nested sequent. We show that, for a range of extensions of tense logic, the two styles of calculi are equivalent, and there is a natural proof theoretic correspondence between display postulates and deep inference. The deep inference calculi enjoy the subformula property and have no display postulates or other structural rules, making them a better framework for proof search.

1. INTRODUCTION

A nested sequent is essentially a tree whose nodes are traditional sequents. It has been used as the syntactic judgment for proof calculi for several tense and modal logics [17, 4, 23, 6], perhaps due to the fact that the tree structure embodies, to some extent, the underlying Kripke frames in those logics. In our setting, the nodes in a nested sequent are traditional single-sided sequents (i.e., multisets of formulae), and the edges connecting the nodes are labelled either with \circ or \bullet (these labels correspond to the modal operator \Box and the tense operator \blacksquare). For example, the trees shown in Figure 1 are a tree representation of nested sequents, where each Γ_i is a multiset of formulae.

There are two natural styles of formalising inference rules on nested sequents. The first is one that conforms with the tradition of sequent calculi, namely, to allow inference rules to act only on formulae or structures that appear at the root sequent. We shall

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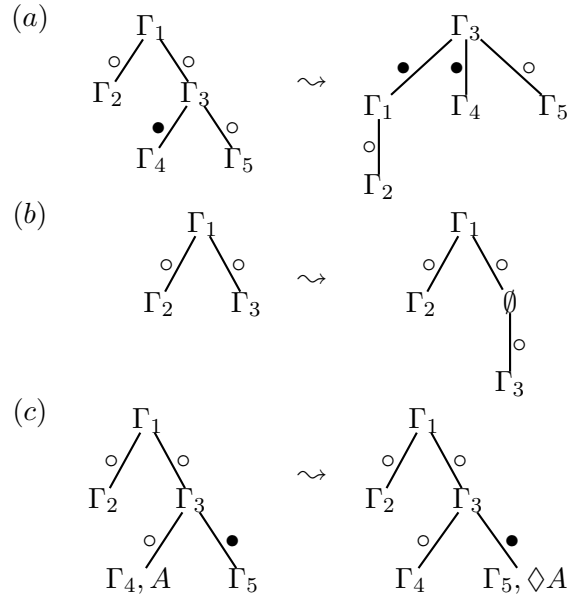


Figure 1: Inference rules seen as operations on trees

refer to this style of inference as *shallow inference*. The second style is to allow inference rules to act on formulae or structures in an arbitrary node in the tree; we call this *deep inference*. Kashima’s work [17] includes inference systems of both kinds. More specifically, Kashima presents two proof systems for tense logic, a shallow proof system **SKt** and a deep-inference system **S2Kt**, and proves, via semantical methods, that they are equivalent. In this paper, we investigate, via proof theoretic methods, the connection between shallow and deep inference systems for a wide range of tense logics extending Kashima’s **SKt** and **S2Kt**.

The primary motivation of our work actually stems from the problem of structuring proof search for display calculi [3]; more specifically Kracht’s formulation of display calculi for extensions of tense logics [18]. We have yet to tackle the proof search problem for Kracht’s display calculi in their full generality. What we show here is that in a more restricted setting of nested sequent calculi, which can be seen as a restricted form of display calculi, one main impediment to proof search, i.e., unrestricted use of structural rules, can be eliminated. In particular, we aim for a uniform design methodology for deep inference calculi *without structural rules*. This design choice sets us apart from similar work by Brünnler and Straßburger [7], where structural rules in the deep inference systems are actually desirable, out of the consideration for modularity (see also the discussion in Section 8).

Display postulates and other structural rules. Kashima’s shallow calculus **SKt** shares an essential feature with Kracht’s display calculi, namely, the presence of the so-called *display postulates* (called the *turn* rules in [17]). Seen as an operation on trees, the display postulates are a rotation operation on trees, allowing one to bring an arbitrary node in a tree to the root, e.g., the transformation shown in Figure 1(a) “displays” the sequent Γ_3 .

An interesting result in Kracht’s work [18] is that one can construct display calculi for extensions of tense logic *modularly*. That is, for every axiom in a certain form called

primitive form, one can construct a structural rule that captures exactly that axiom. Due to the similarity between display calculi and our shallow calculi, Kracht’s approach can be adapted to our setting as well to design modular shallow calculi. Seen as operations on trees, structural rules induced by axioms may involve addition or removal of nodes in the trees, e.g., the transitivity axiom, $\Box A \rightarrow \Box\Box A$, translates to the operation shown in Figure 1(b). In addition to these structural rules, our shallow calculi (and Kracht’s display calculi) also contain contraction and weakening rules, which allow duplication and removal of arbitrary subtrees. A combination of all these structural rules presents a complication in using display calculi or shallow nested-sequent calculi as a framework to structure proof search.

Deep inference and propagation rules. The role of display postulates is really to move a sequent to the root of a nested sequent so that an inference rule may be applied to it. Therefore a natural way to eliminate display postulates is to just allow inference rules to be applied deeply, as already shown by Kashima in his proof of the correspondence between **SKt** and **S2Kt** [17]. However, for extensions of tense logics, deep inference alone is not enough. For example, in the extension with the transitivity axiom, the problem is not so much that one cannot apply rules deeply. Rather, it is more to do with the fact that we extend the tree of sequents with extra nodes. To eliminate the structural rules for transitivity, we need to somehow build in transitivity into *logical rules*. We do this systematically via the so-called *propagation rules*. More specifically, the introduction rules for \Diamond -formulae (and its tense counterparts), reading the rules bottom up, allow propagations of the formulae along certain paths in the nested sequent. As an illustration, consider the instance of a propagation rule needed to absorb the transitivity axiom given in Figure 1(c), where a formula A in one node (where Γ_4 resides) is propagated to another node (where Γ_5 resides). We defer the justification for this rule to Section 6; for now, we just note that one can introduce a \Diamond -prefixed formula across different nodes at arbitrary depth in the tree, not just the top node.

Summary of results. Our main contributions are the following:

- We give a uniform syntactic cut elimination procedure for extensions of **SKt** with what we call *linear structural rules* (Section 3). Our procedure is very similar to Belnap’s general cut elimination for display logics, as it relies on the existence of the “display property” for our shallow calculi. It can be seen as an adaptation of Kracht’s cut elimination for display calculi for tense logics [18] to the setting of nested sequent calculi. Existing works on syntactic cut elimination for nested sequent calculi address only the modal fragment (in the deep inference setting) and for a limited number of extensions, e.g., [4, 5, 7, 23], or only for some extensions of tense logic without negation or implication [25].
- We show that for two classes of axioms, the Scott-Lemmon axioms [20] and *path axioms* (Section 6), the axioms can be modularly turned into linear structural rules, and hence cut admissibility for the shallow systems for these extensions follows from our uniform cut elimination. These two classes of axioms cover most of standard normal modal axioms in the literature, e.g., reflexivity, transitivity, euclideaness, convergence, seriality, etc.
- We give a syntactic proof of the equivalence of **SKt** and **S2Kt** (which we call **DKt** here). Kashima gave a proof of this correspondence via a semantic argument [17]. We further show that, for some extensions of **SKt** with Scott-Lemmon axioms, one can get the corresponding deep inference systems extending **DKt**, *without structural rules*, but

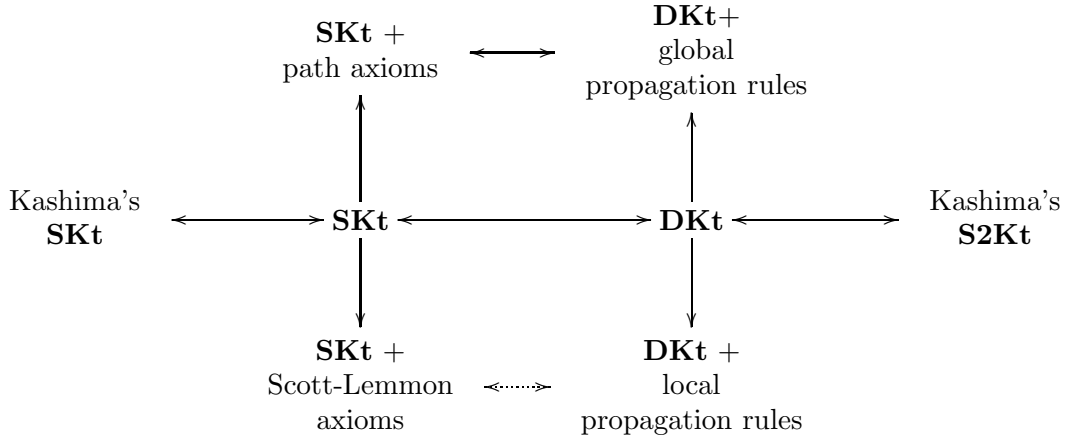


Figure 2: Relationships between proof systems

with *local* propagation rules (Section 5). For the extensions with path axioms, we show how one can derive systematically the corresponding deep inference calculi, also without structural rules, but with *global* propagation rules. By local propagation rules, we mean propagation rules in which formulae may be propagated only along a path of bounded length, whereas global propagation rules do not restrict the length of the path.

- We show that all our deep inference calculi for tense logics enjoy the *separation property*: if one restricts the calculi to their modal fragments, i.e., by omitting rules that mention tense operators, then one gets complete calculi for the modal parts of the tense logics.

The relationships between various proof systems in this paper are summarised in Figure 2. The direction of an arrow denotes inclusion, e.g., Kashima’s **S2Kt** is equivalent to **DKt**. The dashed arrow in the lowest row denotes the fact that the equivalence is only established for some, but not all, Scott-Lemmon axioms. We have not yet explored the connections between path axioms and Scott-Lemmon axioms.

Outline of the paper. Section 2 gives an overview of the syntax and the semantics of tense logic. Section 3 presents the shallow calculus **SKt** and a uniform syntactic cut elimination proof for any extension of **SKt** with linear structural rules. Section 4 presents a deep inference calculus **DKt**, which is similar to Kashima’s **S2Kt**, but without structural rules. We prove that **SKt** and **DKt** are cut-free equivalent, i.e., any cut-free proof in **SKt** can be transformed into a cut-free proof in **DKt** and vice versa. Section 5 presents extensions of **SKt** with Scott-Lemmon axioms. We show that for some extensions, one can design deep inference calculi based on **DKt** extended with some local propagation rules. Section 6 considers extensions of **SKt** with path axioms. We show how these axioms can be captured using global propagation rules in deep inference. We show further that applicability of propagation rules is decidable, by mapping the decision problem into the problem of non-emptiness checking of the intersection of a context-free language and a regular language. Section 7 gives some preliminary results in proof search for **DKt**. Section 8 concludes the paper and discusses related and future work.

This paper is a revised and extended version of an extended abstract presented at the TABLEAUX 2009 conference [13]. We have added the following new material: a uniform

$$\begin{array}{ll}
 (1) & A \rightarrow \Box \blacklozenge A & \overline{A} \vee \Box \blacklozenge A \\
 (2) & A \rightarrow \blacksquare \blacklozenge A & \overline{A} \vee \blacksquare \blacklozenge A \\
 (3) & \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) & \blacklozenge(A \wedge \overline{B}) \vee \blacklozenge \overline{A} \vee \Box B \\
 (4) & \blacksquare(A \rightarrow B) \rightarrow (\blacksquare A \rightarrow \blacksquare B) & \blacklozenge(A \wedge \overline{B}) \vee \blacklozenge \overline{A} \vee \blacksquare B.
 \end{array}$$

Figure 3: Axioms of minimal tense logic. Their nnf are shown on the right hand side.

$$\begin{array}{llll}
 w \Vdash \neg A & \text{iff} & w \not\Vdash A & \\
 w \Vdash A \vee B & \text{iff} & w \Vdash A \text{ or } w \Vdash B & \\
 w \Vdash \Box A & \text{iff} & \forall u. \text{ if } wRu \text{ then } u \Vdash A & \\
 w \Vdash \blacksquare A & \text{iff} & \forall u. \text{ if } uRw \text{ then } u \Vdash A & \\
 w \Vdash A \wedge B & \text{iff} & w \Vdash A \text{ and } w \Vdash B & \\
 w \Vdash \blacklozenge A & \text{iff} & \exists u. wRu \text{ and } u \Vdash A & \\
 w \Vdash \blacklozenge A & \text{iff} & \exists u. uRw \text{ and } u \Vdash A &
 \end{array}$$

Figure 4: Forcing of formulae

cut elimination proof for extensions of **SKt** with linear structural rules, extensions of **SKt** with Scott-Lemmon axioms, a new extension of **DKt**, and a new section (Section 6) on path axioms. However, we have removed the material on proof search for *KtS4* in the conference version, as we have recently discovered that the proof search algorithm outlined in that paper is unsound, although the calculi themselves are sound and complete. We defer the complete treatment of proof search for extensions of **DKt** to future work.

2. TENSE LOGIC

To simplify presentation, we shall consider formulae of tense logic *Kt* which are in negation normal form (nnf), given by the following grammar:

$$A := a \mid \neg a \mid A \vee A \mid A \wedge A \mid \Box A \mid \blacksquare A \mid \blacklozenge A \mid \blacklozenge A$$

where a ranges over atomic formulae and $\neg a$ is the negation of a . We shall denote with \overline{A} the nnf of the negation of A . Implication can then be defined via negation: $A \rightarrow B = \overline{A} \vee B$. The axioms of minimal tense logic *Kt* are all the axioms of propositional logic, plus the axioms in Figure 3.

The theorems of *Kt* are those that are generated from the above axioms and their substitution instances using the following rules:

$$\frac{A \quad \overline{A} \vee B}{B} MP \quad \frac{A}{\Box A} Nec\Box \quad \frac{A}{\blacksquare A} Nec\blacksquare$$

A *Kt-frame* is a pair $\langle W, R \rangle$, with W a non-empty set (of worlds) and $R \subseteq W \times W$. A *Kt-model* is a triple $\langle W, R, V \rangle$, with $\langle W, R \rangle$ a *Kt frame* and $V : Atm \rightarrow 2^W$ a valuation mapping each atom to the set of worlds where it is true.

For a world $w \in W$ and an atom $a \in Atm$, if $w \in V(a)$ then we write $w \Vdash a$ and say a is forced at w ; otherwise we write $w \not\Vdash a$ and say a is rejected at w . Forcing and rejection of compound formulae is defined by mutual recursion in Figure 4. A *Kt-formula* A is valid iff it is forced by all worlds in all models, i.e. iff $w \Vdash A$ for all $\langle W, R, V \rangle$ and for all $w \in W$.

3. SYSTEM **SKt**: A “SHALLOW” CALCULUS

We consider a right-sided proof system for tense logic where the syntactic judgment is a tree of multisets of formulae, called a *nested sequent*. Nested sequents have been used previously in proof systems for modal and tense logics [17, 4, 23].

Definition 3.1. A *nested sequent* is a multiset

$$\{A_1, \dots, A_k, \circ\{\Gamma_1\}, \dots, \circ\{\Gamma_m\}, \bullet\{\Delta_1\}, \dots, \bullet\{\Delta_n\}\}$$

where $k, m, n \geq 0$, and each Γ_i and each Δ_j are themselves nested sequents.

We shall use the following notational conventions when writing nested sequents. We shall remove outermost braces, e.g., we write A, B, C instead of $\{A, B, C\}$. Braces for sequents nested inside $\circ\{\}$ or $\bullet\{\}$ are also removed, e.g., instead of writing $\circ\{\{A, B, C\}\}$, we write $\circ\{A, B, C\}$. The empty (nested) sequent is denoted by \emptyset . When we juxtapose two sequents as in Γ, Δ we mean a sequent resulting from the multiset-union of Γ and Δ . When Δ is a singleton multiset, e.g., $\{A\}$ or $\{\circ\{\Delta'\}\}$, we simply write: Γ, A or $\Gamma, \circ\{\Delta'\}$. Since we shall only be concerned with nested sequents, we shall refer to nested sequents simply as sequents in the rest of the paper.

The above definition of sequents can also be seen as a special case of *structures* in display calculi, e.g., with ‘,’ (comma), \bullet and \circ as structural connectives [12].

A *context* is a sequent with holes in place of formulae. A context with a single hole is written as $\Sigma[]$. Multiple-hole contexts are written as $\Sigma[] \cdots []$, or abbreviated as $\Sigma^k[]$ where k is the number of holes. We write $\Sigma^k[\Delta]$ to denote the sequent that results from filling the holes in $\Sigma^k[]$ uniformly with Δ .

Given a proof system **S**, a *derivation in S* is defined as usual, i.e., as a tree whose nodes are nested sequents such that every node is the conclusion of an inference rule in **S**, and all its child nodes are exactly the premises of the same rule. An *open derivation in S* may additionally contain one or more leaf nodes, called *open leaf nodes*, which are not conclusions of any rules in **S**. We say that a sequent Γ is *derivable from Δ in S* if there is an open derivation of Γ whose open leaf nodes are Δ .

The shallow proof system for *Kt*, called **SKt**, is given in Figure 5. Note that the *id*-rule is restricted to the atomic form, but it is easy to show that the general *id* rule on arbitrary formulae is admissible. **SKt** is basically Kashima’s system for tense logic (also called **SKt**) [17], but with a more general contraction rule (*ctr*), which allows contraction of arbitrary sequents. The general contraction rule is used to simplify our cut elimination proof, and as we shall see in Section 4, it can be replaced by formula contraction. System **SKt** can also be seen as a single-sided version of a display calculus. The rules *rp* and *rf* are called the *residuation rules* [12]. They are an example of *display postulates* commonly found in display calculi, and are used to bring a node in a nested sequent to the top level. The following is an analog of the display property of display calculus. Its proof is straightforward by induction on the size of contexts.

Proposition 3.2. *Let $\Sigma[\Delta]$ be a sequent. Then there exists a sequent Γ such that $\Sigma[\Delta]$ is derivable from Δ, Γ and vice versa, using only the rules *rp* and *rf*.*

3.1. Soundness and completeness. To prove soundness, we first show that each sequent has a corresponding *Kt*-formula, and then show that the rules of **SKt**, reading them top down, preserve validity of the formula corresponding to the premise sequent. Completeness

$$\begin{array}{cccc}
\frac{}{\Gamma, a, \bar{a}} \textit{id} & \frac{\Gamma, A \quad \Delta, \bar{A}}{\Gamma, \Delta} \textit{cut} & \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \wedge & \frac{\Gamma, A, B}{\Gamma, A \vee B} \vee \\
\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta} \textit{ctr} & \frac{\Gamma}{\Gamma, \Delta} \textit{wk} & \frac{\Gamma, \circ\{\Delta\}}{\bullet\{\Gamma\}, \Delta} \textit{rf} & \frac{\Gamma, \bullet\{\Delta\}}{\circ\{\Gamma\}, \Delta} \textit{rp} \\
\frac{\Gamma, \bullet\{A\}}{\Gamma, \blacksquare A} \blacksquare & \frac{\Gamma, \circ\{A\}}{\Gamma, \square A} \square & \frac{\Gamma, \bullet\{\Delta, A\}}{\Gamma, \bullet\{\Delta\}, \blacklozenge A} \blacklozenge & \frac{\Gamma, \circ\{\Delta, A\}}{\Gamma, \circ\{\Delta\}, \diamond A} \diamond
\end{array}$$

Figure 5: System **SKt**

is shown by simulating the Hilbert system for tense logic in **SKt**. The translation from sequents to formulae are given below. In the translation, we assume two logical constants \perp ('false') and \top ('true'). This is just a notational convenience, as the constants can be defined in a standard way, e.g., as $a \wedge \bar{a}$ and $a \vee \bar{a}$ for some fixed atomic proposition a . As usual, the empty disjunction denotes \perp and the empty conjunction denotes \top .

Definition 3.3. The function τ translates an **SKt**-sequent

$$\{A_1, \dots, A_k, \circ\{\Gamma_1\}, \dots, \circ\{\Gamma_m\}, \bullet\{\Delta_1\}, \dots, \bullet\{\Delta_n\}\}$$

into the *Kt*-formula (modulo associativity and commutativity of \vee and \wedge):

$$A_1 \vee \dots \vee A_k \vee \square\tau(\Gamma_1) \vee \dots \vee \square\tau(\Gamma_m) \vee \blacksquare\tau(\Delta_1) \vee \dots \vee \blacksquare\tau(\Delta_n).$$

Lemma 3.4 (Soundness). *Every **SKt**-derivable *Kt* formula is valid.*

Proof. We show that for every rule ρ of **SKt**

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma} \rho$$

the following holds: if for every $i \in \{1, \dots, n\}$, the formula $\tau(\Gamma_i)$ is valid then the formula $\tau(\Gamma)$ is valid.

Since the formula-translation $\tau(\Gamma) \vee a \vee \bar{a}$ of the *id* rule is obviously valid, it then follows that every formula derivable in **SKt** is also valid. We show the soundness of *rf* here; the others are similar or easier: We want to show that if $\tau(\Gamma) \vee \square(\tau(\Delta))$ is valid then $\blacksquare(\tau(\Gamma)) \vee \tau(\Delta)$ is valid. We prove this by contradiction. Suppose $\tau(\Gamma) \vee \square(\tau(\Delta))$ is valid but $\blacksquare(\tau(\Gamma)) \vee \tau(\Delta)$ is not, so there is a model $\langle W, R, V \rangle$ and a world $w \in W$ such that $w \not\models \blacksquare(\tau(\Gamma)) \vee \tau(\Delta)$, which means

$$w \not\models \blacksquare(\tau(\Gamma)) \text{ and } w \not\models \tau(\Delta). \quad (3.1)$$

Since $w \not\models \blacksquare(\tau(\Gamma))$, there must be a world $v \in W$ such that vRw and $v \not\models \tau(\Gamma)$. Now since $\tau(\Gamma) \vee \square(\tau(\Delta))$ is valid, we have $v \models \tau(\Gamma)$ or $v \models \square(\tau(\Delta))$. But because $v \not\models \tau(\Gamma)$, it follows that $v \models \square(\tau(\Delta))$. Since vRw , by definition, we have $w \models \tau(\Delta)$, which contradicts our assumption above in (3.1). \square

$$\begin{array}{c}
\frac{\frac{\frac{\Pi_1}{\Gamma, \bullet\{A\}}}{\circ\{\Gamma\}, A} \quad rf \quad \frac{\frac{\Pi_2}{\circ\{\overline{A}\}, \Delta}}{\overline{A}, \bullet\{\Delta\}} \quad rp}{\circ\{\Gamma\}, \bullet\{\Delta\}} \quad cut \\
(1)
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{\frac{\circ\{\Gamma'\}, A_1 \quad \circ\{\Gamma'\}, A_2}{\circ\{\Gamma'\}, A_1 \wedge A_2} \quad rf \quad \frac{\frac{\overline{A}_1, \overline{A}_2, \bullet\{\Delta'\}}{\overline{A}_1 \vee \overline{A}_2, \bullet\{\Delta'\}} \quad \vee}{\circ\{\overline{A}_1 \vee \overline{A}_2\}, \Delta'} \quad rp}{\Gamma, \bullet\{A_1 \wedge A_2\}} \quad \wedge \quad \frac{\frac{\frac{\overline{A}_1, \overline{A}_2, \bullet\{\Delta'\}}{\overline{A}_1 \vee \overline{A}_2, \bullet\{\Delta'\}} \quad \vee}{\circ\{\overline{A}_1 \vee \overline{A}_2\}, \Delta'} \quad rp}{\circ\{\overline{A}_1 \vee \overline{A}_2\}, \Delta} \quad \vee \\
(2) \qquad \qquad \qquad (3)
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\frac{\frac{\circ\{\circ\{\Gamma'\}\}, \Delta}{\circ\{\Gamma'\}, \bullet\{\Delta\}} \quad rf}{\Gamma', \bullet\{\bullet\{\Delta\}\}} \quad rf}{\Gamma, \bullet\{\bullet\{\Delta\}\}} \quad rp}{\circ\{\Gamma\}, \bullet\{\Delta\}} \quad rp \\
(4)
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{\frac{\frac{\circ\{\Gamma'\}, A_1 \quad \frac{\frac{\circ\{\Gamma'\}, A_2 \quad \overline{A}_1, \overline{A}_2, \bullet\{\Delta'\}}{\overline{A}_1, \circ\{\Gamma'\}, \bullet\{\Delta'\}} \quad cut}{\overline{A}_1, \circ\{\Gamma'\}, \bullet\{\Delta'\}} \quad cut}{\circ\{\Gamma'\}, \circ\{\Gamma'\}, \bullet\{\Delta'\}} \quad ctr}{\circ\{\Gamma'\}, \bullet\{\Delta'\}} \quad rp}{\circ\{\circ\{\Gamma'\}\}, \Delta'} \quad rp}{\circ\{\circ\{\Gamma'\}\}, \Delta} \quad \vee \\
(5)
\end{array}$$

Figure 6: Some derivations in **SKt** illustrating the basic idea of cut elimination

Lemma 3.5 (Completeness). *Every Kt-theorem is SKt-derivable.*

Proof. The proof follows a standard translation from Hilbert systems to Gentzen's systems (see, e.g., [26]). We show here only derivations of Axioms (1) and (3) in Figure 3; the other axioms and rules are not difficult to handle. Double lines abbreviate derivations:

$$\begin{array}{c}
\frac{\frac{\frac{\overline{\circ\{\}, \overline{A}, A}}{\bullet\{\overline{A}, A\}} \quad rf}{\bullet\{\overline{A}\}, \blacklozenge A} \quad \blacklozenge}{\overline{A}, \circ\{\blacklozenge A\}} \quad rp}{\overline{A}, \square\blacklozenge A} \quad \square}{\overline{A} \vee \square\blacklozenge A} \quad \vee \\
id \quad \frac{\frac{\frac{\overline{A, \overline{A}, B, \bullet\{\}}}{\overline{B}, \overline{A}, B, \bullet\{\}} \quad id}{A \wedge \overline{B}, \overline{A}, B, \bullet\{\}} \quad rp}{\circ\{A \wedge \overline{B}, \overline{A}, B\}} \quad \wedge}{\blacklozenge(A \wedge \overline{B}), \blacklozenge\overline{A}, \circ\{B\}} \quad \blacklozenge}{\blacklozenge(A \wedge \overline{B}), \blacklozenge\overline{A}, \square B} \quad \square}{\blacklozenge(A \wedge \overline{B}) \vee \blacklozenge\overline{A} \vee \square B} \quad \vee
\end{array}$$

□

The following theorem is then a simple corollary of the Lemma 3.4 and Lemma 3.5.

Theorem 3.6. *A Kt-formula A is valid iff A is SKt-derivable.*

3.2. Cut elimination. The main difficulty in proving cut elimination for **SKt** is in finding the right cut reduction for some cases involving the rules *rp* and *rf*. For instance, consider the derivation (1) in Figure 6. It is not obvious that there is a cut reduction strategy

that works locally without generalizing the cut rule to, e.g., one which allows cut on any sub-sequent in a sequent. Instead, we shall follow a global cut reduction strategy similar to that used in cut elimination for display logics [3]. The idea is that, instead of permuting the cut rule locally, we trace the cut formula A (in Π_1) and \bar{A} (in Π_2), until they both become principal in their respective proofs, and then apply the cut rule(s) at that point on smaller formulae. Schematically, our simple strategy can be illustrated as follows: Suppose that Π_1 and Π_2 are, respectively, derivation (2) and (3) in Figure 6, that $A = A_1 \wedge A_2$ and there is a single instance in each proof where the cut formula is used. To reduce the cut on A , we first transform Π_1 by uniformly substituting $\bullet\{\Delta\}$ for A in Π_1 (see derivation (4) in Figure 6). We then prove the open leaf $\circ\{\Gamma'\}$, Δ by uniformly substituting $\circ\{\Gamma'\}$ for \bar{A} in Π_2 (see derivation (5) in Figure 6). Notice that the cuts on A_1 and A_2 introduced in the proof above are on smaller formulae than A .

The above simplified explanation implicitly assumes that a uniform substitution of a formula (or formulae) in a derivation results in a well-formed derivation, and that the cut formulae are not contracted. The precise statement of the proof substitution idea becomes more involved once these aspects are taken into account. This will be made precise in the main lemmas in the cut elimination proof.

Note that the proof substitution technique outlined above can actually be applied to proof systems that are more general than **SKt**; what is essentially needed is that the inference rules of the proof systems obey a certain closure property under arbitrary substitutions of structures for formulae. In the following, in anticipation of extensions of **SKt** to be presented in Section 5, we shall prove a more general cut elimination statement, which applies to any extensions of **SKt** with a certain class of structural rules.

Definition 3.7. Let Γ be a nested sequent. We denote with $\mathcal{F}(\Gamma)$ the multiset of all formula occurrences in Γ . A structural rule ρ is said to be *linear* if for every instance of the rule

$$\frac{\Delta}{\Gamma} \rho$$

we have that $\mathcal{F}(\Gamma) = \mathcal{F}(\Delta)$. That is, a linear structural rule does not allow weakening or contraction of formulae occurrences in the premise or conclusion of the rule. We shall assume that each linear rule induces, for each of its instance, a bijection between formula occurrences in the premise and formula occurrences in the conclusion, so that in every instance of the rule, a formula occurrence in the premise can be related to a unique formula occurrence in the conclusion, and vice versa.¹ A linear structural rule ρ is said to be *substitution-closed* if for any instance of the rule as given below left, where A is a formula occurrence shared between the premise and the conclusion, one can obtain another instance of ρ as given below right, for any structure Δ :

$$\frac{\Sigma'[A]}{\Sigma[A]} \rho \quad \frac{\Sigma'[\Delta]}{\Sigma[\Delta]} \rho$$

The substitution-closure property mentioned above is similar to Belnap's condition (C6) for cut elimination for display logics [3]. Note that this requirement for substitution closure rules out context-sensitive linear rules such as the rule shown in the leftmost figure below. To see why, consider the instance of ρ shown in the middle figure below. If one substitutes

¹To guarantee that such a bijection does exist for each instance, we shall restrict to only inference rules which can be represented as finite schemata with no side conditions, as are commonly found in most proof systems.

$\circ\{a\}$ for one of the occurrences of b , say, the first one from the left, then the resulting instance, as shown in the rightmost derivation below, would not be a valid instance of ρ .

$$\frac{\Gamma, \bullet\{\Delta\}, \Delta}{\Gamma, \circ\{\Delta\}, \Delta} \rho \quad \frac{a, \bullet\{b, c\}, b, c}{a, \circ\{b, c\}, b, c} \rho \quad \frac{a, \bullet\{\circ\{a\}, c\}, b, c}{a, \circ\{\circ\{a\}, c\}, b, c}$$

We use the notation $\vdash_S \Gamma$ to denote that the sequent Γ is derivable in the proof system S . We write $\vdash_S \Pi : \Gamma$ when we want to be explicit about the particular derivation Π of Γ . The *cut rank* of an instance of cut is defined as usual as the size of the cut formula. The cut rank of a derivation Π , denoted with $cr(\Pi)$, is the largest cut rank of the cut instances in Π (or zero, if there are no cuts in Π). Given a formula A , we denote with $|A|$ its size. Given a derivation Π , we denote with $|\Pi|$ its height, i.e., the length of a longest branch in the derivation tree of Π .

We shall now give a general cut elimination proof for any extension of **SKt** with substitution-closed linear structural rules. So in the following lemmas and theorem, we shall assume a (possibly empty) set \mathcal{S} of substitution-closed linear structural rules. We denote with **SKt** + \mathcal{S} the proof system obtained by adding the rules in \mathcal{S} to **SKt**.

Lemma 3.8. *If $\vdash_{\mathbf{SKt}+\mathcal{S}} \Pi_1 : \Gamma, a$ and $\vdash_{\mathbf{SKt}+\mathcal{S}} \Pi_2 : \Sigma^k[\bar{a}]$, where $k \geq 1$ and both Π_1 and Π_2 are cut free, then there exists a cut free Π such that $\vdash_{\mathbf{SKt}+\mathcal{S}} \Pi : \Sigma^k[\Gamma]$.*

Proof. By induction on $|\Pi_2|$. For the base cases, the non-trivial case is when Π_2 ends with *id* and \bar{a} is active in the rule, i.e., $\Sigma^k[\bar{a}] = \Sigma_1^{k-1}[\bar{a}], \bar{a}, a$ and Π_2 is as shown below left. Then we construct Π as shown below right.

$$\frac{}{\Sigma_1^{k-1}[\bar{a}], \bar{a}, a} id \quad \frac{\frac{\Pi_1}{\Gamma, a}}{\Sigma_1^{k-1}[\Gamma], \Gamma, a} wk$$

Most of the inductive cases follow straightforwardly from the induction hypothesis. We show here two non-trivial cases involving contraction and a rule in \mathcal{S} :

- Suppose $\Sigma^k[\bar{a}] = \Sigma_1^i[\bar{a}], \Sigma_2^j[\bar{a}]$ and Π_2 ends with a contraction on $\Sigma_2^j[\bar{a}]$, as shown below left. Then Π is constructed as shown below right, where Π_2'' is obtained from the induction hypothesis:

$$\frac{\frac{\Pi_2'}{\Sigma_1^i[\bar{a}], \Sigma_2^j[\bar{a}], \Sigma_2^j[\bar{a}]}}{\Sigma_1^i[\bar{a}], \Sigma_2^j[\bar{a}]} ctr \quad \frac{\frac{\Pi_2''}{\Sigma_1^i[\Gamma], \Sigma_2^j[\Gamma], \Sigma_2^j[\Gamma]}}{\Sigma_1^i[\Gamma], \Sigma_2^j[\Gamma]} ctr$$

- Suppose Π_2 is as shown below left, where $\rho \in \mathcal{S}$. Then Π is constructed as shown below right, where Π_2'' is obtained from the induction hypothesis:

$$\frac{\frac{\Pi_2'}{\Sigma^k[\bar{a}]}}{\Sigma^k[\bar{a}]} \rho \quad \frac{\frac{\Pi_2''}{\Sigma^k[\Gamma]}}{\Sigma^k[\Gamma]} \rho.$$

The substitution closure property of ρ guarantees that the instance of ρ on the right is valid. □

Note that for the substitution of proofs in Lemma 3.8 (and other substitution lemmas to follow) to succeed, one needs to allow contraction on arbitrary structures. Note also that as the rules from \mathcal{S} are closed under substitution of structures for formulae, they do not require any special treatment in the following proofs of substitution lemmas, i.e., in inductive cases involving these rules, the properties being proved can be established by straightforward applications of the inductive hypotheses, so we shall not detail the cases involving these rules.

Lemma 3.9. *Suppose $\vdash_{\mathbf{SKt+S}} \Pi_1 : \Delta, A$ and $\vdash_{\mathbf{SKt+S}} \Pi_2 : \Delta, B$ and $\vdash_{\mathbf{SKt+S}} \Pi : \Sigma^k[\overline{A \vee B}]$, for some $k \geq 1$, where the cut ranks of Π_1 , Π_2 and Π are smaller than $|A \wedge B|$. Then there exists a proof Π' such that $\vdash_{\mathbf{SKt+S}} \Pi' : \Sigma^k[\Delta]$ and $cr(\Pi) < |A \wedge B|$.*

Proof. By induction on $|\Pi|$. Most cases are straightforward. The only non-trivial case is when $\overline{A \vee B}$ is principal in the last rule of Π , i.e., Π is of the form shown below left. The proof Π' is constructed as shown below right, where Ψ' is a cut-free derivation obtained via the induction hypothesis.

$$\frac{\frac{\Psi}{\Sigma_1^{k-1}[\overline{A \vee B}], \overline{A}, \overline{B}}}{\Sigma_1^{k-1}[\overline{A \vee B}], \overline{A \vee B}} \quad \frac{\frac{\frac{\Pi_1}{\Delta, A} \quad \frac{\frac{\Pi_2}{\Delta, B} \quad \frac{\Psi'}{\Sigma_1^{k-1}[\Delta], \overline{A}, \overline{B}}}{\Sigma_1^{k-1}[\Delta], \overline{A}, \Delta} \text{ cut}}{\Sigma_1^{k-1}[\Delta], \Delta, \Delta} \text{ cut}}{\Sigma_1^{k-1}[\Delta], \Delta} \text{ ctr}}$$

□

Lemma 3.10. *Suppose $\vdash_{\mathbf{SKt+S}} \Pi_1 : \Delta, A, B$ and $\vdash_{\mathbf{SKt+S}} \Pi_2 : \Sigma^k[\overline{A \wedge B}]$, for some $k \geq 1$, and the cut ranks of Π_1 and Π_2 are smaller than $|A \vee B|$. Then there exists a proof Π such that $\vdash_{\mathbf{SKt+S}} \Pi : \Sigma^k[\Delta]$ and $cr(\Pi) < |A \vee B|$.*

Proof. This is proved analogously to Lemma 3.9. □

To prove the next two lemmas, we use two derived rules, i.e., $d1$ and $d2$ given below. These two rules are derivable using rp , rf , ctr and wk . They are similar to the so-called “medial rules” used to prove admissibility of structure contraction in [7]. The rule $d1$ is derived as shown in the rightmost derivation below ($d2$ is derived analogously).

$$\frac{\Gamma, \circ\{\Delta_1\}, \circ\{\Delta_2\}}{\Gamma, \circ\{\Delta_1, \Delta_2\}} \text{ d1} \quad \frac{\Gamma, \bullet\{\Delta_1\}, \bullet\{\Delta_2\}}{\Gamma, \bullet\{\Delta_1, \Delta_2\}} \text{ d2} \quad \frac{\frac{\frac{\frac{\Gamma, \circ\{\Delta_1\}, \circ\{\Delta_2\}}{\Delta_2, \bullet\{\Gamma, \circ\{\Delta_1\}\}} \text{ rf}}{\Delta_1, \Delta_2, \bullet\{\Gamma, \circ\{\Delta_1\}\}} \text{ wk}}{\Gamma, \circ\{\Delta_1, \Delta_2\}, \circ\{\Delta_1\}} \text{ rp}}{\bullet\{\Gamma, \circ\{\Delta_1, \Delta_2\}\}, \Delta_1} \text{ rf}}{\frac{\bullet\{\Gamma, \circ\{\Delta_1, \Delta_2\}\}, \Delta_1, \Delta_2}{\Gamma, \circ\{\Delta_1, \Delta_2\}, \circ\{\Delta_1, \Delta_2\}} \text{ wk}} \text{ rp}}{\Gamma, \circ\{\Delta_1, \Delta_2\}} \text{ ctr}$$

Lemma 3.11. *Suppose $\vdash_{\mathbf{SKt+S}} \Pi_1 : \Delta, \circ\{A\}$ and $\vdash_{\mathbf{SKt+S}} \Pi_2 : \Sigma^k[\overline{\diamond A}]$, for some $k \geq 1$, and the cut ranks of Π_1 and Π_2 are smaller than $|\square A|$. Then there exists a proof Π such that $\vdash_{\mathbf{SKt+S}} \Pi : \Sigma^k[\Delta]$ and $cr(\Pi) < |\square A|$.*

Proof. By induction on $|\Pi_2|$. The non-trivial case is when Π_2 ends with \diamond on $\diamond\bar{A}$, as shown below left. The derivation Π in this case is constructed as shown below right. There, the derivation Π' is obtained by applying the induction hypothesis to Π'_2 . Note that by the induction hypothesis, $cr(\Pi') < |\square A|$.

$$\frac{\Pi'_2}{\frac{\Sigma_1^{k-1}[\diamond\bar{A}], \circ\{\Gamma, \bar{A}\}}{\Sigma_1^{k-1}[\diamond\bar{A}], \circ\{\Gamma\}, \diamond\bar{A}}} \diamond \quad \frac{\frac{\frac{\Pi'}{\Sigma_1^{k-1}[\Delta], \circ\{\Gamma, \bar{A}\}}{\bullet\{\Sigma_1^{k-1}[\Delta]\}, \Gamma, \bar{A}}} \text{rf} \quad \frac{\frac{\Pi_1}{\Delta, \circ\{A\}}}{\bullet\{\Delta\}, A} \text{rf}}{\bullet\{\Sigma_1^{k-1}[\Delta]\}, \bullet\{\Delta\}, \Gamma} \text{cut}}{\frac{\bullet\{\Sigma_1^{k-1}[\Delta], \Delta\}, \Gamma}{\Sigma_1^{k-1}[\Delta], \circ\{\Gamma\}, \Delta} \text{d2}}{\Sigma_1^{k-1}[\Delta], \circ\{\Gamma\}, \Delta} \text{rp}}$$

□

Lemma 3.12. *Suppose $\vdash_{\mathbf{SKt+S}} \Pi_1 : \Delta, \circ\{\Delta', A\}$ and $\vdash_{\mathbf{SKt+S}} \Pi_2 : \Sigma^k[\square\bar{A}]$, for some $k \geq 1$, and the cut ranks of Π_1 and Π_2 are smaller than $|\diamond A|$. Then there exists Π such that $\vdash_{\mathbf{SKt+S}} \Pi : \Sigma^k[\Delta, \circ\{\Delta'\}]$ and $cr(\Pi) < |\diamond A|$.*

Proof. By induction on $|\Pi_2|$. The non-trivial case Π_2 is when Π_2 is as given below left. The derivation Π is constructed as shown below right, where Π' is obtained from the induction hypothesis and which satisfies $cr(\Pi') < |\diamond A|$.

$$\frac{\Pi'_2}{\frac{\Sigma_1^{k-1}[\square\bar{A}], \circ\{\bar{A}\}}{\Sigma_1^{k-1}[\square\bar{A}], \square\bar{A}}} \square \quad \frac{\frac{\frac{\Pi'}{\Sigma_1^{k-1}[\Delta, \circ\{\Delta'\}], \circ\{\bar{A}\}}{\bullet\{\Sigma_1^{k-1}[\Delta, \circ\{\Delta'\}]\}, \bar{A}}} \text{rf} \quad \frac{\frac{\Pi_1}{\Delta, \circ\{\Delta', A\}}}{\bullet\{\Delta\}, \Delta', A} \text{rf}}{\bullet\{\Sigma_1^{k-1}[\Delta, \circ\{\Delta'\}]\}, \bullet\{\Delta\}, \Delta'} \text{cut}}{\frac{\bullet\{\Sigma_1^{k-1}[\Delta, \circ\{\Delta'\}], \Delta\}, \Delta'}{\Sigma_1^{k-1}[\Delta, \circ\{\Delta'\}], \Delta, \circ\{\Delta'\}} \text{d2}}{\Sigma_1^{k-1}[\Delta, \circ\{\Delta'\}], \Delta, \circ\{\Delta'\}} \text{rp}}$$

□

Lemma 3.13. *Suppose $\vdash_{\mathbf{SKt+S}} \Pi_1 : \Delta, \bullet\{A\}$ and $\vdash_{\mathbf{SKt+S}} \Pi_2 : \Sigma^k[\blacklozenge\bar{A}]$, for some $k \geq 1$, and the cut ranks of Π_1 and Π_2 are smaller than $|\blacksquare A|$. Then there exists a proof Π such that $\vdash_{\mathbf{SKt+S}} \Pi : \Sigma^k[\Delta]$ and $cr(\Pi) < |\blacksquare A|$.*

Proof. This is proved analogously to Lemma 3.11. □

Lemma 3.14. *Suppose $\vdash_{\mathbf{SKt+S}} \Pi_1 : \Delta, \bullet\{\Delta', A\}$ and $\vdash_{\mathbf{SKt+S}} \Pi_2 : \Sigma^k[\blacklozenge\bar{A}]$, for some $k \geq 1$, and the cut ranks of Π_1 and Π_2 are smaller than $|\blacklozenge A|$. Then there exists Π such that $\vdash_{\mathbf{SKt+S}} \Pi : \Sigma^k[\Delta, \bullet\{\Delta'\}]$ and $cr(\Pi) < |\blacklozenge A|$.*

Proof. This is proved analogously to Lemma 3.12. □

Lemma 3.15. *Let C be a non-atomic formula. Suppose $\vdash_{\mathbf{SKt+S}} \Psi_1 : \Gamma, \bar{C}$ and $\vdash_{\mathbf{SKt+S}} \Psi_2 : \Omega^n[C]$, for some $n \geq 1$, and the cut ranks of Ψ_1 and Ψ_2 are smaller than $|C|$. Then there exists a proof Ψ such that $\vdash_{\mathbf{SKt+S}} \Psi : \Omega^n[\Gamma]$ and $cr(\Psi) < |C|$.*

Proof. By induction on the height of Ψ_2 and case analysis on C . The non-trivial cases are when Ψ_2 ends with an introduction rule on C . That is, we have $\Omega^n[C] = \Omega_1^{n-1}[C], C$ for some context $\Omega_1^{n-1}[\]$. We show the cases where C is either $\square B$, $\diamond B$ or $B_1 \wedge B_2$; the other cases can be treated similarly.

- Suppose $C = \Box B$ and Ψ_2 is the following derivation:

$$\frac{\Psi'_2}{\frac{\Omega_1^{n-1}[\Box B], \circ\{B\}}{\Omega_1^{n-1}[\Box B], \Box B} \Box} \Box$$

By induction hypothesis, we have $\vdash_{\mathbf{SKt+S}} \Psi' : \Omega_1^{n-1}[\Gamma], \circ\{B\}$ and $cr(\Psi') < |C|$. Applying Lemma 3.11 to Ψ' and Ψ_1 (that is, by instantiating A to B , Δ to $\Omega_1^{n-1}[\Gamma]$, and $\Sigma^k \square$ to the context $\Gamma, [\]$), we obtain $\vdash_{\mathbf{SKt+S}} \Psi : \Gamma, \Omega_1^{n-1}[\Gamma] = \Omega^n[\Gamma]$ such that $cr(\Psi) < |\Box B|$.

- Suppose $C = \Diamond B$ and Ψ_2 is the following derivation:

$$\frac{\Psi'_2}{\frac{\Omega_1^{n-1}[\Diamond B], \circ\{\Gamma', B\}}{\Omega_1^{n-1}[\Diamond B], \circ\{\Gamma'\}, \Diamond B} \Diamond} \Diamond$$

By induction hypothesis, we have

$$\vdash_{\mathbf{SKt+S}} \Psi' : \Omega_1^{n-1}[\Gamma], \circ\{\Gamma', B\}.$$

Applying Lemma 3.12 to Ψ' and Ψ_1 (i.e., instantiating A to B , Δ to $\Omega_1^{n-1}[\Gamma]$, Δ' to Γ' , and $\Sigma^k \square$ to the context $\Gamma, [\]$), we obtain $\vdash_{\mathbf{SKt+S}} \Psi : \Gamma, \Omega_1^{n-1}[\Gamma] = \Omega^n[\Gamma]$ such that $cr(\Psi) < |\Diamond B|$.

- Suppose $C = B_1 \wedge B_2$ and Ψ_2 is the following derivation:

$$\frac{\frac{\Theta_1}{\Omega_1^{n-1}[B_1 \wedge B_2], B_1} \quad \frac{\Theta_2}{\Omega_1^{n-1}[B_1 \wedge B_2], B_2}}{\Omega_1^{n-1}[B_1 \wedge B_2], B_1 \wedge B_2} \wedge$$

By induction hypothesis, we have $\vdash_{\mathbf{SKt+S}} \Theta'_1 : \Omega_1^{n-1}[\Gamma], B_1$ and $\vdash_{\mathbf{SKt+S}} \Theta'_2 : \Omega_1^{n-1}[\Gamma], B_2$. Applying Lemma 3.9 to Θ'_1, Θ'_2 and Ψ_1 , we obtain $\vdash_{\mathbf{SKt+S}} \Psi : \Gamma, \Omega_1^{n-1}[\Gamma] = \Omega^n[\Gamma]$ such that $cr(\Psi) < |B_1 \wedge B_2|$.

□

Theorem 3.16. *Cut elimination holds for $\mathbf{SKt} + \mathcal{S}$.*

Proof. Given a derivation with cuts, we remove topmost cuts in succession, using Lemma 3.8 and Lemma 3.15. □

Corollary 3.17. *Cut elimination holds for \mathbf{SKt} .*

4. SYSTEM \mathbf{DKt} : A CONTRACTION-FREE DEEP-SEQUENT CALCULUS

We now consider another sequent system which uses *deep inference*, where rules can be applied directly to any node within a nested sequent. We call this system \mathbf{DKt} , and give its inference rules in Figure 7. Note that there are no structural rules in \mathbf{DKt} , and the contraction rule is absorbed into the logical rules. Notice that, reading the logical rules bottom up, we keep the principal formulae in the premise. This is actually not necessary for some rules (e.g., \blacksquare , \wedge , etc.), but this form of rule allows for a better accounting of formulae in our saturation-based proof search procedure (see Section 7). We also do not include the cut rule in \mathbf{DKt} as it is admissible in \mathbf{DKt} , the translations from \mathbf{SKt} to \mathbf{DKt}

$$\begin{array}{ccc}
\frac{}{\Sigma[a, \bar{a}]} \textit{id} & \frac{\Sigma[A \wedge B, A] \quad \Sigma[A \wedge B, B]}{\Sigma[A \wedge B]} \wedge & \frac{\Sigma[A \vee B, A, B]}{\Sigma[A \vee B]} \vee \\
\frac{\Sigma[\blacksquare A, \bullet\{A\}]}{\Sigma[\blacksquare A]} \blacksquare & \frac{\Sigma[\bullet\{\Delta, A\}, \blacklozenge A]}{\Sigma[\bullet\{\Delta\}, \blacklozenge A]} \blacklozenge_1 & \frac{\Sigma[\circ\{\Delta, \blacklozenge A\}, A]}{\Sigma[\circ\{\Delta, \blacklozenge A\}]} \blacklozenge_2 \\
\frac{\Sigma[\square A, \circ\{A\}]}{\Sigma[\square A]} \square & \frac{\Sigma[\circ\{\Delta, A\}, \diamond A]}{\Sigma[\circ\{\Delta\}, \diamond A]} \diamond_1 & \frac{\Sigma[\bullet\{\Delta, \diamond A\}, A]}{\Sigma[\bullet\{\Delta, \diamond A\}]} \diamond_2
\end{array}$$

Figure 7: The contraction-free deep-inference system **DKt**

and back, to be shown below, do not use the cut rule. A side note on the cut rule: one could introduce a “deep” version of cut:

$$\frac{\Sigma[A] \quad \Sigma[\bar{A}]}{\Sigma[\emptyset]} \textit{cut},$$

just as is done in nested sequent calculi for modal logics in [4, 7]. This form of cut rule can be easily derived from its shallow counterpart (see Figure 5) using the display property (Proposition 3.2). So when we speak of cut admissibility in **DKt**, it applies equally to both the shallow cut and the deep cut above.

The following intuitive observation about **DKt** rules will be useful later: Rules in **DKt** are characterized by propagations of formulae across different nodes in a nested sequent tree. The shape of the tree is not affected by these propagations, and the only change that can occur to the tree is the creation of new nodes (via the introduction rules \blacksquare and \square).

System **DKt** corresponds to Kashima’s **S2Kt** [17], but with the contraction rule absorbed into the logical rules. The modal fragment of **DKt** was also developed independently by Brünnler [4, 7] and Poggiolesi [23]. Kashima shows that **DKt** proofs can be encoded into **SKt**, essentially due to the display property of **SKt** (Proposition 3.2) which allows displaying and undisplaying of any node within a nested sequent. Kashima also shows that **DKt** is complete for tense logic, via semantic arguments. We prove a stronger result: every cut-free **SKt**-proof can be transformed into a **DKt**-proof, hence **DKt** is complete and cut is admissible in **DKt**.

To translate cut-free **SKt**-proofs into **DKt**-proofs, we show that all structural rules of **SKt** are height-preserving admissible in **DKt**.

Definition 4.1. Given a proof system **S** and a rule ρ with premises $\Gamma_1, \dots, \Gamma_n$ and conclusion Γ , ρ is said to be *admissible in S* if the following holds: whenever $\vdash_{\mathbf{S}} \Pi_1 : \Gamma_1, \dots, \vdash_{\mathbf{S}} \Pi_n : \Gamma_n$, then there exists Π such that $\vdash_{\mathbf{S}} \Pi : \Gamma$. In the case where $n = 1$, we say that ρ is *height-preserving admissible in S* if $|\Pi| = |\Pi_1|$.

In the following lemmas, we show a stronger admissibility result for weakening and contraction, i.e., we shall show that the following *deep* versions of weakening and contraction are in fact admissible.

$$\frac{\Sigma[\Gamma]}{\Sigma[\Gamma, \Delta]} \textit{dw} \quad \frac{\Sigma[\Delta, \Delta]}{\Sigma[\Delta]} \textit{dgc}$$

Obviously, the rules *wk* and *ctr* are just instances of the above rules. As we shall see, admissibility of *dgc* follows from admissibility of formula contraction (the rule *dfc* below)

and two distribution rules shown below.

$$\frac{\Sigma[A, A]}{\Sigma[A]} \text{ dfc} \quad \frac{\Sigma[\circ\{\Delta_1\}, \circ\{\Delta_2\}]}{\Sigma[\circ\{\Delta_1, \Delta_2\}]} \text{ mf} \quad \frac{\Sigma[\bullet\{\Delta_1\}, \bullet\{\Delta_2\}]}{\Sigma[\bullet\{\Delta_1, \Delta_2\}]} \text{ mp}$$

The distribution rules mf and mp are usually called the *medial* rules in the deep inference literature (see, e.g., [8, 14, 7]), and, in their various forms, they have been used to reduce general contraction to formulae or atomic contraction in different proof systems for classical, intuitionistic, linear, modal and tense logics. The modal medial rule mf has been used in [7] to show admissibility of contraction for several nested sequent calculi for modal logics. Our proof of admissibility of contraction here is an extension of Brünnler and Straßburger's proof [7] to tense logics.

Lemma 4.2 (Admissibility of weakening). *The rule dw is height-preserving admissible in \mathbf{DKt} .*

Proof. By simple induction on $|\Pi|$. □

The proofs for the following lemmas that concern structural rules that change the shape of the tree of a nested sequent share similarities. That is, the only interesting cases in the proofs are those that concern propagation of formulae across different nodes in a nested sequent. We show here an interesting case in the proof for the admissibility of display postulates.

Lemma 4.3 (Admissibility of display postulates). *The rules rp and rf are both height-preserving admissible in \mathbf{DKt} .*

Proof. We show here admissibility of rp , the other rule can be dealt with similarly. Consider the rp rule in Figure 5. Suppose that $\vdash_{\mathbf{DKt}} \Pi : \Gamma, \bullet\{\Delta\}$. We shall construct a derivation Π' for the nested sequent $\circ\{\Gamma\}, \Delta$ by induction on $|\Pi|$. The non-trivial cases are when there is an exchange of formulae between Γ and Δ . We show one case below; the others can be done analogously. Suppose Π is as shown below left, where $\Gamma = \Gamma', \blacklozenge A$. Then Π' is as shown below right where Π'_1 is obtained from the induction hypothesis:

$$\frac{\frac{\Pi_1}{\Gamma', \blacklozenge A, \bullet\{A, \Delta\}}}{\Gamma', \blacklozenge A, \bullet\{\Delta\}} \blacklozenge_1 \quad \frac{\frac{\Pi'_1}{\circ\{\Gamma', \blacklozenge A\}, A, \Delta}}{\circ\{\Gamma', \blacklozenge A\}, \Delta} \blacklozenge_2$$

By the induction hypothesis $|\Pi_1| = |\Pi'_1|$, hence we also have $|\Pi| = |\Pi'|$. □

To show admissibility of general contraction, we first show that formula contraction, mp and mf are all height-preserving admissible in \mathbf{DKt} .

Lemma 4.4. *The rules dfc , mf and mp are height-preserving admissible in \mathbf{DKt} .*

Proof. Height-preserving admissibility of dfc can be proved by simple induction on the height of the derivation of its premise. We show here height-preserving admissibility of mf ; height-preserving admissibility of mp can be proved analogously.

So suppose we have $\vdash_{\mathbf{DKt}} \Pi : \Sigma[\circ\{\Delta_1\}, \circ\{\Delta_2\}]$. We show by induction on $|\Pi|$ that there exists Π' such that $\vdash_{\mathbf{DKt}} \Pi' : \Sigma[\circ\{\Delta_1, \Delta_2\}]$ and $|\Pi| = |\Pi'|$. We show here two non-trivial cases:

- Suppose Π ends with \blacklozenge_1 that moves a formula into $\circ\{\Delta_1\}$ when read upwards. That is,

$$\Sigma[\circ\{\Delta_1\}, \circ\{\Delta_2\}] = \Sigma'[\blacklozenge A, \circ\{\Delta_1\}, \circ\{\Delta_2\}]$$

and Π is as shown below left. Then Π' is constructed as shown below right, where Ψ' is obtained by applying the induction hypothesis to Ψ .

$$\frac{\frac{\Psi}{\Sigma'[\diamond A, \circ\{A, \Delta_1\}, \circ\{\Delta_2\}]} \diamond_1}{\Sigma'[\diamond A, \circ\{\Delta_1\}, \circ\{\Delta_2\}]} \diamond_1 \qquad \frac{\frac{\Psi'}{\Sigma'[\diamond A, \circ\{A, \Delta_1, \Delta_2\}]} \diamond_1}{\Sigma'[\diamond A, \circ\{\Delta_1, \Delta_2\}]} \diamond_1$$

Since $|\Psi'| = |\Psi|$, it follows that $|\Pi'| = |\Pi|$.

- Suppose Π ends with \blacklozenge_2 that moves a formula out from $\circ\{\Delta_1\}$. That is, $\Delta_1 = \blacklozenge A, \Delta'_1$ and Π is as shown below left. Then Π' is constructed as shown below right, where Ψ' is obtained from the induction hypothesis. It is easy to see that $|\Pi'| = |\Pi|$.

$$\frac{\frac{\Psi}{\Sigma[A, \circ\{\blacklozenge A, \Delta'_1\}, \circ\{\Delta_2\}]} \blacklozenge_2}{\Sigma[\circ\{\blacklozenge A, \Delta'_1\}, \circ\{\Delta_2\}]} \blacklozenge_2 \qquad \frac{\frac{\Psi'}{\Sigma[A, \circ\{\blacklozenge A, \Delta_1, \Delta_2\}]} \blacklozenge_2}{\Sigma[\circ\{\blacklozenge A, \Delta'_1, \Delta_2\}]} \blacklozenge_2$$

□

Lemma 4.5 (Admissibility of contraction). *The rule dgc is height-preserving admissible in \mathbf{DKt} .*

Proof. Suppose $\vdash_{\mathbf{DKt}} \Pi : \Sigma[\Delta, \Delta]$. We need to show that there exists Π' such that $\vdash_{\mathbf{DKt}} \Pi' : \Sigma[\Delta]$ and $|\Pi| = |\Pi'|$. We do this by induction on the size of Δ . If Δ is the empty set then it is straightforward. If Δ is a formula, then it is an instance of dfc which is height-preserving admissible by Lemma 4.4. The other cases follow from the induction hypothesis and Lemma 4.4. Consider, for instance, the case where $\Delta = \circ\{\Delta'\}$. Then by Lemma 4.4 we have a proof Ψ , with $|\Psi| = |\Pi|$, such that $\vdash_{\mathbf{DKt}} \Psi : \Sigma[\circ\{\Delta', \Delta'\}]$. Note that since Δ' is of a smaller size than $\circ\{\Delta'\}$, we can apply the induction hypothesis to Ψ and obtain a proof Π' , with $|\Pi'| \leq |\Pi|$, such that $\vdash_{\mathbf{DKt}} \Pi' : \Sigma[\circ\{\Delta'\}]$. □

Theorem 4.6. *For every sequent Γ , $\vdash_{\mathbf{SKt}} \Gamma$ if and only if $\vdash_{\mathbf{DKt}} \Gamma$.*

Proof. The forward direction, that is, showing that $\vdash_{\mathbf{SKt}} \Gamma$ implies $\vdash_{\mathbf{DKt}} \Gamma$, follows from admissibility of the structural rules of \mathbf{SKt} in \mathbf{DKt} (Lemma 4.2 – Lemma 4.5).

For the converse, we use the display property of \mathbf{SKt} (Proposition 3.2) to simulate the deep-inference rules of \mathbf{DKt} . We show here the derivations for the rules \diamond_1 and \blacklozenge_2 (the other cases are similar):

$$\begin{array}{c} \Sigma[\circ\{\Delta, A\}, \diamond A] \\ \vdots \\ \frac{\Delta', \circ\{\Delta, A\}, \diamond A}{\Delta', \circ\{\Delta\}, \diamond A, \diamond A} \diamond \\ \vdots \\ \Sigma[\circ\{\Delta\}, \diamond A] \end{array} \qquad \begin{array}{c} \Sigma[\circ\{\Delta, \blacklozenge A\}, A] \\ \vdots \\ \frac{\Delta', A, \circ\{\Delta, \blacklozenge A\}}{\bullet\{\Delta', A\}, \Delta, \blacklozenge A} rf \\ \frac{\bullet\{\Delta'\}, \blacklozenge A, \Delta, \blacklozenge A}{\bullet\{\Delta'\}, \Delta, \blacklozenge A} \blacklozenge \\ \frac{\bullet\{\Delta'\}, \Delta, \blacklozenge A}{\Delta', \circ\{\Delta, \blacklozenge A\}} ctr \\ \vdots \\ \Sigma[\circ\{\Delta, \blacklozenge A\}] \end{array}$$

where the dotted part of the derivation is obtained from applying Proposition 3.2. □

A consequence of Theorem 4.6 is that the general contraction rule in **SKt** can be replaced by formula contraction. This can be proved as follows: take a cut-free proof in **SKt**, translate it to **DKt** and then translate it back to **SKt**. Since general contraction is admissible in **DKt**, and since the translation from **DKt** to **SKt** does not use general contraction (only formula contraction), we can effectively replace the general contraction in **SKt** with formula contraction.

An interesting feature of **DKt** is that in a proof of a sequent, the ‘colour’ of a (formula or structural) connective does not change when moving from premise to conclusion or vice versa. Let us call a formula (a sequent, a rule) *purely modal* if it contains no black connectives. It is easy to see that if a purely modal formula (sequent) is provable in **DKt**, then it is provable using only purely modal rules. Let $\mathbf{DK} = \{id, \wedge, \vee, \Box, \Diamond_1\}$, i.e., it is the set of purely modal rules of **DKt**. The above observation leads to the following “separation” result:

Theorem 4.7. *For every modal formula A , $\vdash_{\mathbf{DK}} A$ iff A is a theorem of K .*

Proof. (\Rightarrow) Suppose $\vdash_{\mathbf{DK}} A$. Since **DK** is a subsystem of **DKt**, we must have $\vdash_{\mathbf{DKt}} A$, and then $\vdash_{\mathbf{SKt}} A$. By the soundness of **SKt**, A is Kt -valid. But all purely modal Kt -valid formulae are also K -valid. Thus purely modal A is also a theorem of K .

(\Leftarrow) Suppose A is a theorem of K . But the theorems of K are also theorems of Kt , hence A is derivable in **SKt**. This derivation may contain cuts, but by cut elimination we know that A is also cut-free derivable in **SKt**. The cut-free **SKt**-derivation of a purely modal formula cannot contain any instances of the rules \blacksquare or \blacklozenge since these introduce non-modal connectives into their conclusion. Thus, the only way to create an occurrence of \bullet on our way up from the end-sequent is to use rp . By Theorem 4.6, the cut-free **SKt**-derivation of A can be transformed into a (cut-free) derivation of A in **DKt**. Moreover, the transformation given in the proof removes all applications of rp without creating black structural or logical connectives. For example, an **SKt** derivation of $a, \bar{a}, \bullet\{\Delta\}$ is converted to a **DKt** derivation of $(\circ\{a, \bar{a}\}, \Delta) = \Sigma[a, \bar{a}]$. Hence the transformed derivation is actually a derivation in **DK**. \square

This completeness result for **DK** is known from [4]; what we show here is how it can be derived as a consequence of completeness of **DKt**.

5. PROOF SYSTEMS FOR SOME EXTENSIONS OF TENSE LOGIC

We now consider extensions of tense logic with a class of axioms that subsumes a range of standard normal modal axioms, e.g., reflexivity, transitivity, euclideaness, etc. These axioms, called Scott-Lemmon axioms [20], are formulae of the form:

$$G(h, i, j, k) : \quad \Diamond^h \Box^i A \rightarrow \Box^j \Diamond^k A$$

where $h, i, j, k \geq 0$ and $\Diamond^n A$ (likewise, $\Box^n A$) denotes the formula A prefixed with n -occurrences of \Diamond (resp. \Box). For example, the axiom for transitivity, $\Box A \rightarrow \Box \Box A$, is an instance of Scott-Lemmon axiom scheme with $h = 0$, $i = 1$, $j = 2$ and $k = 0$.

In the following subsection, we show that, for each set **SL** of Scott-Lemmon axioms, there is a shallow system that modularly extends **SKt** with **SL** for which cut elimination holds. By modular extension we mean that the rules of the extended systems are the rules of **SKt** plus a set of structural rules that are derived directly from the modal axioms (in fact, they are in one-to-one correspondence). However, there does not appear to be

a systematic way to derive the corresponding deep-inference systems for these extensions. In subsequent subsections, we give deep-inference systems for two well-known extensions of Kt , i.e., Kt extended with axioms for $S4$ and $S5$, and an extension of Kt with the axiom of uniqueness $CD : \diamond A \rightarrow \Box A$. Again, as with **DKt**, the rules for the deep-inference systems are characterized by propagations of formulae across different nodes in the nested sequents. However, the design of the rules for the deep system is not as modular as its shallow counterpart, since it needs to take into account the closure of the axioms.

A nice feature of the deep inference systems shown below is that they satisfy the same separation property as with **DKt**: the purely modal subset of each deep-inference system is sound and complete with respect to its modal fragment. That is, we obtain the deep-inference systems for $S4$, $S5$ and $K + CD$ “for free” simply by dropping all the tense rules.

5.1. Extending SKt with Scott-Lemmon axioms. One way to extend **SKt** with Scott-Lemmon axioms is to simply add those axiom schemes as inference rules without premise. However, the resulting system would not satisfy cut elimination. Instead, we shall follow an approach that absorbs those axioms into structural rules without breaking cut elimination. In the display calculus setting, Kracht [18] has shown that a class of axioms, called *primitive axioms*, can be turned into structural rules in a systematic way and the display calculus for tense logic extended with those structural rules also satisfies cut elimination. A *primitive axiom* is an axiom of the form $A \rightarrow B$ where both A and B are built using propositional variables, \wedge , \vee , \diamond , and \blacklozenge . We shall follow Kracht’s approach in absorbing Scott-Lemmon’s axioms into structural rules. However, the main problem is that Scott-Lemmon axioms, in the form shown earlier, are not strictly speaking primitive axioms. But as we shall see later, they have equivalent representations in primitive form. A *primitive Scott-Lemmon axiom* is a formula of the form

$$P(h, i, j, k) : \quad \blacklozenge^h \diamond^j A \rightarrow \diamond^i \blacklozenge^k A.$$

Definition 5.1. Let \mathbf{S} be a set of axiom schemes whose members are formulae of the form $F \rightarrow G$. An *axiomatic extension of SKt* with \mathbf{S} is the proof system obtained by adding to **SKt** the inference rule

$$\frac{}{\overline{F, G}}$$

for each $F \rightarrow G \in \mathbf{S}$. We denote with **SKtAxS** the axiomatic extension of **SKt** with axioms \mathbf{S} .

In the following, we shall use the notation $\circ^n\{\Delta\}$ to denote the sequent

$$\circ \underbrace{\{\dots \circ \{\Delta\} \dots\}}_n.$$

The notation $\bullet^n\{\Delta\}$ is defined similarly.

Lemma 5.2. *For any $h, i, j, k \geq 0$, the axiomatic extension of **SKt** with $G(h, i, j, k)$ is equivalent to the axiomatic extension of **SKt** with $P(h, i, j, k)$.*

Proof. We give a syntactic proof of this lemma, i.e., we show that the axiom $G(h, i, j, k)$ is derivable in **SKt** extended with axiom $P(h, i, j, k)$, and vice versa. The axiom rules corresponding to $G(h, i, j, k)$ and $P(h, i, j, k)$ are, respectively,

$$\frac{}{\Box^h \diamond^i \overline{A}, \Box^j \diamond^k A} SL \quad \text{and} \quad \frac{}{\blacklozenge^h \Box^j \overline{A}, \diamond^i \blacklozenge^k A} PSL.$$

In the following derivation, we make use of the fact that deep inference rules of **DKt** are derivable in **SKt**, so we shall freely mix deep and shallow inference rules (including residuation rules). We shall also make use of derived rules that allow one to go from a formula to its sequent counterpart, e.g., replacing $\Box A$ with $\circ\{A\}$, etc., which could easily be done using appropriate cuts. So we shall also assume the following deep inference rules:

$$\frac{\Sigma[\Box A]}{\Sigma[\circ\{A\}]} \equiv \frac{\Sigma[\blacksquare A]}{\Sigma[\bullet\{A\}]} \equiv$$

The primitive form of Scott-Lemmon axiom can then be derived as follows:

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{\circ^j\{\overline{A}, A, \circ^k\{\}\}}{id}}{\circ^j\{\overline{A}, \circ^k\{\diamond^k A\}}}{\diamond_2}}{\circ^j\{\overline{A}, \Box^k \diamond^k A\}}{\Box}}{\circ^j\{\overline{A}\}, \diamond^j \Box^k \diamond^k A}{\diamond_1}}{\Box^h \diamond^i \diamond^k A, \Box^j \diamond^k \blacksquare^k \overline{A}} SL}{\circ^j\{\overline{A}\}, \Box^h \diamond^i \diamond^k A}{\circ^j\{A\}, \circ^h\{\diamond^i \diamond^k A\}} \equiv}{\bullet^h\{\circ^j\{\overline{A}\}\}, \diamond^i \diamond^k A}{\blacksquare^h \Box^j \overline{A}, \diamond^i \diamond^k A} \blacksquare; \Box} rf}{\circ^j\{\overline{A}\}, \diamond^j \Box^k \diamond^k A}{\circ^j\{A\}, \circ^h\{\diamond^i \diamond^k A\}} \equiv}{\bullet^h\{\circ^j\{\overline{A}\}\}, \diamond^i \diamond^k A}{\blacksquare^h \Box^j \overline{A}, \diamond^i \diamond^k A} \blacksquare; \Box} cut$$

Note that in the derivation above, to simplify presentation, we do not keep the principal formula of a rule in the premise as we would normally do in **DKt**.

It is not difficult to see that the converse also holds, i.e., assuming $P(h, i, j, k)$ (i.e., the rule PSL), one can derive the axiom $G(h, i, j, k)$, using cuts, rp and other modal/tense introduction rules. We leave this as an exercise to the reader. \square

Having shown the equivalence of the axioms $G(h, i, j, k)$ and $P(h, i, j, k)$, we shall use the latter to design a cut-free extension of **SKt** with Scott-Lemmon axioms. For each $P(h, i, j, k)$, we define a corresponding structural rule as follows:

$$\frac{\Gamma, \circ^i\{\bullet^k\{\Delta\}\}}{\Gamma, \bullet^h\{\circ^j\{\Delta\}\}} sl(h, i, j, k)$$

Definition 5.3. Let \mathbf{S} be a set of axioms and let $\rho(\mathbf{S})$ be the corresponding structural rules for axioms in \mathbf{S} . The *structural extension* of **SKt** with \mathbf{S} is the proof system obtained by adding the structural rules $\rho(\mathbf{S})$ to **SKt**. We denote with **SKtS** the structural extension of **SKt** with \mathbf{S} .

Proposition 5.4. For any set of Scott-Lemmon axioms \mathbf{SL} , the proof systems **SKtAxSL** and **SKtSL** are equivalent.

Proof. The following two derivations show how one can derive an axiom $P(h, i, j, k)$ using its structural rule counterpart, and vice versa.

$$\frac{\frac{\frac{\frac{\frac{\frac{\circ^i\{\bullet^k\{\overline{A}, A\}}}{id}}{\circ^i\{\bullet^k\{\overline{A}\}\}, \diamond^i \diamond^k A}{\diamond_1; \diamond_1}}{\bullet^h\{\circ^j\{\overline{A}\}\}, \diamond^i \diamond^k A}{\blacksquare^h \Box^j \overline{A}, \diamond^i \diamond^k A} \blacksquare; \Box} sl(h, i, j, k)}{\Gamma, \circ^i\{\bullet^k\{\Delta\}\}} \frac{\frac{\frac{\frac{\frac{\frac{\blacksquare^h \Box^j \tau(\Delta), \diamond^i \diamond^k \tau(\Delta)}{axiom}}{\Gamma, \Box^i \blacksquare^k \tau(\Delta)}{\Gamma, \blacksquare^h \Box^j \tau(\Delta)} \equiv}{\Gamma, \bullet^h\{\circ^j\{\Delta\}\}} \equiv}{\Gamma, \circ^i\{\bullet^k\{\Delta\}\}} \Gamma, \circ^i\{\bullet^k\{\Delta\}\}} cut$$

$$\begin{array}{ccc}
\frac{\Sigma[\blacklozenge A, A]}{\Sigma[\blacklozenge A]} T_a & \frac{\Sigma[\blacklozenge A, \bullet\{\blacklozenge A, \Delta\}]}{\Sigma[\blacklozenge A, \bullet\{\Delta\}]} 4_a & \frac{\Sigma[\blacklozenge A, \circ\{\blacklozenge A, \Delta\}]}{\Sigma[\blacklozenge A, \circ\{\Delta\}]} 4_c \\
\frac{\Sigma[\blacklozenge A, A]}{\Sigma[\blacklozenge A]} T_b & \frac{\Sigma[\circ\{\Delta, \blacklozenge A\}, \blacklozenge A]}{\Sigma[\circ\{\Delta, \blacklozenge A\}]} 4_b & \frac{\Sigma[\bullet\{\Delta, \blacklozenge A\}, \blacklozenge A]}{\Sigma[\bullet\{\Delta, \blacklozenge A\}]} 4_d
\end{array}$$

Figure 8: Additional propagation rules for **DS4**

Note that in presenting the derivations, we adopt the same simplification steps as we did in Lemma 5.2. \square

As noted earlier, **SKtAxSL** does not have cut elimination, as typical for axiomatic extensions of sequent calculi, although one could perhaps show that applications of the cut rule can be limited to those that cut directly with the axioms. But we shall show that the “pure” sequent calculus **SKtSL** does enjoy true cut elimination. This is a simple consequence of Theorem 3.16, as the rules in $\rho(\mathbf{SL})$ are substitution-closed linear rules.

Theorem 5.5. *For any set of Scott-Lemmon axioms **SL**, cut elimination holds for **SKtSL**.*

In the following subsections, we consider three instances of **SKtSL**, i.e., extensions of **SKt** with axioms for S_4 , S_5 , and the axiom of uniqueness. We give deep inference systems for these logics that are equivalent to their shallow counterparts. These are by no means an exhaustive list of logics for which the correspondence between deep and shallow systems holds; they are meant as an illustration of the kind of methods used to eliminate structural rules via propagation rules. For the extensions with S_4 and S_5 , the proofs of the correspondence are not very different from the proof of the correspondence between **SKt** and **DKt**, so we shall only state the correspondence results and omit the proofs. The interested reader can consult the doctoral thesis of the second author [24] for details. We shall present a more general framework in Section 6, in which this correspondence can be proved uniformly for a class of axiomatic extensions of **SKt**.

5.2. A deep-inference system for modal tense logic KtS4. Consider an extension of **SKt** with the axioms for reflexivity and transitivity (given in primitive form): $T : A \rightarrow \blacklozenge A$ and $4 : \blacklozenge\blacklozenge A \rightarrow \blacklozenge A$. Their corresponding structural rules are:

$$\frac{\Gamma, \circ\{\Delta\}}{\Gamma, \Delta} T_f \quad \frac{\Gamma, \circ\{\Delta\}}{\Gamma, \circ\{\circ\{\Delta\}\}} 4_f.$$

Using residuation, we can also derived the tense counterparts of the rule T_f and 4_f , with the structural connective \circ replaced by \bullet :

$$\frac{\Gamma, \bullet\{\Delta\}}{\circ\{\Gamma\}, \Delta} T_f \quad \frac{\frac{\Gamma, \bullet\{\Delta\}}{\circ\{\Gamma\}, \Delta} rf}{\circ\{\circ\{\Gamma\}\}, \Delta} 4_f \quad \frac{\circ\{\circ\{\Gamma\}\}, \Delta}{\circ\{\Gamma\}, \bullet\{\Delta\}} rp \quad \frac{\circ\{\Gamma\}, \bullet\{\Delta\}}{\Gamma, \bullet\{\bullet\{\Delta\}\}} rp$$

As with the design of **DKt**, in designing a deep inference system for **KtS4**, we aim to get rid of all structural rules. This is achieved via propagation rules for \blacklozenge -formulae, and by residuation, also for \blacklozenge -formulae. The propagation rules needed are given in Figure 8.

$$\frac{\Sigma[\blacklozenge A, \circ\{\blacklozenge A, \Delta\}]}{\Sigma[\blacklozenge A, \circ\{\Delta\}]} \ 5_a \quad \frac{\Sigma[\circ\{\Delta, \blacklozenge A\}, \blacklozenge A]}{\Sigma[\circ\{\Delta, \blacklozenge A\}]} \ 5_b \quad \frac{\Sigma[\blacklozenge A, \bullet\{\blacklozenge A, \Delta\}]}{\Sigma[\blacklozenge A, \bullet\{\Delta\}]} \ 5_c \quad \frac{\Sigma[\bullet\{\Delta, \blacklozenge A\}, \blacklozenge A]}{\Sigma[\bullet\{\Delta, \blacklozenge A\}]} \ 5_d$$

 Figure 9: Additional propagation rules for **DS5**

Definition 5.6. We denote with **SS4** the proof system obtained by adding to **SKt** the structural rules T_f and 4_f . System **DS4** denotes **DKt** plus the propagation rules given in Figure 8.

The purely modal rules of **DS4**, i.e., T_b and 4_c , coincide with Brünnler’s rules for T and 4 in [4]. The rules of **DS4** can be shown to be derivable in **SS4**.

Theorem 5.7. *For every Γ , we have $\vdash_{\mathbf{SS4}} \Gamma$ if and only if $\vdash_{\mathbf{DS4}} \Gamma$.*

As with **DKt**, if we restrict **DS4** to its purely modal fragment, we obtain a sound and complete proof system for modal logic $S4$. Let **DKS4** be **DK** extended with T_b and 4_c . The proof of the following theorem is similar to the proof of Theorem 4.7.

Theorem 5.8 (Separation). *For every modal formula A , $\vdash_{\mathbf{DKS4}} A$ iff A is a theorem of $S4$.*

5.3. A deep-inference system for modal tense logic S5. We can obtain **KtS5** from **SS4** by adding the symmetry axiom $B : A \rightarrow \Box\blacklozenge A$. The corresponding primitive form of B is $\blacklozenge A \rightarrow \blacklozenge A$, and its corresponding structural rule is

$$\frac{\Gamma, \bullet\{\Delta\}}{\Gamma, \circ\{\Delta\}} B$$

The additional propagation rules, on top of those for **DS4**, needed to absorb this structural rule and those of **SS4** are given in Figure 9.

Definition 5.9. System **SS5** is **SS4** plus the rule B . System **DS5** is **DS4** plus the propagation rules given in Figure 9.

Note that as a consequence of symmetry, the forward-looking and the backward-looking modal operators (and their structural counterparts) collapse. Hence, the propagation of diamond-formulae becomes ‘colour-blind’, i.e., \blacklozenge behaves exactly as \blacklozenge in any context. This simplifies significantly the proof of admissibility of structural rules of **SS5** in **DS5**, in particular, admissibility of B .

Theorem 5.10. *For every Γ , we have $\vdash_{\mathbf{SS5}} \Gamma$ if and only if $\vdash_{\mathbf{DS5}} \Gamma$.*

Note that **DS5** captures $S5 = KT4B$ rather than $S5 = KT45$. It is also possible to formulate deep inference rules that correspond directly to axiom 5, but one would need a form of global propagation rule (see Section 6). Again, as with **DS4**, the separation property also holds for **DS5**. Let **DKS5** be the restriction of **DS5** to the purely modal fragment.

Theorem 5.11 (Separation). *For every modal formula A , $\vdash_{\mathbf{DKS5}} A$ iff A is a theorem of $S5$.*

5.4. A deep inference system for an extension of Kt with the axiom of uniqueness. We now consider extending Kt with the axiom $CD : \diamond A \rightarrow \Box A$. Its primitive form is $\blacklozenge \lozenge A \rightarrow A$ and its corresponding structural rule is

$$\frac{\Gamma, \Delta}{\Gamma, \bullet\{\circ\{\Delta\}\}} U.$$

The propagation rules needed to absorb this structural rules are as follows:

$$\frac{\Sigma[A, \bullet\{\Gamma, \circ\{A, \Delta\}\}]}{\Sigma[A, \bullet\{\Gamma, \circ\{\Delta\}\}]} u_1 \quad \frac{\Sigma[\circ\{\Delta_1, A\}, \circ\{A, \Delta_2\}]}{\Sigma[\circ\{\Delta_1, A\}, \circ\{\Delta_2\}]} u_2 \quad \frac{\Sigma[A, \bullet\{\Gamma, \circ\{A, \Delta\}\}]}{\Sigma[\bullet\{\Gamma, \circ\{A, \Delta\}\}]} u_3$$

Definition 5.12. System **SSU** is **SKt** plus the rule U . System **DKtU** is **DKt** plus the propagation rules u_1, u_2 and u_3 .

Lemma 5.13. *Every rule of **DKtU** is derivable in **SSU**.*

Proof. Since all the rules of **DKt** are derivable in **SKt**, which is a subset of **SSU**, it is enough to show that the additional propagation rules u_1, u_2 and u_3 are derivable in **SSU**. Figure 10 shows the derivations of u_1 (the left figure) and u_2 (the right figure). The rule u_3 can be derived similarly, i.e., using u_1 and appropriate applications of residuation. In the derivation of u_1 , we use implicitly Proposition 3.2 to display nested structures, and the fact that deep inference rules \diamond_1 and \blacklozenge_1 , and the deep weakening rule are derivable in **SKt**. \square

Theorem 5.14. *For every Γ , we have $\vdash_{\mathbf{SSU}} \Gamma$ if and only if $\vdash_{\mathbf{DKtU}} \Gamma$.*

Proof. Lemma 5.13 shows one direction; it remains to show the other, i.e., that every cut-free derivation of **SSU** can be transformed into a derivation in **DKtU**. As with the case with **DS4** and **DS5**, we need to first prove admissibility of all structural rules. This can be done by straightforward induction on the height of derivations and case analyses on the last rules of the derivations. There are numerous tedious cases to consider, but none are difficult; we leave them as an exercise for the reader. \square

By restricting to the purely modal fragment of **DKtU**, we get a sound and complete proof system for modal logic $K + CD$. Let **DKU** be the modal fragment of **DKtU**, i.e., **DK** plus the rule u_2 .

Theorem 5.15 (Separation). *For every modal formula A , $\vdash_{\mathbf{DKU}} A$ iff A is a theorem of the modal logic $K + CD$.*

6. PATH AXIOMS AND GLOBAL PROPAGATION RULES

We now consider extensions of Kt with a class of axioms which we call *path axioms*. As the name suggests, these axioms can be seen as describing paths in a tree of sequents along which formulae can propagate. We show that Kt extended with path axioms can be formulated in both the shallow calculus and the deep calculus. For the latter, the formulation of the propagation rules is derived naturally from the (transitive closure of) axioms.

Before we proceed, it will be helpful to draw a distinction between a formula and a *schematic formula*. We have so far blurred this distinction when we discuss axioms (which are schematic formulae) and their instances. By a schematic formula, we mean syntactic expressions composed using logical connectives and *meta variables*. We shall denote meta variables with X, Y and Z . A formula scheme can be instantiated by substituting its meta

$$\begin{array}{c}
 \frac{\overline{A, A} \textit{id}}{\bullet\{\circ\overline{A}\}, A} U \quad \frac{\frac{\frac{\frac{\Sigma[A, \bullet\{\Gamma, \circ\{A, \Delta\}]}{\Psi, A, \bullet\{\Gamma, \circ\{\Delta, A\}}] \textit{rf}; \textit{rp}}{\Psi, A, \bullet\{\Gamma, \diamond A, \circ\{\Delta\}}] \diamond_1; \textit{wk}}{\Psi, A, \blacklozenge\{A, \bullet\{\Gamma, \circ\{\Delta\}\}}] \blacklozenge_1; \textit{wk}}{\Psi, A, A, \bullet\{\Gamma, \circ\{\Delta\}}] \textit{ctr}} \blacksquare; \square}{\Psi, A, \bullet\{\Gamma, \circ\{\Delta\}}] \textit{rf}; \textit{rp}} \textit{cut} \\
 \hline
 \frac{\Psi, A, A, \bullet\{\Gamma, \circ\{\Delta\}}}{\Psi, A, \bullet\{\Gamma, \circ\{\Delta\}}] \textit{ctr}}{\Sigma[A, \bullet\{\Gamma, \circ\{\Delta\}]} \textit{rf}; \textit{rp}}
 \end{array}
 \tag{1}$$

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{\Sigma[\circ\{\Delta_1, A\}, \circ\{A, \Delta_2\}]}{\Psi, \circ\{\Delta_1, A\}, \circ\{A, \Delta_2\}}] \textit{rp}; \textit{rf}}{\Delta_1, A, \bullet\{\Psi, \circ\{A, \Delta_2\}}] \textit{rf}}}{\Delta_1, A, \bullet\{\Psi, \circ\{\Delta_2\}}] \textit{rp}} u_1 \\
 \hline
 \frac{\Psi, \circ\{\Delta_1, A\}, \circ\{\Delta_2\}}{\Sigma[\circ\{\Delta_1, A\}, \circ\{\Delta_2\}]} \textit{rp}; \textit{rf}
 \end{array}
 \tag{2}$$

 Figure 10: Derivations of the rules u_1 and u_2 .

variables with (concrete) formulae or other formulae schemes. By axioms, we usually mean schematic formulae whose (concrete) instances are admitted as theorems of the logic. In the following, we shall make explicit this distinction between formulae and schematic formulae. We shall also use the notation $\langle ? \rangle$ (possibly with subscripts) to denote a diamond-operator of either color, and $[?]$ to denote its de Morgan dual.

Definition 6.1. A *path axiom* is a schematic formula for the form $\langle ? \rangle_1 \cdots \langle ? \rangle_n X \rightarrow \langle ? \rangle X$ where $n \geq 0$, and each of $\{\langle ? \rangle, \langle ? \rangle_1, \dots, \langle ? \rangle_n\}$ is either a \diamond , or a \blacklozenge .

The class of path axioms includes any instance of primitive Scott-Lemmon axiom $P(h, i, j, k)$ where $i + k = 1$. By Lemma 5.2, these are equivalent to the following instances of Scott-Lemmon axioms:

$$\diamond^h \square^j X \rightarrow \square^j X \quad \diamond^h X \rightarrow \square^j \diamond X.$$

Hence, it subsumes most standard axioms such as reflexivity ($\square X \rightarrow X$), transitivity ($\diamond \diamond X \rightarrow \diamond X$), symmetry ($X \rightarrow \square \diamond X$), and euclideaness ($\diamond X \rightarrow \square \diamond X$).

To each path axiom, $\langle ? \rangle_1 \cdots \langle ? \rangle_n X \rightarrow \langle ? \rangle X$, we define a corresponding structural rule as shown below left, where \star is the structural connective for $[?]$ and each \star_i is the structural connective for $[?]_i$. For example, the structural rule for the axiom $\diamond \blacklozenge \diamond X \rightarrow \diamond X$ is as given below right.

$$\frac{\Gamma, \star\{\Delta\}}{\Gamma, \star_1\{\cdots \star_n\{\Delta\}\cdots\}} \rho \quad \frac{\Gamma, \circ\{\Delta\}}{\Gamma, \circ\{\bullet\{\circ\{\Delta\}\}\}}.$$

Given a set of axioms \mathbf{P} , we denote with $\rho(\mathbf{P})$ the set of structural rules corresponding to axioms in \mathbf{P} . As with Scott-Lemmon axioms, axiomatic and structural extensions of \mathbf{SKt} with path axioms are equivalent. The proof of the following proposition is similar to the proof of Proposition 5.4.

Proposition 6.2. *For any set \mathbf{P} of path axioms, the proof systems \mathbf{SKtAxP} and \mathbf{SKtP} are equivalent.*

As a corollary of Theorem 3.16, cut elimination holds for \mathbf{SKtP} .

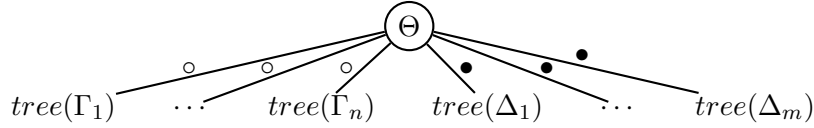
Theorem 6.3. *Cut elimination holds for \mathbf{SKtP} , for any set \mathbf{P} of path axioms.*

6.1. Propagation rules for path axioms. A straightforward way to incorporate a path axiom, say, $\diamond\diamond X \rightarrow \diamond X$ in the deep inference system **DKt** is to simply use it as a rule, by replacing $\diamond\diamond X$ with $\diamond X$ (reading the rule top down), i.e.,

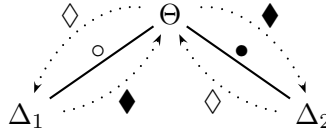
$$\frac{\Sigma[\diamond\diamond X]}{\Sigma[\diamond X]}.$$

Despite its appealing simplicity, adding such a rule will destroy the subformula property, and as our main goal is to design proof-search friendly calculi, such an introduction rule must be ruled out. What we propose here is essentially the same, but instead of putting the formula $\diamond\diamond X$ in the premise, we consider all its possible interactions with the surrounding context ($\Sigma[\]$) to decompose it to X . This would involve propagating X to different subcontexts in $\Sigma[\]$, depending on the axiom. The main challenge here is then to design a sound and complete set of propagation rules for the axiom.

To understand the intuitive idea behind propagation rules for path axioms, it is helpful to view a nested sequent as a tree of traditional sequents. Following Kashima [17], we define a mapping from sequents to trees as follows. A *node* is a multiset of formulae. A *tree* is a node with 0 or more children, where each child is a tree, and each child is labelled as either a \circ -child, or a \bullet -child. Given a sequent $\Xi = \Theta, \circ\{\Gamma_1\}, \dots, \circ\{\Gamma_n\}, \bullet\{\Delta_1\}, \dots, \bullet\{\Delta_m\}$, where Θ is a multiset of formulae and $n \geq 0$ and $m \geq 0$, the tree $tree(\Xi)$ represented by Ξ is:



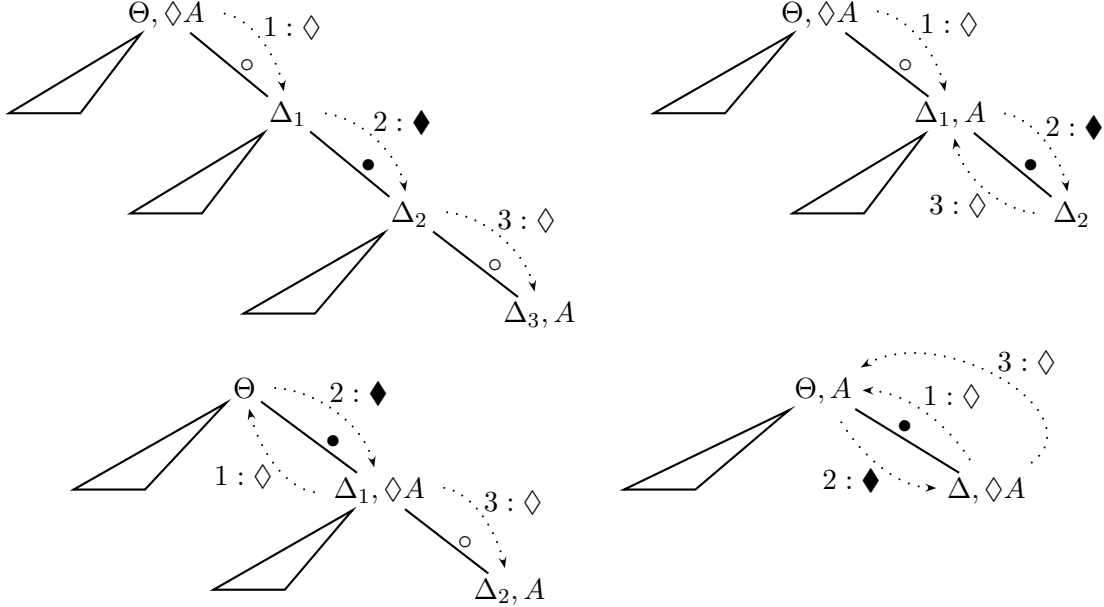
In **DKt**, a \diamond - or a \blacklozenge -prefixed formula can navigate up and down a sequent tree, depending on where it is positioned in the tree. The rule \diamond_1 allows a formula $\diamond A$ to propagate its subformula A down the tree along an edge labelled by \circ , and \diamond_2 allows the same formula to propagate A up the tree along an edge labelled by \bullet . Similarly, \blacklozenge_1 allows $\blacklozenge A$ to propagate A down an \bullet -edge and \blacklozenge_2 allows it to propagate A up an \circ -edge. Graphically, one can represent these movements by assigning two kinds of diamond-labelled directed edges to each edge in a sequent tree, which encode the kinds of diamond-prefixed formulae that can propagate along the directed edges. The four movements mentioned previously can thus be represented as the dotted lines in the following graph:



For example, the “diamond paths” from the node labelled by Δ_1 to Δ_2 characterise the diamond prefixes needed to propagate a formula from Δ_1 to Δ_2 ; they include formulae such as $\blacklozenge\blacklozenge A$ (one goes up to the root and then down to Δ_2), or $\blacklozenge\blacklozenge\blacklozenge\blacklozenge A$ (i.e., one does a “loop” from Δ_1 to Θ and back to Δ_1 , before proceeding to Δ_2), etc.

In proof search, a path axiom such as $\diamond\diamond X \rightarrow \diamond X$ can be read as an instruction for propagating a formula $\diamond A$: replace $\diamond A$ with $\diamond\diamond A$ and propagate along the diamond path $\diamond\diamond$. Depending on where the formula $\diamond A$ is located in a sequent tree, there are several possible moves that correspond to the path $\diamond\diamond$. Some of these are given in Figure 11.

In designing the propagation rules for a set of path axioms, in order to get completeness, one needs to take into account two things: arbitrary compositions of the axioms and their


 Figure 11: Some propagation scenarios for axiom $\diamond\diamond\diamond A \rightarrow \diamond A$.

interactions with the residuation axioms. An axiom such as $\diamond\diamond\diamond X \rightarrow \diamond X$ not only specifies a set of possible propagations for $\diamond A$, but also specifies, via residuation, propagations for $\blacklozenge A$. It is easy in this case to show that $\blacklozenge\diamond\diamond X \rightarrow \blacklozenge X$ is a consequence of that axiom.

In the following, when $\langle ? \rangle$ denotes an diamond operator (\blacklozenge or \diamond), $\langle ? \rangle^{-1}$ denotes its tense or modal counterpart. That is, if $\langle ? \rangle = \diamond$ then $\langle ? \rangle^{-1}$ denotes \blacklozenge and vice versa.

Definition 6.4. Let F be the path axiom $\langle ? \rangle_1 \cdots \langle ? \rangle_n X \rightarrow \langle ? \rangle X$. The *inverted version* of F , denoted by $I(F)$, is the schematic formula $\langle ? \rangle_n^{-1} \cdots \langle ? \rangle_1^{-1} X \rightarrow \langle ? \rangle^{-1} X$.

Obviously, we have $I(I(F)) = F$. A path axiom can be shown equivalent to its inverted version.

Lemma 6.5. *Let F be a path axiom. Then F is equivalent to $I(F)$.*

Proof. Since $I(I(F)) = F$ and $I(F)$ itself is a path axiom, it is enough to show one direction, i.e., F implies $I(F)$. We first note that the following are theorems of tense logic (they are, in fact, the axioms of residuation):

$$X \rightarrow \square\blacklozenge X \quad X \rightarrow \blacksquare\diamond X.$$

There are two cases to consider:

- $F = \langle ? \rangle_1 \cdots \langle ? \rangle_n X \rightarrow \diamond X$. Then $I(F) = \langle ? \rangle_n^{-1} \cdots \langle ? \rangle_1^{-1} X \rightarrow \blacklozenge X$. By contraposition, we have that F implies $\square X \rightarrow [\?]_1 \cdots [\?]_n X$. By instantiating this axiom scheme with $\blacklozenge X$, we have $\square\blacklozenge X \rightarrow [\?]_1 \cdots [\?]_n \blacklozenge X$. Since $X \rightarrow \square\blacklozenge X$, we also have $X \rightarrow [\?]_1 \cdots [\?]_n \blacklozenge X$. Note that since $[\?]_i$ is the de Morgan dual of $\langle ? \rangle_i$, its residual must be $\langle ? \rangle_i^{-1}$. Therefore, by residuation, we have

$$\langle ? \rangle_n^{-1} \cdots \langle ? \rangle_1^{-1} X \rightarrow \blacklozenge X.$$

- $F = \langle ? \rangle_1 \cdots \langle ? \rangle_n X \rightarrow \blacklozenge X$. This is similar to the previous case, except that we compose with the axiom $X \rightarrow \blacksquare\diamond X$. □

Definition 6.6. Let F and G be the following path axioms:

$$\langle ? \rangle_{F_1} \cdots \langle ? \rangle_{F_m} X \rightarrow \langle ? \rangle_F X \qquad \langle ? \rangle_{G_1} \cdots \langle ? \rangle_{G_n} X \rightarrow \langle ? \rangle_G X.$$

F is said to be *composable with G at position i* if $\langle ? \rangle_F = \langle ? \rangle_{G_i}$. We denote by $F \triangleright^i G$ the composition of F with G at i , i.e., the formula:

$$\langle ? \rangle_{G_1} \cdots \langle ? \rangle_{G_{i-1}} \langle ? \rangle_{F_1} \cdots \langle ? \rangle_{F_m} \langle ? \rangle_{G_{i+1}} \cdots \langle ? \rangle_{G_n} \rightarrow \langle ? \rangle_G X.$$

We say that F is *composable with G* if F is composable with G at some position i . We denote with $F \triangleright G$ the set of all compositions of F with G , i.e.,

$$F \triangleright G = \{F \triangleright^i G \mid F \text{ composable with } G \text{ at } i\}.$$

Notice that composition of axioms are basically just modus ponens, so the compositions of F and G are obviously logical consequences of F and G .

Lemma 6.7. *If F is composable with G at i , then $F \triangleright^i G$ is a logical consequence of F and G .*

Definition 6.8. Let \mathbf{P} be a set of path axioms. The *completion of \mathbf{P}* , written \mathbf{P}^* , is the smallest set of path axioms containing \mathbf{P} and satisfying the following conditions:

- (1) It contains the identity axioms $\diamond X \rightarrow \diamond X$ and $\blacklozenge X \rightarrow \blacklozenge X$.
- (2) It is closed under *composition*, i.e., if $F, G \in \mathbf{P}^*$ and F is composable with G , then $F \triangleright G \subseteq \mathbf{P}^*$.

Alternatively, we can characterise \mathbf{P}^* via a monotone operator:

$$\mathcal{C}(S) = \bigcup \{F \triangleright G \mid F, G \in S \text{ and } F \text{ is composable with } G\}.$$

Now define an n -th iteration of \mathcal{C} as follows:

$$\begin{aligned} \mathcal{C}^0(S) &= \emptyset \\ \mathcal{C}^{n+1}(S) &= S \cup \mathcal{C}(\mathcal{C}^n(S)). \end{aligned}$$

Then it can be shown that (see [1])

$$\mathbf{P}^* = \bigcup_{n < \omega} \mathcal{C}^n(\mathbf{P} \cup \{\diamond X \rightarrow \diamond X, \blacklozenge X \rightarrow \blacklozenge X\}).$$

That is, every element of the set \mathbf{P}^* can be obtained via a finite number of compositions using axioms in the set $\mathbf{P} \cup \{\diamond X \rightarrow \diamond X, \blacklozenge X \rightarrow \blacklozenge X\}$. We shall use this fact in the proofs involving the completion of \mathbf{P} .

In the following, we lift the operator I to a set of axioms, i.e., $I(\mathbf{P}) = \{I(F) \mid F \in \mathbf{P}\}$.

Lemma 6.9. *Let \mathbf{P} be a set of path axioms. If $I(\mathbf{P}) \subseteq \mathbf{P}$ then for every $F \in \mathbf{P}^*$ we have $I(F) \in \mathbf{P}^*$.*

Proof. By induction on the formation of the set \mathbf{P}^* and Definition 6.6. □

To define the propagation rules, we need to define the notion of a path between two nodes in a tree. This is given in the following.

Definition 6.10. Let Γ be a nested sequent, and let N be the set of nodes of $tree(\Gamma)$. The *propagation graph* $PG(\Gamma)$ for Γ is a directed graph such that the set of nodes of $PG(\Gamma)$ is N , its edges are labelled with \blacklozenge or \blacklozenge and are defined as follows:

- For each node $n \in N$, and each \circ -child n_1 of n , there is exactly one edge (n, n_1) labelled with \blacklozenge , and exactly one edge (n_1, n) labelled with \blacklozenge .
- For each node $n \in N$, and each \bullet -child n_1 of n , there is exactly one edge (n, n_1) labelled with \blacklozenge , and exactly one edge (n_1, n) labelled with \blacklozenge .

A *labelled path* (or simply, a path) in a propagation graph is defined as usual, i.e., as a sequence of nodes and diamonds, separated by semicolons,

$$n_1; \langle ? \rangle_1; n_2; \langle ? \rangle_2; \cdots; n_{k-1}; \langle ? \rangle_{k-1}; n_k$$

such that each (n_i, n_{i+1}) is a $\langle ? \rangle_i$ -labelled edge in $PG(\Gamma)$. We use π to range over paths in a propagation graph. If π is a path then $\langle \pi \rangle$ denotes the sequence of labels (i.e., \blacklozenge or \blacklozenge) that occur along that path.

We are now ready to define the set of propagation rules for a set of axioms. But first we introduce a notational convention for writing contexts. Note that since a context is just a structure with a hole \square in place of a formula, it also has a tree representation. In a single-hole context, the hole \square occupies a unique node in the tree. We shall write $\Sigma[\square]_i$ when we want to be explicit about the particular node i where the hole is located. This notation extends to multiple-hole contexts, e.g., $\Sigma[\square]_i[\square]_j$ denotes a two-hole context where the first hole is located at node i and the second at node j in $tree(\Sigma[\square]_i[\square]_j)$.

Definition 6.11. Let \mathbf{P} be a set of path axioms. The set of *propagation rules for \mathbf{P}* , written $Prop(\mathbf{P})$, consists of rules of the form:

$$\frac{\Sigma[\langle ? \rangle A]_i[A]_j}{\Sigma[\langle ? \rangle A]_i[\emptyset]_j}$$

if there is a path π from i to j in $PG(\Gamma)$ such that $\langle \pi \rangle X \rightarrow \langle ? \rangle X \in (\mathbf{P} \cup I(\mathbf{P}))^*$.

We denote with \mathbf{SKtP} the structural extension of \mathbf{SKt} with \mathbf{P} and \mathbf{DKtP} the extension of \mathbf{DKt} with propagation rules $Prop(\mathbf{P})$.

Notice that by definition, the rule \blacklozenge_1 , \blacklozenge_2 , \blacklozenge_1 and \blacklozenge_2 are just instances of propagation rules, i.e., they are propagation rules for the identity axiom $\blacklozenge X \rightarrow \blacklozenge X$ and $\blacklozenge X \rightarrow \blacklozenge X$. So in the following proofs, we do not explicitly do case analyses on instances of these rules, as they are subsumed by the more general cases involving the propagation rules.

Lemma 6.12. *For any set of path axioms \mathbf{P} and any structure Γ , if $\vdash_{\mathbf{DKtP}} \Gamma$ then $\vdash_{\mathbf{SKtP}} \Gamma$.*

Proof. Since \mathbf{SKt} is a subset of \mathbf{SKtP} , derivations of \mathbf{DKt} rules in \mathbf{SKtP} are done as in Theorem 4.6. It remains to show derivations of the propagation rules. It is enough to show that each instance of each axiom in $(\mathbf{P} \cup I(\mathbf{P}))^*$ is derivable in \mathbf{SKtP} . This in effect would allow us to derive the following rule (via cut and Proposition 3.2):

$$\frac{\Sigma[\langle ? \rangle_1 \cdots \langle ? \rangle_n A]}{\Sigma[\langle ? \rangle A]}$$

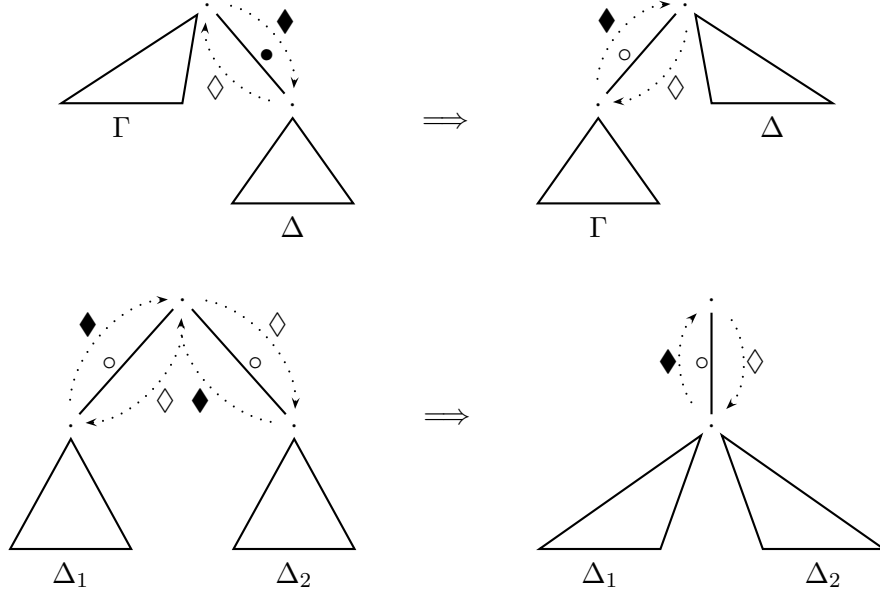


Figure 12: Preservation of propagation paths in residuation and medial rules

for each axiom $\langle ? \rangle_1 \cdots \langle ? \rangle_n A \rightarrow \langle ? \rangle A$, which would then allow us to mimick the propagation rule for that axiom. The derivation of the axioms of $(\mathbf{P} \cup I(\mathbf{P}))^*$ follows straightforwardly from Lemma 6.5, Lemma 6.7 Definition 6.8 and Lemma 6.9. \square

Lemma 6.13. *The rules dw , dgc , rp and rf are height-preserving admissible in \mathbf{DKtP} , for any set of path axioms \mathbf{P} .*

Proof. As height-preserving admissibility of these rules have been proved for \mathbf{DKt} , the new cases are those that interact with the propagation rules in $Prop(\mathbf{P})$. That is, we need to prove these for the cases where the derivation of the premise of the rules, say Π , ends with a propagation rule:

$$\frac{\Pi_1 \quad \Sigma[\langle ? \rangle A]_i[A]_j}{\Sigma[\langle ? \rangle A]_i[\emptyset]_j}.$$

Since the propagation rule only requires the existence of a path between node i and j , it is sufficient to show that a path still exists between those nodes in the modified structure. This is trivial for weakening. For the residuation rule rp (the case with rf is similar), suppose that Π is a derivation of the premise of rp , i.e.,

$$\Sigma[\langle ? \rangle A]_i[\emptyset]_j = \Gamma, \bullet\{\Delta\}$$

for some Γ and Δ . We need to show that there exists a derivation Π' of $\circ\{\Gamma\}, \Delta$. It is enough to show that the propagation graph of $\circ\{\Gamma\}, \Delta$ is identical to the propagation graph of $\Gamma, \bullet\{\Delta\}$; hence any path that exists in the latter also exists in the former, and therefore any propagation that applies to the latter also applies to the former. The fact that the propagation graphs of both structures coincide can be easily seen in the graphs in the upper row in Figure 12: the only change caused by residuation is confined to the root of

the sequent tree, so one needs only to check that the nodes affected by these changes are still connected with the same labelled edges.

To prove admissibility of dgc , as with the case of **DKt**, we need to prove admissibility of formula contraction dfc and the medial rules mf and mp , as in Lemma 4.4.

Admissibility of dfc can be proved by a simple induction on the height of derivation. We show here a proof of admissibility of mf ; admissibility of mp can be proved similarly. So suppose $\vdash_{\mathbf{DKtP}} \Pi : \Sigma[\circ\{\Delta_1\}, \circ\{\Delta_2\}]$. We need to show that there exists Π' such that $\vdash_{\mathbf{DKtP}} \Pi' : \Sigma[\circ\{\Delta_1, \Delta_2\}]$ and $|\Pi'| = |\Pi|$. The proof in this case is similar to the proof of the admissibility of residuation: one shows that the modified structure still preserves the existence of a path between two nodes where propagation happens. Since the differences between $tree(\circ\{\Delta_1\}, \circ\{\Delta_2\})$ and $tree(\circ\{\Delta_1, \Delta_2\})$ are confined to the top three nodes in the trees (see the graphs in the lower row of Figure 12), we need only to show that labelled edges between the top three nodes in the propagation graph for $\circ\{\Delta_1\}, \circ\{\Delta_2\}$ are preserved in their corresponding nodes in the propagation graph for $\circ\{\Delta_1, \Delta_2\}$. This is shown in the graphs in the lower row in Figure 12. \square

Lemma 6.14. *Let \mathbf{P} be a set of path axioms. Every structural rule in $\rho(\mathbf{P})$ is admissible in **DKtP**.*

Proof. Let $\langle ? \rangle_1 \cdots \langle ? \rangle_k X \rightarrow \langle ? \rangle X$ be an axiom in \mathbf{P} and let ρ be its corresponding structural rule:

$$\frac{\Gamma, \star\{\Delta\}}{\Gamma, \star_1\{\cdots \star_k \{\Delta\} \cdots\}} \rho$$

Let Π be a **DKtP**-derivation of $\Gamma, \star\{\Delta\}$. We show by induction on the height of Π that there exists a **DKtP**-derivation Π' of $\Gamma, \star_1\{\cdots \star_k \{\Delta\} \cdots\}$. Let n_1 denote the root node of the tree $tree(\Gamma, \star\{\Delta\})$ and let n_2 denote its child that is the root of its subtree Δ . So graphically, the nested sequent $\Gamma, \star\{\Delta\}$ can be represented schematically as the tree on the left in Figure 13. The tree for $\Gamma, \star_1\{\cdots \star_k \{\Delta\} \cdots\}$ replaces the node n_2 in $tree(\Gamma, \star\{\Delta\})$ with k new nodes. As k could be 0, node n_1 and node n_2 could possibly be identified in the conclusion of the rule ρ . The only interesting cases are when Π ends with a propagation rule that propagates a $\langle ? \rangle A$ formula across node n_1 to n_2 or the reverse. So suppose ρ propagates a $\langle ? \rangle A$ formula along the following path:

$$\pi_1; n_1; \langle ? \rangle; n_2; \pi_2.$$

This means that $\langle \pi_1 \rangle \langle ? \rangle \langle \pi_2 \rangle X \rightarrow \langle ? \rangle X$ is a member of $(\mathbf{P} \cup I(\mathbf{P}))^*$. Since the set $(\mathbf{P} \cup I(\mathbf{P}))^*$ is closed under axiom composition, we also have that $\langle \pi_1 \rangle \langle ? \rangle_1 \cdots \langle ? \rangle_k \langle \pi_2 \rangle X \rightarrow \langle ? \rangle X$. The latter implies that the following

$$\pi_1; n_1; \langle ? \rangle_1; \cdots; \langle ? \rangle_k; n_2; \pi_2$$

is a path in the propagation graph of $\Gamma, \star_1\{\cdots \star_k \{\Delta\} \cdots\}$, so the propagation of $\langle ? \rangle A$ that applies to $\Gamma, \star\{\Delta\}$ can also be applied to $\Gamma, \star_1\{\cdots \star_k \{\Delta\} \cdots\}$. The other case where the propagation passes from n_2 to n_1 can be proved symmetrically, since the set $(\mathbf{P} \cup I(\mathbf{P}))^*$ is closed under residuation. This is represented graphically in Figure 13. \square

Theorem 6.15. *For any set of path axioms \mathbf{P} and any nested sequent Γ , $\vdash_{\mathbf{SKtP}} \Gamma$ if and only if $\vdash_{\mathbf{DKtP}} \Gamma$.*

Proof. This follows from Lemma 6.12, Lemma 6.13 and Lemma 6.14. \square

Each path axiom $\langle ? \rangle_1 \cdots \langle ? \rangle_n X \rightarrow \langle ? \rangle_{n+1} X$ defines a production rule as follows:

$$C(\langle ? \rangle_{n+1}) \rightarrow C(\langle ? \rangle_1) \cdots C(\langle ? \rangle_n).$$

If A is a path axiom, we write $G(A)$ to denote its associated production rule defined as above. For example, the axiom $\diamond \diamond X \rightarrow \diamond X$ defines the production rule $F \rightarrow FPF$.

We recall that a context-free grammar is defined by a tuple (N, T, Pr, S) of a set of non-terminal symbols N , a set of terminal symbol T , a set of production rules Pr , and a start symbol $S \in N$. We shall write $F \rightarrow^* s$ to denote a derivation of the sequence s of symbols from the symbol F .

Definition 6.17. Let \mathbf{P} be a finite set of path axioms. Define two context-free grammars generated from \mathbf{P} as follows:

- (1) Let $L_\diamond(\mathbf{P})$ be the grammar $(\{F, P\}, \{\diamond, \blacklozenge\}, Pr, F)$ where Pr is the smallest set of production rules such that:
 - (a) $F \rightarrow \diamond$ and $P \rightarrow \blacklozenge$ are in Pr .
 - (b) For each axiom in $A \in \mathbf{P} \cup I(\mathbf{P})$, $G(A) \in Pr$.
- (2) Let $L_\blacklozenge(\mathbf{P})$ be the same grammar as $L_\diamond(\mathbf{P})$ except that the start symbol is P instead of F .

The following lemma follows immediately from Definition 6.17.

Lemma 6.18. *Let \mathbf{P} be a finite set of path axioms. Then $\langle ? \rangle_1 \cdots \langle ? \rangle_n X \rightarrow \diamond X \in (\mathbf{P} \cup I(\mathbf{P}))^*$ if and only if $\langle ? \rangle_1 \cdots \langle ? \rangle_n \in L_\diamond(\mathbf{P})$. Similarly, $\langle ? \rangle_1 \cdots \langle ? \rangle_n X \rightarrow \blacklozenge X \in (\mathbf{P} \cup I(\mathbf{P}))^*$ if and only if $\langle ? \rangle_1 \cdots \langle ? \rangle_n \in L_\blacklozenge(\mathbf{P})$.*

Note that the propagation graph of a nested sequent can be seen as essentially a finite state automaton, minus the initial and final states.

Definition 6.19. Let Γ be a nested sequent, and let n_1 and n_2 be two nodes in $tree(\Gamma)$. The (n_1, n_2) -path automaton of Γ , written $Path(\Gamma, n_1, n_2)$, is the directed graph $PG(\Gamma)$ with starting state n_1 and final state n_2 .

Lemma 6.20. *Let Γ be a nested sequent and let n_1 and n_2 be two nodes in $tree(\Gamma)$. Then for every π , π is a path from n_1 to n_2 if and only if $\langle \pi \rangle \in Path(\Gamma, n_1, n_2)$.*

Theorem 6.21. *Let \mathbf{P} be a finite set of path axioms and let Γ be $\Sigma[\langle ? \rangle A]_i[\emptyset]_j$. Then a formula occurrence $\diamond A$ at node i can be propagated to node j in the proof system \mathbf{DKtP} if and only if $L_\diamond(\mathbf{P}) \cap Path(\Gamma, i, j) \neq \emptyset$. Similarly, a formula occurrence $\blacklozenge A$ at node i can be propagated to node j in the proof system \mathbf{DKtP} if and only if $L_\blacklozenge(\mathbf{P}) \cap Path(\Gamma, i, j) \neq \emptyset$.*

Proof. Straightforward from Lemma 6.18 and Lemma 6.20. □

Theorem 6.22. *Let \mathbf{P} be a finite set of path axioms. Let Γ be a nested sequent. The problem of checking whether there is a propagation rule in \mathbf{DKtP} that is (bottom-up) applicable to Γ is decidable. Moreover, assuming \mathbf{P} is fixed, the complexity of the decision problem is PTIME in the size of Γ .*

Proof. By Theorem 6.21, this decision problem reduces to the problem of checking emptiness of the intersection of a regular language and a context-free language, which is itself a context-free language (see [11], Chapter 3). Let \mathcal{A} be the finite state automaton encoding paths in Γ and let n be its size. Let \mathcal{G} be the context free grammar generated from the axiom \mathbf{P} (i.e., it is either $L_\diamond(\mathbf{P})$ or $L_\blacklozenge(\mathbf{P})$). In [11], the intersection of \mathcal{A} and \mathcal{G} is done by

constructing another context-free grammar \mathcal{G}' . More specifically, for each production rule of \mathcal{G} , say $V \rightarrow \alpha_1\alpha_2 \cdots \alpha_m$, where V is a non-terminal of \mathcal{G} and α_i is either a terminal or a non-terminal of \mathcal{G} , one constructs n^{m+1} production rules of the same length for \mathcal{G}' . So the size of \mathcal{G}' is bounded by $O(l \times k \times n^{k+1})$ where l is the number of production rules in \mathcal{G} and k is the maximum length of the production rules of \mathcal{G} . Since the construction of each production rule of \mathcal{G}' from a production rule of \mathcal{G} length m takes $O(m)$ -time, the time complexity of the construction of \mathcal{G}' is also bounded by $O(l \times k \times n^{k+1})$. See [11] for the details of the construction of \mathcal{G}' . If we assume that \mathbf{P} is fixed, then obviously l and k are constants, the grammar \mathcal{G}' is computable in PTIME in the size of Γ , and its size is also polynomial in the size of Γ . Since emptiness checking of a context-free language is decidable in PTIME (see e.g., [22]), it follows that the problem of checking the applicability of the propagation rules is also decidable in PTIME. \square

In some cases, the propagation rules for a given set of axioms can be characterised by simple regular expressions. We give some examples below. In the following, we shall use the symbols $+$ and $*$ to denote the union operation and the Kleene-star operations on regular languages. We shall be concerned only with regular languages generated by the alphabets $\{\diamond, \blacklozenge\}$.

Example 6.23. *Transitivity.* Consider the case where $\mathbf{P} = \{\diamond\diamond X \rightarrow \diamond X\}$. It is easy to see that in this case, we have $L_\diamond(\mathbf{P}) = \diamond\diamond^*$ and $L_\blacklozenge(\mathbf{P}) = \blacklozenge\blacklozenge^*$. If one adds the axiom of reflexivity, then we get the logic *KtS4* and the propagation paths are characterised by $L_\diamond(\mathbf{P}) = \diamond^*$ and $L_\blacklozenge(\mathbf{P}) = \blacklozenge^*$. In other words, the propagation rules for *KtS4* are characterised by movements along paths of diamonds of arbitrary length and of the same color.

Example 6.24. *Euclideaness.* Consider the case of where $\mathbf{P} = \{\blacklozenge\diamond X \rightarrow \diamond X\}$. Note that the inverted version of the (primitive form) of the axiom 5 is $\blacklozenge\diamond X \rightarrow \blacklozenge X$. We claim that the paths allowed by \mathbf{P} can be characterised as follows:

$$L_\diamond(\mathbf{P}) = \diamond + (\blacklozenge(\blacklozenge + \diamond)^*\diamond) \quad L_\blacklozenge(\mathbf{P}) = \blacklozenge + (\blacklozenge(\blacklozenge + \diamond)^*\diamond).$$

We prove this claim for the characterisation of $L_\diamond(\mathbf{P})$, the other case is similar. First, we show that $L_\diamond(\mathbf{P}) \subseteq \diamond + (\blacklozenge(\blacklozenge + \diamond)^*\diamond)$. By definition, the production rules of $L_\diamond(\mathbf{P})$ are

$$F \rightarrow PF, P \rightarrow PF, F \rightarrow \diamond, \text{ and } P \rightarrow \blacklozenge.$$

It is clear that members of $L_\diamond(\mathbf{P})$ are either of the form \diamond or $\blacklozenge s \diamond$, for some sequence of diamonds s . But obviously, $s \in (\blacklozenge + \diamond)^*$, so we indeed have $L_\diamond(\mathbf{P}) \subseteq \diamond + (\blacklozenge(\blacklozenge + \diamond)^*\diamond)$. For the other direction, suppose that $s \in \diamond + (\blacklozenge(\blacklozenge + \diamond)^*\diamond)$. We show by induction on the length of s that $s \in L_\diamond(\mathbf{P})$. The case where $s = \diamond$ is trivial. So suppose $s = \blacklozenge s' \diamond$ for some $s' \in (\blacklozenge + \diamond)^*$. The case where s' is the empty string is trivial; there remain two cases to consider:

- $s' = \diamond t$ for some t . By the induction hypothesis, we have that $\blacklozenge t \diamond \in L_\diamond(\mathbf{P})$. Note that the first \blacklozenge in this sequence can only be a result of the production rule $P \rightarrow \blacklozenge$, so we have that $F \rightarrow^* Pt \diamond \rightarrow \blacklozenge t \diamond$. Now, the sequence s is then generated as follows:

$$F \rightarrow^* Pt \diamond \rightarrow PFt \diamond \rightarrow P \diamond t \diamond \rightarrow \blacklozenge \diamond t \diamond = s.$$

- $s' = \blacklozenge t$ for some t . By the induction hypothesis, we have $\blacklozenge t \diamond \in L_\diamond(\mathbf{P})$, that is, we have $F \rightarrow^* \blacklozenge t \diamond$. The sequence s is then derived as follows:

$$F \rightarrow PF \rightarrow^* P \blacklozenge t \diamond \rightarrow \blacklozenge \blacklozenge t \diamond = s.$$

- Function Prove (Sequent Ξ) : Bool
- (1) Let $T = tree(\Xi)$
 - (2) If the *id* rule is applicable to any node in T , return *True*
 - (3) Else if there is some node $\Theta \in T$ that is not saturated
 - (a) If $A \vee B \in \Theta$ and $A \notin \Theta$ or $B \notin \Theta$ then let Ξ_1 be the premise of the \vee rule applied to $A \vee B \in \Theta$. Return $Prove(\Xi_1)$.
 - (b) If $A \wedge B \in \Theta$ and $A \notin \Theta$ and $B \notin \Theta$ then let Ξ_1 and Ξ_2 be the premises of the \wedge rule applied to $A \wedge B \in \Theta$. Return *True* iff $Prove(\Xi_1) = True$ and $Prove(\Xi_2) = True$.
 - (4) Else if there is some node $\Theta \in T$ that is not realised, i.e. some $B = \Box A$ ($B = \blacksquare A$) is not realised
 - (a) Let Ξ_1 be the premise of the \Box (\blacksquare) rule applied to $B \in \Theta$. Return $Prove(\Xi_1)$.
 - (5) Else if there is some node Θ that is not propagated
 - (a) Let ρ be the rule corresponding to the requirement of Definition 7.3 that is not met, and let Ξ_1 be the premise of ρ . Return $Prove(\Xi_1)$.
 - (6) Else return *False*

Figure 14: Proof search strategy for **DKt**

The above characterisation of the propagation rules for axiom 5 basically says that a formula such as $\diamond A$ can be propagated along paths of the following form: it is either \diamond , or it must start with \blacklozenge , followed by any path of arbitrary length, and end with \diamond . Using this characterisation, one can replace the generic propagation rule for \diamond -formulae in $Prop(\mathbf{P})$ (see Definition 6.11), with more specific rules in the following (in addition to the rules \diamond_1 and \diamond_2 in Figure 7):

$$\frac{\Sigma[\circ\{\diamond A, \Delta\}][\circ\{\Gamma, A\}]}{\Sigma[\circ\{\diamond A, \Delta\}][\circ\{\Gamma\}]} p5_a \quad \frac{\Sigma[\circ\{\diamond A, \Delta\}][\bullet\{\Gamma\}, A]}{\Sigma[\circ\{\diamond A, \Delta\}][\bullet\{\Gamma\}]} p5_b$$

$$\frac{\Sigma[\diamond A, \bullet\{\Delta\}][\circ\{\Gamma, A\}]}{\Sigma[\diamond A, \bullet\{\Delta\}][\circ\{\Gamma\}]} p5_c \quad \frac{\Sigma[\diamond A, \bullet\{\Delta\}][\bullet\{\Gamma\}, A]}{\Sigma[\diamond A, \bullet\{\Delta\}][\bullet\{\Gamma\}]} p5_d$$

In the purely modal setting, the propagation rule $p5_a$ in Example 6.24 above is similar to that considered by Brünnler [7]:²

$$\frac{\Sigma[\circ\{\Gamma\}][\diamond A]}{\Sigma[\circ\{\Gamma, \diamond A\}][\emptyset]} \overset{\diamond}{5}$$

But notice that, unlike our propagation rules, Brünnler's rule allows propagation of $\diamond A$ without introducing the connective \diamond .

Example 6.25. *S5.* If one adds the axiom of reflexivity $X \rightarrow \diamond X$ to the set \mathbf{P} in the previous example, one gets the logic *S5*. In this case, the propagation rules admit a very simple characterisation: the formula $\diamond A$ (likewise, $\blacklozenge A$) in a node u in a tree of sequents can be propagated to *any* node in the tree, i.e., we have $L_{\diamond}(\mathbf{P}) = L_{\blacklozenge}(\mathbf{P}) = (\diamond + \blacklozenge)^*$.

7. PROOF SEARCH IN **DKt**

We now present a preliminary result in proof search for **DKt**. This section is meant to serve as a preview of our planned future work in designing more general proof search strategies for a wide range of deep inference calculi discussed in the previous section.

We shall be working with the tree representation of nested sequents as discussed in the previous section. However, since contraction is admissible in **DKt** and its extensions discussed so far, we shall consider a node as a set rather than a multiset. While traditional tableaux methods operate on a single node at a time, our proof search strategies will consider the whole tree. Our proof search strategy is based on a saturation procedure familiar from the tableaux setting. In the following, given a tree T of sequents and a node u in T , we denote with $S(u)$ the set of formulae at node u .

Definition 7.1. A set of formulae Θ is *saturated* iff it satisfies:

- (1) If $A \vee B \in \Theta$ then $A \in \Theta$ and $B \in \Theta$.
- (2) If $A \wedge B \in \Theta$ then $A \in \Theta$ or $B \in \Theta$.
- (3) For every propositional variable p , $p \in \Theta$ implies $\neg p \notin \Theta$, and $\neg p \in \Theta$ implies $p \notin \Theta$.

A node u in a tree T is *saturated* iff $S(u)$ is saturated.

Definition 7.2. Given a tree T and a node u in T , a formula $\Box A \in S(u)$ ($\blacksquare A \in S(u)$) is *realised* iff there exists a \circ -child (\bullet -child) v of u in T with $A \in S(v)$.

Definition 7.3. Given a tree T and a node u in T , we say u is *propagated* iff:

- \diamond_1 :: for every $\diamond A \in S(u)$ and for every \circ -child v of u , we have $A \in S(v)$;
- \blacklozenge_1 :: for every $\blacklozenge A \in S(u)$ and for every \bullet -child v of u , we have $A \in S(v)$;
- \diamond_2 :: for every \bullet -child v of u and for every $\diamond A \in S(v)$, we have $A \in S(u)$;
- \blacklozenge_2 :: for every \circ -child v of u and for every $\blacklozenge A \in S(v)$, we have $A \in S(u)$.

Figure 14 gives a proof search strategy for **DKt**. The application of a rule deep inside a sequent can be viewed as focusing on a particular node of the tree. The rules of **DKt** can then be viewed as operations on the tree encoded in the sequent. In particular, Step 3 saturates a node locally, Step 4 appends new nodes to the tree, and Step 5 moves \diamond (\blacklozenge) prefixed formulae between neighbouring nodes.

The degree of a formula is the maximum number of nested modalities:

$$\begin{aligned} \deg(p) &= 0 \\ \deg(A\#B) &= \max(\deg(A), \deg(B)) \text{ for } \# \in \{\wedge, \vee\} \\ \deg(\#A) &= 1 + \deg(A) \text{ for } \# \in \{\Box, \diamond, \blacksquare, \blacklozenge\}. \end{aligned}$$

The degree of a set of formulae is the maximum degree over all its members. We write $sf(A)$ for the subformulae of A , and define the set of subformulae of a set Θ as $sf(\Theta) = \bigcup_{A \in \Theta} sf(A)$. For a sequent Ξ we define $sf(\Xi)$ as below:

$$\begin{aligned} \Xi &= \Theta, \circ\{\Gamma_1\}, \dots, \circ\{\Gamma_n\}, \bullet\{\Delta_1\}, \dots, \bullet\{\Delta_m\} \\ sf(\Xi) &= sf(\Theta) \cup sf(\Gamma_1) \cup \dots \cup sf(\Gamma_n) \cup sf(\Delta_1) \cup \dots \cup sf(\Delta_m). \end{aligned}$$

Theorem 7.4. *Function Prove terminates for any input sequent Ξ .*

Proof. Let $m = |sf(\Xi)|$, $d = \deg(sf(\Xi)) \leq m$ and $T = tree(\Xi)$. The saturation process for each node in T is bounded by m . Therefore after at most m moves at each node, Step 3 is

²This is not the exact form of the rule given in [7], but it describes the same rule.

no longer applicable to this node. T is finitely branching, since new nodes are only created for unrealised box formulae. Therefore after at most m moves at each node, Step 4 is no longer applicable to this node. The depth of T is bounded by d , since each node u in T at distance k from the root of T has $\text{degree}(S(u)) \leq d - k$. Since \diamond - and \blacklozenge -prefixed formulae are only propagated to nodes that do not already contain these formulae, after at most m propagation moves into each node, Step 5 is no longer applicable to this node. \square

We now show that the procedure `Prove` is sound and complete with respect to **DKt**. A typical semantic completeness proof would construct a countermodel from a failed proof search. In the following proofs, however, we shall use purely proof theoretic arguments without reference to semantics, unlike, say, completeness proof for a similar procedure for modal logics in [4].

Lemma 7.5. *Let Ξ be a sequent such that each node in $\text{tree}(\Xi)$ is saturated, realised, and propagated. Then Ξ is not derivable in **DKt**.*

Proof. We prove this by contradiction: Assume that Ξ has a derivation, therefore it also has a shortest derivation, say Π . We show that one can construct an even shorter derivation, hence contradicting the assumption. This is done by exploiting the fact that $\text{tree}(\Xi)$ is saturated, realised and propagated, and Lemma 4.4 (essentially, height-preserving admissibility of contraction). We show that every attempt to apply a rule to Ξ will lead to a duplication of formulae or create unnecessary structures (in the sense of the medial rules). We show here one case involving the rule \square ; the others are similar.

Suppose Π ends with the rule \square . In this case we have $\Xi = \Sigma[\square A, \circ\{A, \Delta\}]$, for some context $\Sigma[\]$ and some sequent Δ , such that the rule \square is applied to $\square A$ in the context $\Sigma[\]$. Note that $\circ\{A, \Delta\}$ must also be in the same context since every node of $\text{tree}(\Xi)$ is realised. Π in this case takes the form:

$$\frac{\Pi'}{\Sigma[\square A, \circ\{A\}, \circ\{A, \Delta\}]} \square$$

Applying Lemma 4.4 to Π' , we get a derivation Π_1 of $\Sigma[\square A, \circ\{A, A, \Delta\}]$ such that $|\Pi_1| = |\Pi'|$, and applying the same lemma to Π_1 , we get another derivation Π_2 of $\Sigma[\square A, \circ\{A, \Delta\}]$ with $|\Pi_2| = |\Pi_1| = |\Pi'| < |\Pi|$.

Since Π cannot end with any of the rules of **DKt**, this obviously contradicts the assumption that it is a derivation in **DKt**. It then follows that Ξ is not derivable in **DKt**. \square

Theorem 7.6. *Let Ξ be a sequent. Then $\vdash_{\mathbf{DKt}} \Xi$ if and only if `Prove`(Ξ) returns *True*.*

Proof. Soundness of the `Prove` procedure is obvious since each of Step 1 – Step 5 are just applications of **DKt**-rules. By Theorem 7.4, `Prove`(Ξ) always terminates and returns either *True* or *False*. To show completeness, we show that if `Prove`(Ξ) returns *False* then Ξ is not derivable in **DKt**. Note that each rule of **DKt** is invertible, hence Step 1 – Step 5 in `Prove` preserves provability of the original sequent. If `Prove`(Ξ) returns *false*, this can only be the case if Step 6 is reached, i.e., the systematic bottom-up applications of the rules of **DKt** produce a sequent such that every node in the tree of the sequent is saturated, realised, and propagated. By Lemma 7.5, such a sequent would not be derivable, and since all other steps of `Prove` preserves derivability, it follows that Ξ is not derivable either in **DKt**. \square

8. CONCLUSION AND RELATED WORK

This work started out as an attempt to manage proof search in display calculi, in particular, display calculi for tense logics by Kracht [18]. Due to the high-degree of non-determinism in display calculi, our approach was to first consider a restricted form of display calculi with good properties, in particular, it should allow one to prove cut elimination in a uniform manner as in display calculi, but also close enough to traditional sequent calculi, so that traditional proof search methods, e.g., those based on saturation of sequents, can be applied. We have turned to nested sequent calculi for tense logics, as originally studied by Kashima [17], as a compromise; nested sequents are more restricted than display sequents, but they still allow an important property, i.e., the display property, to be proved. The display property is essentially what makes it possible to prove cut elimination uniformly. More interestingly, our re-formulation of tense logics in nested sequent calculi allows us to observe an important connection between display postulates and structural rules (in the shallow calculi) and deep inference and propagation rules (in the deep inference calculi). We exploit this connection to get rid of all structural rules, which are the main obstacle to proof search, in the deep inference calculi. We have shown a preliminary result in structuring proof search for **DKt**. In the future, we hope to extend this to other extensions of **DKt**. We need to emphasize that our work is first and foremost a proof theoretic investigation of a proof search framework. Whether or not an efficient decision procedure can be built on top of our framework is an important question, but one which is out of the scope of the present paper.

Related work. Areces and Bernardi [2] appear to be the first to have noticed the connection between deep inference and residuation in display logic in the context of categorial grammar, although they do not give an explicit proof of this correspondence. Lamarche [19] proposes an approach to eliminating display postulates by moving to a more general theory of contexts in which reversible structural rules like display postulates are treated as part of the algebraic definition of contexts, and gives a cut elimination procedure for substructural logics defined using this more general notion of contexts. Brünnler [4, 5, 7] and Poggiolesi [23] have given deep inference calculi for the modal logic K and some extensions. Sadrzadeh and Dyckhoff [25] have given a syntactic cut elimination procedure for some extensions of positive tense logic, i.e., tense logic without negation or implication. Brünnler has recently shown that the deep-inference-based cut elimination technique for K [4] can be extended to prove cut elimination for Kashima’s **S2Kt**.³ In his proof, a crucial step is a proof of the admissibility of a “deep” version of residuation:

$$\frac{\Sigma[\bullet\{\circ\{\Delta\}, \Gamma\}]}{\Sigma[\Delta, \bullet\{\Gamma\}]} \quad \frac{\Sigma[\circ\{\bullet\{\Delta\}, \Gamma\}]}{\Sigma[\Delta, \circ\{\Gamma\}]}$$

More recently, Brünnler and Straßburger [7] have shown how one can extend, modularly, their deep inference calculus for modal logic K with several standard axioms of normal modal logics. It is worth noting that their formulation of these extensions allow for structural rules to be present in the deep inference systems, contrary to our approach. In our setting, modular extensions of tense logic are easily achieved in the shallow setting. There is, however, a catch: our modularity result does not imply modularity in the modal fragments.

³K. Brünnler. Personal communication.

This is because our modularity result relies on the display property, which in turn relies on the presence of both modal and tense structural connectives.

Indrzejczak [16] and Trzesicki [27] have given cut-free sequent-like calculi for tense logic. In each such calculus there is a rule (or rules) which allow us to “return” to previously seen worlds when the rules are viewed from the perspective of counter-model construction. However, Trzesicki’s calculus has a large degree of non-determinism and is therefore not suitable for proof search. In contrast, our system **DKt** admits a simple proof search strategy and termination argument. Indrzejczak’s calculus is suitable for proof search but lacks a natural notion of a cut rule and cut elimination. It is also possible to give proof calculi for many modal and tense logics using semantic methods such as labelled deduction [21] and graph calculi [9], but we prefer purely syntactic methods since they can potentially be applied to logics with more complicated semantics such as substructural logics.

Future work. The immediate future work is to devise a terminating proof strategy for each extension of **DKt** with path axioms. For extensions that include transitivity, e.g., *KtS4*, one would need to perform loop checking as in Heuerding’s proof calculus for *S4* [15] to ensure termination. Although we have shown that one can compile any set of path axioms into a complete set of propagation rules, it will be more desirable if one can do it using only local propagation rules. Another interesting avenue for future work is to investigate compositions of path axioms with other axioms. For instance, a composition of path axioms with the seriality axiom ($\Box A \rightarrow \Diamond A$) will allow us to capture all fifteen basic modal logics. Another problem is to find a complete set of propagation rules for the confluence axiom ($\Diamond \Box A \rightarrow \Box \Diamond A$). It is also interesting to see whether the connection between deep inference and display postulates can be extended to calculi with more complex binary residuation principles like those in substructural logics [2]. Another interesting direction is the addition of (first-order) quantifiers. An approach to this would be to consider quantifiers as modal operators, with appropriate display postulates, such as the ones developed in [28].

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