TOWARDS A PROOF THEORY OF GÖDEL MODAL LOGICS*

GEORGE METCALFE a AND NICOLA OLIVETTI b

- ^a Mathematics Institute, University of Bern, Sidlerstrasse 5, Bern 3012, Switzerland e-mail address: george.metcalfe@math.unibe.ch
- b LSIS-UMR CNRS 6168, Université Paul Cézanne, Campus de Saint Jérôme, Avenue Escadrille Normandie-Niemen, 13397 Marseille Cedex 20, France e-mail address: nicola.olivetti@univ-cezanne.fr

ABSTRACT. Analytic proof calculi are introduced for box and diamond fragments of basic modal fuzzy logics that combine the Kripke semantics of modal logic K with the many-valued semantics of Gödel logic. The calculi are used to establish completeness and complexity results for these fragments.

1. Introduction

A broad spectrum of concepts spanning necessity, knowledge, belief, obligation, and spatiotemporal relations have been investigated in the field of modal logic (see, e.g., [10]), while notions relating to truth degrees such as vagueness, possibility, and uncertainty have received careful attention in the study of fuzzy logics (see, e.g., [19, 25]). Relatively little attention, however, has been paid to logics combining these approaches, that is, to modal fuzzy logics. Ideally, a systematic development of these logics would provide a unified approach to a range of topics considered in the literature such as fuzzy belief [22, 17], spatial reasoning in the presence of vagueness [27], fuzzy similarity measures [18], and fuzzy description logics, which may be understood, analogously to classical description logics, as multi-modal fuzzy logics (see, e.g., [30, 21, 6]).

Fuzzy modal logics developed for particular applications are typically situated quite high up in the spectrum of modal logics, e.g., at the level of the logic S5 (see, e.g., [19]) or based on Zadeh's minimal fuzzy logic (see, e.g., [32]). On the other hand, general approaches dealing with many-valued modal logics, such as [14, 15], have concentrated mainly on the finite-valued case. In particular, Priest [26] and Bou et al. [7, 8] have provided frameworks for studying many-valued modal logics but their results so far (e.g., for axiomatizations and decidability) relate mostly to finite-valued modal logics. The general strategy, followed also in this paper, is to consider logics based either on standard Kripke frames or Kripke frames

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^{*} A precursor to this paper, covering only the box fragment of the logics, has appeared as [24].

where the accessibility relation between worlds is many-valued (fuzzy). Propositional connectives are interpreted using the given (fuzzy) logic at individual worlds, while the values of modal formulas $\Box A$ and $\Diamond A$ are calculated using the infima and suprema, respectively, of values of A at accessible (to some degree) worlds. Validity can then be defined as usual as truth (e.g., taking the value 1) at all worlds of all models. Let us emphasize, however, that this approach diverges significantly from certain other developments in the literature. In particular, intuitionistic and intermediate logics extended with modalities, as investigated in, e.g., [28, 31], make use of two accessibility relations for Kripke models, one for the modal operator and another for the intuitionistic connectives. Also, the modalities added to fuzzy logics in works such as [20, 11] represent truth stressers such as "very true" or "classically true" and, unlike the modalities considered here, may be interpreted as unary functions on the real unit interval.

In this paper, we narrow our focus on the fuzzy side to propositional Gödel logic G, the infinite-valued version of a family of finite-valued logics introduced by Gödel in [16], axiomatized by Dummett in [12] by adding the prelinearity schema $(A \to B) \lor (B \to A)$ to an axiomatization of propositional intuitionistic logic. Aside from being an important fuzzy and intermediate logic, there are good practical reasons to focus on G in modal contexts. As noted in [7], G is the only fuzzy logic whose modal analogues with a fuzzy accessibility relation admit the schema $\Box(A \to B) \to (\Box A \to \Box B)$ (roughly speaking, since G, unlike other fuzzy logics, admits both weakening and contraction). Also, a multimodal variant of this logic (restricted, however, to finite models) has already been proposed as the basis for a fuzzy description logic in [6]. Caicedo and Rodríguez have already provided axiomatizations for the box and diamond fragments of a Gödel modal logic based on fuzzy Kripke frames in [9], observing that the box fragment is also characterized by standard Kripke frames and does not have the finite model property, while, conversely, the diamond fragment has the finite model property but is not characterized by standard Kripke frames.

In this work, we introduce (the first) analytic proof systems for fragments of the Gödel modal logic studied by Caicedo and Rodríguez and also for the "other" diamond fragment based on standard Kripke frames, the broader aim being to initiate a general investigation into the proof theory of modal fuzzy logics (e.g., as undertaken for propositional and firstorder fuzzy logics in [25]). A wide range of proof systems have been developed for Gödel logic, including the sequent calculi of Sonobe [29] and Dyckhoff [13], decomposition systems of Avron and Konikowska [3], graph-based methods of Larchey-Wendling [23], and goaldirected systems of Metcalfe et al. [25]. Here we extend the sequent of relations calculus of Baaz and Fermüller [4], which provides a particularly elegant and suitable framework for investigating computational properties such as complexity and proof search, and also the hypersequent calculus of Avron [2], which is better suited to extensions to the firstorder level and other logics. More precisely, we provide sequent of relations calculi for all three fragments and a hypersequent calculus for the box fragment. We are then able to use these calculi to obtain PSPACE-completeness results for the fragments and constructive completeness proofs for Hilbert-style axiomatizations. In the final section, we discuss connections with related work on fuzzy description logics, modal intermediate logics, and other proof frameworks, and consider the problems involved with extending this approach to the full logics.

Figure 1: The Hilbert System HG

2. Gödel Logic

2.1. Syntax and Semantics. We define Gödel logic G based on a language \mathcal{L}_{G} consisting of a fixed countably infinite set Var of (propositional) variables, denoted p,q, binary connectives \to , \wedge , \vee , and constants \bot , \top . We call variables and constants atoms, denoted a,b. The set of formulas $\mathrm{Fm}_{\mathcal{L}_{\mathsf{G}}}$, with arbitrary members denoted A,B,C,\ldots , is defined inductively as usual, and the complexity of a formula A, denoted |A|, is defined as the number of connectives occurring in A. We let $\neg A =_{\mathrm{def}} A \to \bot$ and $A \leftrightarrow B =_{\mathrm{def}} (A \to B) \wedge (B \to A)$. We use $\Gamma, \Pi, \Sigma, \Delta, \Theta$ to stand for finite multisets of formulas, written $[A_1, \ldots, A_n]$, letting $[A_1, \ldots, A_n]$ denote the empty multiset of formulas and $\Gamma \uplus \Delta$ the multiset sum of Γ and Γ . We write Γ and Γ with Γ and Γ with Γ and Γ and Γ with Γ and Γ

The standard semantics of Gödel logic is characterized by the Gödel t-norm min and its residuum \rightarrow_{G} , defined on the real unit interval [0, 1] by

$$x \to_{\mathsf{G}} y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise.} \end{cases}$$

More precisely, a G-valuation is a function $v : \operatorname{Fm}_{\mathcal{L}_G} \to [0, 1]$ satisfying

$$v(\bot) = 0$$

$$v(\top) = 1$$

$$v(A \to B) = v(A) \to_{\mathsf{G}} v(B)$$

$$v(A \land B) = \min(v(A), v(B))$$

$$v(A \lor B) = \max(v(A), v(B)).$$

A formula A is G-valid, written $\models_{\mathsf{G}} A$, iff v(A) = 1 for all G-valuations v.

Sometimes it will be helpful to consider extensions of the language \mathcal{L}_{G} with (finitely many) constants, denoted c,d, where G -valuations satisfy $v(c)=r_c$ for each additional constant c for some fixed $r_c \in [0,1]$. In such cases, \top and \bot are considered together alongside the extra constants, with $v(r_{\top})=1$ and $v(r_{\bot})=0$.

A standard axiomatization HG for Gödel logic in the language \mathcal{L}_{G} is provided in Figure 1. HG is complete with respect to the standard semantics, and also with respect to both Gödel algebras, defined as Heyting algebras obeying the prelinearity law $\top = (x \to y) \lor (y \to x)$, and linearly ordered (intuitionistic) Kripke models. Analytic proof systems for Gödel logic,

¹Note that the real number r_c does not appear in the language, rather it is fixed at the same time as the constant is added to the language.

Axioms and Structural Rules

$$\frac{\mathcal{S} \mid \mathcal{A} \leq A \text{ (ID)}}{\mathcal{S} \mid A \leq A} \text{ (ID)} \qquad \frac{\mathcal{S} \mid \mathcal{A} \leq B \mid \mathcal{A} \leq \bot}{\mathcal{S}} \text{ (CS)}$$

$$\frac{\mathcal{S} \mid A \leq B \mid \mathcal{T} \leq B}{\mathcal{S} \mid A \leq B} \text{ (WL)} \qquad \frac{\mathcal{S} \mid A \leq B \mid A \leq \bot}{\mathcal{S} \mid A \leq B} \text{ (WR)}$$

$$\frac{\mathcal{S}}{\mathcal{S} \mid A \leq B} \text{ (EW)} \qquad \frac{\mathcal{S} \mid A \triangleleft_1 B \mid C \triangleleft_2 D \mid A \leq D \quad \mathcal{S} \mid A \triangleleft_1 B \mid C \triangleleft_2 D \mid C \triangleleft_1 B}{\mathcal{S} \mid A \triangleleft_1 B \mid C \triangleleft_2 D} \text{ (COM)}$$

Logical Rules

$$\frac{S \mid A \triangleleft C \mid B \triangleleft C}{S \mid A \land B \triangleleft C} \; (\land \triangleleft) \qquad \qquad \frac{S \mid C \triangleleft A \quad S \mid C \triangleleft B}{S \mid C \triangleleft A \land B} \; (\triangleleft \land)$$

$$\frac{S \mid A \triangleleft C \quad S \mid B \triangleleft C}{S \mid A \lor B \triangleleft C} \; (\lor \triangleleft) \qquad \qquad \frac{S \mid C \triangleleft A \mid C \triangleleft B}{S \mid C \triangleleft A \lor B} \; (\triangleleft \lor)$$

$$\frac{S \mid B \triangleleft A \quad S \mid B \triangleleft C}{S \mid A \rightarrow B \triangleleft C} \; (\rightarrow \triangleleft) \qquad \qquad \frac{S \mid A \triangleleft B \mid C \triangleleft B}{S \mid C \triangleleft A \lor B} \; (\triangleleft \lor)$$

$$\frac{S \mid T \triangleleft C \mid B \triangleleft A \quad S \mid B \triangleleft C}{S \mid A \rightarrow B \triangleleft C} \; (\rightarrow \triangleleft) \qquad \qquad \frac{S \mid A \triangleleft B \mid C \triangleleft B}{S \mid C \triangleleft A \rightarrow B} \; (\triangleleft)$$

Figure 2: The Sequent of Relations Calculus SG

where structures appearing in a derivation are constructed from subformulas of the formula to be proved, have been defined in a number of different frameworks. Below we consider two of the most useful of these frameworks, sequents of relations and hypersequents.

2.2. Sequents of Relations. Sequents of relations, consisting of sets of pairs of formulas ordered by the relations \leq and <, were introduced by Baaz and Fermüller in [4] as a proof-theoretic framework for Gödel and other "projective" logics. More formally, a sequent of relations \mathcal{S} for a language \mathcal{L} is a finite (possibly empty) set of ordered triples, written

$$A_1 \triangleleft_1 B_1 \mid \ldots \mid A_n \triangleleft_n B_n$$

where $A_i, B_i \in \operatorname{Fm}_{\mathcal{L}}$ and $\triangleleft_i \in \{<, \leq\}$ for $i = 1 \dots n$. A sequent of relations is called *atomic* if it contains only atoms.

For Gödel logic, the syntactic symbol $\triangleleft \in \{\leq, <\}$ is interpreted as the corresponding relation over [0,1]. That is, for a sequent of relations \mathcal{S} for \mathcal{L}_{G} ,

 $\models_{\mathsf{G}} \mathcal{S}$ iff for all G -valuations $v, v(A) \triangleleft v(B)$ for some $(A \triangleleft B) \in \mathcal{S}$, and we say in this case that \mathcal{S} is G -valid.

Alternatively, we can define the following formula interpretation:

$$I_S(\{A_i < B_i\}_{i=1}^n \mid \{C_j \le D_j\}_{j=1}^m) = \bigwedge_{i=1}^n (B_i \to A_i) \to \bigvee_{j=1}^m (C_j \to D_j).$$

It follows easily using the deduction theorem for Gödel logic that $\models_{G} S$ iff $\models_{G} I_{S}(S)$.

The sequent of relations calculus SG for Gödel logic displayed in Figure 2 consists of logical rules taken from [4] together with additional axioms and structural rules based on similar calculi for G presented in [25]. The rule (COM) (the only rule with more than one noncontext relation in the conclusion) reflects the linearity of the truth values in Gödel logic,

while the rule (EW) is not strictly necessary for completeness, but is useful for constructing derivations.

Note also that the following helpful axioms and rules for the constants and negation are derivable:

Given some fixed notion of L-validity, we say that a rule is L-sound if whenever the premises are L-valid, then so is the conclusion. Conversely, a rule is L-invertible if whenever the conclusion is L-valid, then so are the premises. The key observation for SG is that, unlike several other calculi for Gödel logic, the logical rules are not only G-sound, but also G-invertible [4]. Since upwards applications of the logical rules terminate (a standard argument), this means that the question of the G-validity of a sequent of relations can be reduced to the question of the G-validity of atomic sequents of relations.

Moreover, the following lemma provides a perspicuous and useful characterization of G-valid atomic sequents of relations.

Lemma 2.1. An atomic sequent of relations S (for \mathcal{L}_{G} possibly with additional constants) is G -valid iff there exists $(a_i \triangleleft_{i+1}) \in S$ for $i = 1 \dots n$ such that one of the following holds:

- (1) $a_1 = a_{n+1}$ and \triangleleft_i is \leq for some $i \in \{1, \ldots, n\}$
- (2) $a_1 = \bot$ and \triangleleft_i is \leq for some $i \in \{1, ..., n\}$
- (3) $a_{n+1} = \top$ and \triangleleft_i is \leq for some $i \in \{1, \ldots, n\}$
- (4) $a_1 = c$, $a_{n+1} = d$, and $r_c < r_d$
- (5) $a_1 = c$, $a_{n+1} = d$, $r_c = r_d$, and a_i is $a_i = c$ for some $a_i \in \{1, ..., n\}$.

Proof. It is clear that S is G-valid if any of the above conditions are met. For the other direction, we proceed by induction on the number of different variables occurring in S. Note first that if one of $a \le a$, $a \le \top$, $\bot \le a$, c < d for $r_c < r_d$, or $c \le d$ for $r_c = r_d$ occurs in S, then we are done. This takes care of the base case. For the inductive step, we fix a variable q occurring in S, and define:

$$\begin{split} \mathcal{S}_{<} &=_{\operatorname{def}} & \{a < b \mid \{a < q, q < b\} \subseteq \mathcal{S}\} \\ \mathcal{S}_{\leq} &=_{\operatorname{def}} & \{a \leq b \mid \{a \triangleleft_{1} q, q \triangleleft_{2} b\} \subseteq \mathcal{S}, \leq \in \{\triangleleft_{1}, \triangleleft_{2}\}\} \\ \mathcal{S}' &=_{\operatorname{def}} & \{a \triangleleft_{b} \in \mathcal{S} \mid a \neq q, b \neq q\} \cup \mathcal{S}_{<} \cup \mathcal{S}_{<}. \end{split}$$

 \mathcal{S}' has fewer different variables than \mathcal{S} . So if \mathcal{S}' is G-valid, then applying the induction hypothesis to \mathcal{S}' , we have $(a_i \triangleleft_i a_{i+1}) \in \mathcal{S}'$ for $i = 1 \dots n$, satisfying one of the above conditions. But then by replacing the inequalities $a_i \triangleleft_i a_{i+1}$ that occur in $\mathcal{S}_{<}$ or \mathcal{S}_{\leq} appropriately by $a_i \triangleleft' q$ and $q \triangleleft'' a_{i+1}$, we get that one of the conditions holds for \mathcal{S} . Hence it is sufficient to show that \mathcal{S}' is G-valid. Suppose otherwise, i.e., that there exists a G-valuation v such that $v(a) \triangleleft v(b)$ does not hold for any $a \triangleleft b \in \mathcal{S}'$. We show for a contradiction that \mathcal{S} is not G-valid. Let

$$x = \min\{v(a) \mid a \triangleleft q \in \mathcal{S}\}$$
 and $y = \max\{v(b) \mid q \triangleleft b \in \mathcal{S}\}.$

Note first that $x \geq y$. Otherwise it follows that for some a, b, we have $\{a \triangleleft_1 q, q \triangleleft_2 b\} \subseteq \mathcal{S}$ and v(a) < v(b). But then $(a \triangleleft b) \in \mathcal{S}'$ so $v(a) \geq v(b)$, a contradiction. Hence there are two

cases. If x > y, then we extend v such that x > v(q) > y. For any $(a \triangleleft q)$ or $(q \triangleleft b)$ in \mathcal{S} , we have $v(a) \geq x > v(q) > y \geq v(b)$. So \mathcal{S} is not G-valid, a contradiction. Now suppose that x = y and extend v such that v(q) = x. We must have atoms a_0 , b_0 such that $(a_0 < q)$ and $(q < b_0)$ are in \mathcal{S} and $v(a_0) = v(b_0) = v(q)$. Now consider any $(a \triangleleft_1 q)$ or $(q \triangleleft_2 b)$ in \mathcal{S} . Since $(a \triangleleft_1 b_0)$ and $(a_0 \triangleleft_2 b)$ are in \mathcal{S}' , $v(a) \triangleleft_1 v(q) = v(b_0)$ and $v(a_0) = v(q) \triangleleft_2 v(b)$ cannot hold. So \mathcal{S} is again not G-valid, a contradiction.

Let us call an atomic sequent of relations \mathcal{S} saturated if whenever \mathcal{S} occurs as the conclusion of a logical rule, (COM), (CS), (WL), or (WR), then \mathcal{S} also occurs as one of the premises. In other words, the sequent of relations is "closed" under applications of these rules. Then saturated G-valid atomic sequents of relations have the following property:

Lemma 2.2. Every saturated G-valid atomic sequent of relations S (for \mathcal{L}_{G} possibly with additional constants) contains either $(a \leq a)$ or $(c \triangleleft d)$ where $r_{c} \triangleleft r_{d}$.

Proof. By the previous lemma, each saturated G-valid atomic sequent of relations S contains $(a_i \triangleleft_i a_{i+1})$ for $i = 1 \dots n$ such that one of the conditions (1)-(5) holds. The claim then follows by a simple induction on n, where the base case makes use of (WL) and (WR), and the inductive step makes use of (COM).

Note that in the case of sequents of relations for \mathcal{L}_G without extra constants, every saturated G-valid atomic sequent of relations, and hence every G-valid atomic sequent of relations, is derivable in SG. Moreover, since, as mentioned above, the logical rules are G-sound and G-invertible, and reduce G-valid sequents of relations to G-valid atomic sequents of relations, it follows that:

Theorem 2.3. For any sequent of relations S for \mathcal{L}_{G} : $\vdash_{SG} S$ iff $\models_{G} S$ iff $\models_{G} I_{S}(S)$. \square

2.3. A Hypersequent Calculus. Hypersequents were introduced by Avron in [1] as a generalization of Gentzen sequents that allow disjunctive or parallel forms of reasoning. Instead of a single sequent, there is a collection of sequents that can be "worked on" simultaneously. More precisely, a (single-conclusion) sequent S for a language \mathcal{L} is (defined here as) an ordered pair consisting of a finite multiset Γ of \mathcal{L} -formulas and a multiset Δ containing at most one \mathcal{L} -formula, written $\Gamma \Rightarrow \Delta$. A (single-conclusion) hypersequent \mathcal{G} for \mathcal{L} is a finite (possibly empty) multiset of sequents for \mathcal{L} , written $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ or sometimes, for short, as $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n$.

We interpret sequents and hypersequents for Gödel logic as follows (recalling that $\Lambda[] =_{\text{def}} \top$ and $V[] =_{\text{def}} \bot$):

$$I_H(\Gamma \Rightarrow \Delta) =_{\operatorname{def}} \bigwedge \Gamma \to \bigvee \Delta$$

 $I_H(S_1 \mid \dots \mid S_n) =_{\operatorname{def}} I_H(S_1) \vee \dots \vee I_H(S_n).$

Hypersequent calculi admitting cut-elimination have been defined for a wide range of fuzzy logics (for details see [25]). In particular, the first example of such a system (modulo a few inessential changes) was the calculus GG defined for Gödel logic by Avron in [2]. This calculus, displayed in Figure 3, can be viewed as a direct extension of a sequent calculus for intuitionistic logic. Namely, the axioms, weakening, contraction, cut, and logical rules, are obtained from standard sequent rules simply by adding a hypersequent context \mathcal{G} to the premises and conclusion. The external weakening and contraction rules, (EW) and (EC), reflect the interpretation of "|" as a meta-level disjunction, while the communication rule

$$\frac{\mathcal{G} \mid A \Rightarrow A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \text{ (L\Rightarrow)} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Gamma}{\mathcal{G} \mid \Gamma \Rightarrow \Gamma} \text{ (\RightarrowT$)}$$

$$\frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}} \text{ (EC)} \qquad \frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}} \text{ (EW)} \qquad \frac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2 \Rightarrow \Delta_2} \text{ (COM)}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \text{ (WL)} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow}{\mathcal{G} \mid \Gamma \Rightarrow A} \text{ (WR)} \qquad \frac{\mathcal{G} \mid \Gamma, A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \text{ (CL)}$$
I Rules
$$\mathcal{G} \mid \Gamma \Rightarrow A \quad \mathcal{G} \mid \Gamma \Rightarrow A \quad \mathcal{G} \mid \Gamma \Rightarrow A$$

Logical Rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A \quad \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta} \quad (\to \Rightarrow) \qquad \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow B}{\mathcal{G} \mid \Gamma, A \to B} \quad (\Rightarrow \to)$$

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \land B \Rightarrow \Delta} \quad (\wedge \Rightarrow)_{1} \qquad \qquad \frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \land B \Rightarrow \Delta} \quad (\wedge \Rightarrow)_{2}$$

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \lor B \Rightarrow \Delta} \quad (\vee \Rightarrow) \qquad \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A \quad \mathcal{G} \mid \Gamma \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow A \land B} \quad (\Rightarrow \land)$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A}{\mathcal{G} \mid \Gamma \Rightarrow A \lor B} \quad (\Rightarrow \lor)_{1} \qquad \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow A \lor B} \quad (\Rightarrow \lor)_{2}$$

Cut Rule

$$\frac{\mathcal{G} \mid \Gamma_1, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma_2 \Rightarrow A}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta} \text{ (CUT)}$$

Figure 3: The Gentzen System GG

(COM), corresponding to the prelinearity axiom schema $(A \to B) \lor (B \to A)$, is the crucial ingredient in extending the system beyond intuitionistic logic.

Example 2.4. Consider the following derivation in GG:

$$\frac{\overline{p \Rightarrow p} \text{ (ID)} \quad \overline{q \Rightarrow q} \text{ (ID)}}{\frac{p \Rightarrow q \mid q \Rightarrow p}{p \Rightarrow q \mid q \Rightarrow p} \text{ (\Rightarrow)}}$$

$$\frac{\frac{p \Rightarrow q \mid q \Rightarrow p}{p \Rightarrow q \mid \Rightarrow q \rightarrow p} \text{ (\Rightarrow)}}{\Rightarrow p \rightarrow q \mid \Rightarrow (p \rightarrow q) \lor (q \rightarrow p)} \text{ (\Rightarrow)}_{2}$$

$$\frac{\Rightarrow p \rightarrow q \mid \Rightarrow (p \rightarrow q) \lor (q \rightarrow p)}{\Rightarrow (p \rightarrow q) \lor (q \rightarrow p)} \text{ (\Rightarrow)}_{1}$$

$$\Rightarrow (p \rightarrow q) \lor (q \rightarrow p) \text{ (EC)}$$

Notice that the hypersequent $(p \Rightarrow q \mid q \Rightarrow p)$ two lines down might be read as just a "hypersequent translation" of the prelinearity axiom $(p \rightarrow q) \lor (q \rightarrow p)$.

Theorem 2.5 ([2]).

- (1) For any hypersequent \mathcal{G} for \mathcal{L}_{G} : $\models_{\mathsf{G}} I_{\mathsf{H}}(\mathcal{G})$ iff $\vdash_{\mathsf{GG}} \mathcal{G}$.
- (2) GG admits cut-elimination.

For future reference, we observe that the following rules for the defined negation $\neg A =_{\text{def}} A \to \bot$ are derivable in GG:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A}{\mathcal{G} \mid \Gamma, \neg A \Rightarrow} (\neg \Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow}{\mathcal{G} \mid \Gamma \Rightarrow \neg A} (\Rightarrow \neg)$$

While the logical rules of the sequent of relations calculus SG are invertible, with consequent advantages for establishing complexity and interpolation results and building efficient proof systems, the hypersequent calculus GG does not have this property. The virtue of the system lies rather with its close connection to sequent calculi for intuitionistic logic and the existence of relatively straightforward cut-elimination proofs. In particular, this means that GG, unlike SG, can be easily extended to the first-order level or with propositional quantifiers, and used to prove, for example, completeness, Herbrand theorems, and Skolemization results (see, e.g., [5, 25]).

3. Adding Modalities

3.1. Syntax and Semantics. We extend our language \mathcal{L}_{G} to a modal language $\mathcal{L}_{\Box\Diamond}$ by adding the unary operators \Box and \Diamond , obtaining a set of formulas $\mathrm{Fm}_{\mathcal{L}_{\Box\Diamond}}$. For a finite multiset $\Gamma = [A_1, \ldots, A_n]$ of $\mathcal{L}_{\Box\Diamond}$ -formulas and $\star \in \{\Box, \Diamond\}$, we let $\star \Gamma =_{\mathrm{def}} [\star A_1, \ldots, \star A_n]$. Gödel modal logics are then defined, following similar ideas proposed in [14, 15, 19, 9, 7], as generalizations of the modal logic K where connectives behave locally at individual worlds as in Gödel logic. In particular, GK and GK^F are Gödel modal logics based on, respectively, standard Kripke frames and Kripke frames with fuzzy accessibility relations.

A fuzzy Kripke frame is a pair $F = \langle W, R \rangle$ where W is a non-empty set of worlds and $R: W \times W \to [0,1]$ is a binary fuzzy accessibility relation on W. If $Rxy \in \{0,1\}$ for all $x,y \in W$, then R is called crisp and F is called simply a (standard) Kripke frame. In this case, we often write $R \subseteq W^2$ and Rxy or $(x,y) \in R$ to mean Rxy = 1. A Kripke model for GK^F is then a 3-tuple $K = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a fuzzy Kripke frame and $V : \mathsf{Var} \times W \to [0,1]$ is a mapping, called a valuation, extended to $V : \mathsf{Fm}_{\mathcal{L}_{\square} \Diamond} \times W \to [0,1]$ as follows:

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\begin{array}{rcl} V(\bot,x) & = & 0 \\ V(\top,x) & = & 1 \\ V(A \to B,x) & = & V(A,x) \to_{\mathsf{G}} V(B,x) \\ V(A \land B,x) & = & \min(V(A,x),V(B,x)) \\ V(A \lor B,x) & = & \max(V(A,x),V(B,x)) \\ V(\Box A,x) & = & \inf\{Rxy \to_{\mathsf{G}} V(A,y) \mid y \in W\} \\ V(\Diamond A,x) & = & \sup\{\min(V(A,y),Rxy) \mid y \in W\}. \end{array}
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A Kripke model for GK satisfies the extra condition that $\langle W, R \rangle$ is a standard Kripke frame. In this case, the conditions for \square and \lozenge may also be read as:

```
V(\Box A, x) = \inf\{\{1\} \cup \{V(A, y) \mid Rxy\}\}\
V(\Diamond A, x) = \sup\{\{0\} \cup \{V(A, y) \mid Rxy\}\}.
```

 $A \in \operatorname{Fm}_{\square \lozenge}$ is valid in $\langle W, R, V \rangle$ if V(A, x) = 1 for all $x \in W$. A is L- valid for $L \in \{\mathsf{GK}, \mathsf{GK}^\mathsf{F}\}$, written $\models_\mathsf{L} A$, if A is valid in all Kripke models $\langle W, R, V \rangle$ for L .

An important feature of Gödel logic and, more particularly, Kripke models for GK^F , is that only the *order* of truth values matters. The following lemma is proved by a straightforward induction on formula complexity.

Lemma 3.1. Let $K = \langle W, R, V \rangle$ be a Kripke model for GK^F and $h : [0,1] \to [0,1]$ an order-automorphism of the real unit interval. Define $K' = \langle W, R', V' \rangle$ where R'xy = h(Rxy) for all $x, y \in W$ and V'(p, x) = h(V(p, x)) for each $p \in \mathsf{Var}$ and $x \in W$. Then V'(A, x) = h(V(A, x)) for all $A \in \mathsf{Fm}_{\square \lozenge}$ and $x \in W$.

Moreover, for Kripke models for GK (which, recall, have a crisp accessibility relation), we can shift the values of all formulas above a certain threshold to 1, while preserving the values of formulas below that threshold.

Lemma 3.2. Let $K = \langle W, R, V \rangle$ be a Kripke model for GK and $\lambda \in (0, 1]$. Define $K' = \langle W, R, V' \rangle$ where $V'(p, x) = \lambda \to_{\mathsf{G}} V(p, x)$ for each $p \in \mathsf{Var}$ and $x \in W$. Then $V'(A, x) = \lambda \to_{\mathsf{G}} V(A, x)$ for all $A \in \mathsf{Fm}_{\square \lozenge}$ and $x \in W$.

Proof. We proceed by induction on |A|. The base cases are immediate. Suppose that A is $B \to C$. Then, using the induction hypothesis for the second step:

$$V'(B \to C, x) = V'(B, x) \to_{\mathsf{G}} V'(C, x)$$

$$= (\lambda \to_{\mathsf{G}} V(B, x)) \to_{\mathsf{G}} (\lambda \to_{\mathsf{G}} V(C, x))$$

$$= \lambda \to_{\mathsf{G}} (V(B, x) \to_{\mathsf{G}} V(C, x))$$

$$= \lambda \to_{\mathsf{G}} V(B \to C, x).$$

If A is $\Diamond B$, then, again using the induction hypothesis for the second step:

$$V'(\lozenge B, x) = \sup(\{0\} \cup \{V'(B, y) \mid Rxy\})$$

$$= \sup(\{0\} \cup \{\lambda \rightarrow_{\mathsf{G}} V(B, y) \mid Rxy\})$$

$$= \lambda \rightarrow_{\mathsf{G}} \sup(\{0\} \cup \{V(B, y) \mid Rxy\})$$

$$= \lambda \rightarrow_{\mathsf{G}} V(\lozenge B, x).$$

Cases for the other connectives are very similar.

3.2. Box and Diamond Fragments. Due to the difficulty of dealing with the full language (see Section 6), we focus in this work on the "box" and "diamond" fragments of GK and GK^F based on restrictions of the language $\mathcal{L}_{\Box\Diamond}$ to the single modality sublanguages \mathcal{L}_{\Box} and \mathcal{L}_{\Diamond} , respectively. These fragments are worthy of investigation since they already contain enough extra expressive power to deal with certain modal fuzzy notions. Indeed the general approach of [7, 8] begins with a treatment of just the addition of the box operator, reflecting the fact that in (classical) modal logics, typically just one of the dual modalities is considered primitive.

We use proof-theoretic methods to establish decidability and complexity results for these fragments, and completeness for the axiomatizations:

• HGK_{\square} is HG extended with

$$(K_{\square}) \quad \square(A \to B) \to (\square A \to \square B)$$

$$(Z_{\square}) \quad \neg \neg \square A \to \square \neg \neg A$$

$$\frac{A}{\square A} \text{ (NEC)}_{\square}$$

HGK_◊ is HG extended with

$$\begin{array}{ll} (K_{\diamondsuit}) & \diamondsuit(A \vee B) \to (\lozenge A \vee \lozenge B) \\ (Z_{\diamondsuit}) & \lozenge \neg \neg A \to \neg \neg \lozenge A \\ (F_{\diamondsuit}) & \neg \lozenge \bot \\ \hline \frac{(A \to B) \vee C}{(\lozenge A \to \lozenge B) \vee \lozenge C} \text{ (NEC)}_{\diamondsuit} \\ \end{array}$$

• $\mathsf{HGK}^\mathsf{F}_\lozenge$ is HGK_\lozenge with $(\mathtt{NEC})_\lozenge$ replaced by

$$\frac{A \to B}{\Diamond A \to \Diamond B} \text{ (NEC)}_{\Diamond}^*$$

It is proved in [9] that HGK_\square is complete with respect to the box fragments of both GK and GK^F (in other words, these fragments coincide and there is no need to define an alternative $\mathsf{HGK}_\square^\mathsf{F}$), and that $\mathsf{HGK}_\lozenge^\mathsf{F}$ is complete with respect to the diamond fragment of GK^F . These results, plus completeness for HGK_\lozenge with respect to the diamond fragment of GK , will follow from our proof-theoretic investigations, as will decidability and complexity results for all three fragments. First, however, we note the following interesting feature of Gödel modal logics.

Theorem 3.3. The box and diamond fragments of GK do not have the finite model property. Proof. Following [9], consider the \mathcal{L}_{\square} -formula

$$A = \Box \neg \neg p \to \neg \neg \Box p.$$

A is valid in all Kripke models for GK with a finite number of worlds, but not in the Kripke model $\langle \mathbb{N}, R, V \rangle$ where Rxy holds for all $x, y \in \mathbb{N}$ and V(p, x) = 1/(x+1) for all $x \in \mathbb{N}$. Now consider the \mathcal{L}_{\Diamond} -formula:

$$B = (\lozenge p \to \lozenge q) \to ((\lozenge q \to \bot) \lor \lozenge (p \to q)).$$

We claim first that B is valid in all Kripke models for GK with a finite number of worlds. Consider a world x. If no worlds are accessible to x, then $V(\Diamond q, x) = 0$ and V(B, x) = 1. Otherwise, since the model is finite, we can choose an accessible world y such that $V(\Diamond q, x) = V(q, y)$. If V(p, y) > V(q, y), then $V(\Diamond p, x) > V(\Diamond q, x)$, so also V(B, x) = 1. If $V(p, y) \leq V(q, y)$, then $V(p \to q, y) = 1$, so also V(B, x) = 1.

On the other hand, consider a Kripke model $\langle \mathbb{N}, R, V \rangle$ for GK where Rxy holds iff x=0 and $y \geq 1$, $V(p,x) = \frac{1}{2}$ and $V(q,x) = \frac{1}{2} - \frac{1}{x+2}$ for all $x \in \mathbb{N}$. Then $V(\Diamond q,0) = V(\Diamond p,0) = V(\Diamond p,0) = V(\Diamond p,0) = \frac{1}{2}$.

This contrasts with the following result of Caicedo and Rodríguez.

Theorem 3.4 ([9]). The diamond fragment of GK^F has the finite model property.

3.3. Sequents of Relations. We extend the definition of validity for sequents of relations from G to $L \in \{GK, GK^F\}$ as follows

$$\models_{\mathsf{L}} \mathcal{S}$$
 iff for all Kripke models $\langle W, R, V \rangle$ for L and $x \in W$, $V(A, x) \triangleleft V(B, x)$ for some $(A \triangleleft B) \in \mathcal{S}$

and say in this case that S is L-valid.

We call a sequent of relations propositional if it contains no occurrences of modalities, and identify the modal part of a sequent of relations S as the relations in S made up of box formulas, diamond formulas, and constants, calling S purely modal if it coincides with its modal part. We say that a formula occurs in a sequent of relations if it occurs as the left or right side of one of the relations. A sequent of relations is said to be propositionally L-valid for $L \in \{GK, GK^F\}$ if the sequent of relations obtained by replacing each occurrence of a box formula $\Box A$ with a new variable p_A and diamond formula $\Diamond A$ with a new variable q_A is G-valid.

It will also be helpful to adopt the following notation for sets of relations:

Note that we always restrict expressions $\Gamma \triangleleft \Delta$ to cases where either Γ or Δ has at most one element. We remark, moreover, that Γ and Δ can also be considered sets of formulas rather than multisets without changing the meaning of the notation.

Example 3.5. For instance,

$$[\Box(p \to q), r \to \Box \bot] \le \Box p \mid [] \le \Box(p \land q) \mid \Box(p \to \bot) < [p, r]$$

stands for the sequent of relations

$$\square(p \to q) \le \square p \mid r \to \square \bot \le \square p \mid \top \le \square(p \land q) \mid \square(p \to \bot)$$

The logical rules, (COM), (WL), and (WR) are L-invertible for $L \in \{GK, GK^F\}$. Hence applying the logical rules upwards to an L-valid sequent of relations terminates with L-valid sequent of relations containing only modal formulas and atoms. Since sequents of relations are sets of pairs of formulas, there is a finite number that can be obtained by applying the rules backwards to any given sequent of relations. Hence:

Lemma 3.6. Every sequent of relations S is derivable from a set of saturated sequents of relations $\{S_1, \ldots, S_n\}$ using the logical rules, (COM), (CS), (WL), and (WR); moreover, S is L-valid for $L \in \{GK, GK^F\}$ iff S_i is L-valid for $i = 1 \ldots n$.

4. The Box Fragment

4.1. The Sequent of Relations Calculus SGK_{\square} . For convenience, let us assume for this section that all notions (formulas, rules, etc.) refer exclusively to the sublanguage \mathcal{L}_{\square} . We define the calculus SGK_{\square} as the extension of SG (for \mathcal{L}_{\square}) with the modal rule

$$\frac{\Gamma \leq B \mid \Pi \leq \bot}{\mathcal{S} \mid \Box \Gamma \leq \Box B \mid \Box \Pi \leq \bot} \ (\Box)$$

We note that it will emerge that this calculus is sound and complete for both GK and GK^F , and hence that the box fragments of these logic coincide.

Example 4.1. Consider the following derivation of a sequent of relations corresponding to the axiom $\Box(p \to q) \to (\Box p \to \Box q)$:

$$\frac{p \leq q \mid q
$$\frac{p \leq q \mid q < p}{p \leq q \mid T \leq q \mid q < p} \text{ (EW)} \qquad \frac{p \leq q \mid q \leq q}{p \leq q \mid q \leq q} \text{ (ID)}$$

$$\frac{p \leq q \mid p \rightarrow q \leq q}{\square p \leq \square q \mid \square (p \rightarrow q) \leq \square q} \text{ (\square)}$$

$$\frac{\square p \leq \square q \mid \square (p \rightarrow q) \leq \square q}{\square (p \rightarrow q) \leq \square p \rightarrow \square q} \text{ (\square)}$$

$$\frac{\square (p \rightarrow q) \leq \square p \rightarrow \square q \mid T \leq \square p \rightarrow \square q}{\square (p \rightarrow q) \rightarrow (\square p \rightarrow \square q)} \text{ (\square)}$$$$

We prove soundness with respect to Kripke models for GK^F , recalling that these include the Kripke models for GK as a special case.

Theorem 4.2. If $\vdash_{\mathsf{SGK}_{\square}} \mathcal{S}$, then $\models_{\mathsf{GK}^{\mathsf{F}}} \mathcal{S}$.

Proof. The proof is a standard induction on the height of a derivation in SGK_\square . Let us just check that the rule (\square) preserves validity in Kripke models for GK^F . Note first that it suffices, using the equivalence in GK^F of ($\square A_1 \to B$) \vee ($\square A_2 \to B$) and $\square (A_1 \wedge A_2) \to B$, to check that $\models_{\mathsf{GK}^\mathsf{F}} A \leq B \mid C \leq \bot$ implies $\models_{\mathsf{GK}^\mathsf{F}} \square A \leq \square B \mid \square C \leq \bot$. Suppose, contrapositively, that $\not\models_{\mathsf{GK}^\mathsf{F}} \square A \leq \square B \mid \square C \leq \bot$. I.e., for some Kripke model $K = \langle W, R, V \rangle$ for GK^F and $x \in W \colon V(\square A, x) > V(\square B, x)$ and $V(\square C, x) > 0$. So for some $y \in W \colon Rxy \to_{\mathsf{G}} V(A, y) > Rxy \to_{\mathsf{G}} V(B, y)$ and $Rxy \to_{\mathsf{G}} V(C, y) > 0$. But this implies that V(A, y) > V(B, y) and V(C, y) > 0. So $\not\models_{\mathsf{GK}^\mathsf{F}} A \leq B \mid C \leq \bot$ as required.

Completeness is of course more complicated. The challenge is to show that a GK -valid saturated sequent of relations $\mathcal S$ is derivable in SGK_\square . This will also show that a GK^F -valid saturated sequent of relations $\mathcal S$ is derivable in SGK_\square . Our strategy will be to use Lemmas 4.3 and 4.4 to show that either $\mathcal S$ is derivable in SG or the modal part of $\mathcal S$ is itself GK -valid. For the latter case, $\mathcal S$ is derivable using (\square) from a less complex sequent of relations shown to be GK -valid in Lemma 4.5. An inductive argument will then complete the proof.

Lemma 4.3. If S is saturated and GK-valid, then either S is propositionally valid or the modal part of S is GK-valid.

Proof. Proceeding contrapositively, suppose that a saturated sequent of relations S is not propositionally valid and the modal part S_{\square} of S is not GK-valid. Hence for some Kripke model $K = \langle W, R, V \rangle$ for GK and $x \in W$: $V(A, x) \not\wedge V(B, x)$ for each $(A \triangleleft B) \in S^{\square}$. For each $\square A$ occurring in S, add a constant c_A to the language so that $r_{c_A} = V(\square A, x)$. Let S^P be S with each $\square A$ occurring in S replaced by c_A .

Claim: S^P is not G-valid.

Observe that the result follows from this claim. Let $v: \operatorname{Fm}_{\mathcal{L}_{\mathsf{G}}} \to [0,1]$ be the propositional counter-valuation for \mathcal{S}^P . Define $K' = \langle W \cup \{x_0\}, R', V' \rangle$ where:

- (1) $R' = R \cup \{(x_0, y) \mid (x, y) \in R\}$
- (2) V' is V extended with $V'(p, x_0) = v(p)$ for each $p \in Var$.

Then $V'(\Box A, x_0) = V(\Box A, x) = r_{c_A}$ for all $\Box A$ occurring in \mathcal{S} . So, since v is a countervaluation for \mathcal{S}^P , we have that \mathcal{S} is not GK -valid.

Proof of claim. Suppose that \mathcal{S}^P is GK-valid. Then since \mathcal{S}^P is saturated, by Lemma 2.2, \mathcal{S}^P contains either $(a \leq a)$, or $(c \triangleleft d)$ for constants c, d such that $r_C \triangleleft r_D$; i.e., we have one of the following situations:

- (i) S^P contains $a \leq a$ or $\bot \leq a$ or $a \leq \top$ or $\bot < \top$. But then S is propositionally valid, a contradiction.
- (ii) S^P contains $c_C \triangleleft c_D$ or $c_C < \top$ or $\bot < c_D$. But since S_{\square} is not L-valid, we must have, respectively that $r_{c_C} \not \triangleleft r_{c_D}$ or $r_{c_C} = 1$ or $0 = r_{c_D}$, a contradiction.
- (iii) S^P contains $a \le c_D$ and $r_{c_D} = 1$. But then by (WL), S contains $T \le \square D$ and $r_{c_D} < 1$, a contradiction.
- (iv) S^P contains $c_C \leq a$ and $r_{c_C} = 0$. But then by (WR), S contains $\square C \leq \bot$ and $r_{c_C} > 0$, a contradiction.

Lemma 4.4. Let $S \mid S'$ be saturated, purely modal, and GK-valid, where S' consists only of relations of the form A < B. Then S is GK-valid.

Proof. Let us define (COM)' as (COM) restricted to instances where \triangleleft_1 is \leq , and say that a sequent of relations \mathcal{S} is *semi-saturated* if whenever \mathcal{S} occurs as the conclusion of a logical rule, (COM)', (CS), (WL), or (WR), then \mathcal{S} also occurs as one of the premises. It is then sufficient to prove the following:

Claim. If $S \mid A < \Box B$ is semi-saturated, purely modal, and GK-valid, then S is GK-valid.

We just notice that if $S \mid A < \Box B$ is semi-saturated, then S is also semi-saturated. Hence we can apply the claim repeatedly, noting that the case where \bot appears on the right is trivial and when \top appears on the right, we can replace \top with $\Box \top$ and apply the claim.

Proof of claim. Proceeding contrapositively, suppose that $\not\models_{\mathsf{GK}} \mathcal{S}$. Then there is a Kripke model $K = \langle W, R, V \rangle$ for GK and $x \in W$ such that $V(C, x) \not\prec V(D, x)$ for all $(C \triangleleft D) \in \mathcal{S}$. Moreover, if $V(A, x) \geq V(\Box B, x)$, then $\not\models_{\mathsf{GK}} \mathcal{S} \mid A < \Box B$ as required, so assume that $(\star) V(A, x) < V(\Box B, x)$.

Since $S \mid A < \square B$ is semi-saturated, for each $(C \leq D) \in S$:

either
$$(C \leq \Box B) \in \mathcal{S}$$
 and so $V(C, x) > V(\Box B, x)$
or $(A \leq D) \in \mathcal{S}$ and so $V(A, x) > V(D, x)$.

In particular:

$$(\star\star)$$
 $V(A,x) \leq V(D,x) < V(C,x) \leq V(\Box B,x)$ is not possible.

We have two cases. If $V(\Box B,x)=1$, then from the above either/or distinction, for each $(C \leq D) \in \mathcal{S}$: $(A \leq D) \in \mathcal{S}$ and V(A,x) > V(D,x). Hence also, since \mathcal{S} is non-empty (containing at least $\top \leq \bot$), V(A,x) > 0. Now, using Lemma 3.2, we obtain a Kripke model $K' = \langle W, R, V' \rangle$ for GK such that $V'(C,y) = V(A,x) \to_{\mathsf{G}} V(C,y)$ for all $y \in W$ and $C \in \mathrm{Fm}_{\mathcal{L}_{\square}}$. In particular, $V'(A,x) = V'(\Box B,x) = 1$. Moreover, for all $(C \triangleleft D) \in \mathcal{S}$, we have $V'(C,x) = V(A,x) \to_{\mathsf{G}} V(C,x) \geq V(C,x) \not\preceq V(D,x) = V(A,x) \to_{\mathsf{G}} V(D,x) = V'(D,x)$. Hence $\not\models_{\mathsf{GK}} \mathcal{S} \mid A < \Box B$ as required.

Now suppose that $V(\Box B, x) < 1$. Using Lemma 3.1, we choose a suitable automorphism of [0,1] and define for each $i \in \mathbb{Z}^+$, a Kripke model $K_i = \langle W_i, R_i, V_i \rangle$ for GK such that:

(1) $\langle W_i, R_i \rangle$ is a copy of $\langle W, R \rangle$ with distinct worlds for each $i \in \mathbb{Z}^+$ where x_i is the corresponding copy of x.

(2) For all formulas E satisfying $V_i(A, x_i) < V_i(E, x_i) \le V_i(\Box B, x_i)$:

$$V_i(A, x_i) < V_i(E, x_i) < V(A, x_i) + 1/i.$$

Now we define a model $K' = \langle W', R', V' \rangle$ such that:

- $(1) W' = \{x_0\} \cup \bigcup_{i \in \mathbb{Z}^+} W_i$
- (2) $R' = \{(x_0, y) \mid (x_i, y) \in V_i \text{ for some } i \in \mathbb{Z}^+\} \cup \bigcup_{i \in \mathbb{Z}^+} R_i$
- (3) $V'(p,y) = V_i(p,y)$ for all $y \in W_i$ and $V'(p,x_0) = 0$.

But then:

$$V'(\Box B, x_0) = \inf\{\{1\} \cup \{V'(B, y) \mid R'x_0y\}\}\$$

= \inf\{V_i(\Bigcup_B, x_i) \ | i \in \mathbb{Z}^+\}
= \bigcup V'(A, x_0).

Clearly, if $V(C,x) \geq V(D,x)$ for some $(C < D) \in \mathcal{S}$ where C,D are box formulas or constants, then $V'(C,x_0) \geq V'(D,x_0)$. Moreover, if $(C \leq D) \in \mathcal{S}$ where C,D are box formulas or constants, then using $(\star\star)$, it follows that $V'(C,x_0) > V'(D,x_0)$. Hence $\not\models_{\mathsf{GK}} \mathcal{S} \mid A < \Box B$ as required.

Lemma 4.5. If $\models_{\mathsf{GK}} \{ \Box A_i \leq \Box B_i \}_{i=1}^n \mid \Box C \leq \bot$, then

$$\models_{\mathsf{GK}} \bigwedge_{j \in J} A_j \leq \bigwedge_{j \in J} B_j \mid C \leq \bot$$

for some $\emptyset \subset J \subseteq \{1, \ldots, n\}$.

Proof. We argue by contraposition; i.e., suppose that:

$$\not\models_{\mathsf{GK}} \bigwedge_{j \in J} A_j \leq \bigwedge_{j \in J} B_i \mid C \leq \bot \quad \text{ for all } \emptyset \subset J \subseteq \{1, \dots, n\}.$$

In particular for $i = 1 \dots n$:

$$\not\models_{\mathsf{GK}} A_i \wedge \ldots \wedge A_n \leq B_i \wedge \ldots \wedge B_n \mid C \leq \bot$$
.

So for each i = 1 ... n, there exists a Kripke model $K_i = \langle W_i, R_i, V_i \rangle$ for GK and $x_i \in W_i$ (with each W_i distinct) such that:

$$V_i(A_i \wedge \ldots \wedge A_n, x_i) > V_i(B_i \wedge \ldots \wedge B_n, x_i)$$
 and $V_i(C, x_i) > 0$.

Moreover, using Lemma 3.1, we can assume without loss of generality that

$$V_i(B_i, x_i) \leq V_i(B_k, x_i)$$
 and so $V_i(A_k, x_i) > V_i(B_i, x_i)$ for $k = i \dots n$.

Now, again using Lemma 3.1, we define iteratively $K'_i = \langle W_i, R_i, V'_i \rangle$ for $i = n \dots 1$ such that for $j = i \dots n$:

- (i) $V'_i(B_j, x_j) < V_k(A_j, x_k)$ for $k = 1 \dots i 1$.
- (ii) $V'_{j}(B_{j}, x_{j}) < V'_{k}(A_{j}, x_{k})$ for $k = i \dots n$.
- (iii) $V'_{j}(C, x_{j}) > 0.$

Let us deal with step i, supposing that we have already dealt with steps $n \dots i+1$. We choose an order automorphism h of [0,1] scaling the interval $[0,V_i(B_i,x_i)]$ to a smaller interval $[0,V_i'(B_i,x_i)]$ so that $V_i'(B_i,x_i)=h(V(B_i,x_i))$ satisfies $V_i'(B_i,x_i)< V_k(A_i,x_k)$ for $k=1\dots i-1$ and $V_i'(B_i,x_i)< V_k'(A_i,x_k)$ for $k=i\dots n$. This is possible since we need only force the value $V_i'(B_i,x_i)$ to be suitably small. Also clearly $V_i'(C,x_i)>0$. Now consider $j\in\{i+1,\dots,n\}$. By construction, we already have $V_j'(B_j,x_j)< V_k(A_j,x_k)$ for $k=1\dots i-1$, $V_j'(B_j,x_j)< V_k'(A_j,x_k)$ for $k=i+1\dots n$, and $V_j'(C,x_j)>0$. So

it remains only to show that $V'_j(B_j, x_j) < V'_i(A_j, x_i)$. Note first that using step j > i, we have that $V'_j(B_j, x_j) < V_i(A_j, x_i)$. But $V_i(A_j, x_i) > V_i(B_i, x_i)$, so we can assume that $V'_i(A_j, x_i) = V_i(A_j, x_i)$. Hence $V'_i(B_j, x_j) < V'_i(A_j, x_i)$ as required.

Finally, we define a model $K = \langle W, R, V \rangle$ where for a new world x_0 :

- (1) $W = W_1 \cup \ldots \cup W_n \cup \{x_0\}.$
- (2) $R = R_1 \cup \ldots \cup R_n \cup \{(x_0, x_1), \ldots, (x_0, x_n)\}\}.$
- (3) $V(p,x) = V'_i(p,x)$ for all $x \in W_i$ and $V(p,x_0) = 0$.

But then for $i = 1 \dots n$:

$$V(\Box B_i, x_0) \le V'_i(B_i, x_i) < V'_j(A_i, x_j)$$
 for $j = 1 \dots n$.

So $V(\Box B_i, x_0) < V(\Box A_i, x_0)$. Since also, using (iii), $V(\Box C, x_0) > 0$, we have $\not\models_{\mathsf{GK}} \{\Box A_i \leq \Box B_i\}_{i=1}^n \mid \Box C \leq \bot$ as required.

Theorem 4.6. If $\models_{\mathsf{GK}} \mathcal{S}$, then $\vdash_{\mathsf{SGK}_{\square}} \mathcal{S}$.

Proof. We prove the theorem by induction on the modal degree of the sequent of relations \mathcal{S} : the maximal complexity of a boxed subformula occurring in \mathcal{S} . If the modal degree is 0, then \mathcal{S} is propositional and SGK_{\square} -derivable. Otherwise, by Lemma 3.6 (since working upwards, the rules do not increase modal degree), we can assume that \mathcal{S} is both GK -valid and saturated. If \mathcal{S} is propositionally GK -valid, then it is derivable. Otherwise, by Lemmas 4.3 and 4.4, \mathcal{S} is of the form:

$$S' \mid \{ \Box A_i \le \Box B_i \}_{i=1}^n \mid \{ \Box C_j \le \bot \}_{i=1}^m$$

and $\models_{\mathsf{GK}} \{ \Box A_i \leq \Box B_i \}_{i=1}^n \mid \{ \Box C_j \leq \bot \}_{j=1}^m$. But then by Lemma 4.5:

$$\models_{\mathsf{GK}} \bigwedge_{i \in I} A_i \leq \bigwedge_{i \in I} B_i \mid \{C_j \leq \bot\}_{j=1}^m \quad \text{ for some } \emptyset \subset I \subseteq \{1, \dots, n\}.$$

Let us assume without loss of generality that $I = \{1, ..., n\}$. Then also:

$$\models_{\mathsf{GK}} \{A_i \leq B_k\}_{i=1}^n \mid \{C_j \leq \bot\}_{j=1}^m \text{ for } k = 1 \dots n.$$

So by the induction hypothesis and an application of (\square) :

$$\vdash_{\mathsf{SGK}} \mathcal{S}' \mid \{ \Box A_i \leq \Box B_k \}_{i=1}^n \mid \{ \Box C_j \leq \bot \}_{j=1}^m \text{ for } k = 1 \dots n.$$

But then S is derivable by repeated applications of (COM).

Corollary 4.7.
$$\vdash_{\mathsf{SGK}_{\square}} \mathcal{S} \text{ iff } \models_{\mathsf{GK}^{\mathsf{F}}} \mathcal{S} \text{ iff } \models_{\mathsf{GK}} \mathcal{S}.$$

4.2. Consequences. We can use the above characterization of SGK_{\square} to give an alternative proof of completeness of the axiomatization HGK_{\square} with respect to both standard and fuzzy Kripke frames, recalling that this result was first obtained by Caicedo and Rodríguez in [9].

Theorem 4.8.
$$\vdash_{\mathsf{HGK}_{\square}} A \text{ iff } \models_{\mathsf{GK}^{\mathsf{F}}} A \text{ iff } \models_{\mathsf{GK}} A.$$

Proof. We can easily show that the axioms of HGK_□ are GK^F-valid and that the rules (MP) and (NEC)_□ preserve GK^F-validity. So $\vdash_{\mathsf{HGK}_{\square}} A$ implies $\models_{\mathsf{GK}^{\mathsf{F}}} A$, which, since all Kripke frames are fuzzy Kripke frames, implies $\models_{\mathsf{GK}} A$. On the other hand, if $\models_{\mathsf{GK}} A$, then by Theorem 4.6, $\vdash_{\mathsf{SGK}_{\square}} \top \leq A$. Hence it suffices to show that $\vdash_{\mathsf{SGK}_{\square}} \mathcal{S}$ implies $\vdash_{\mathsf{HGK}_{\square}} I_S(\mathcal{S})$. I.e., we need that for each rule $\mathcal{S}_1, \ldots, \mathcal{S}_n / \mathcal{S}$ of SGK_{\square} , whenever $\vdash_{\mathsf{HGK}_{\square}} I_S(\mathcal{S}_i)$ for $i = 1 \ldots n$, also $\vdash_{\mathsf{HGK}_{\square}} I_S(\mathcal{S})$. This is straightforward for the logical rules and easy for the

axioms and structural rules, so let us just consider the case of (\Box) . Using the derivability $\vdash_{\mathsf{HGK}_{\Box}} ((\Box A_1 \to B) \lor (\Box A_2 \to B)) \leftrightarrow (\Box (A_1 \land A_2) \to B)$, it is enough to show that $\vdash_{\mathsf{HGK}_{\Box}} (A \to B) \lor \neg C$ implies $\vdash_{\mathsf{HGK}_{\Box}} (\Box A \to \Box B) \lor \neg \Box C$. Note first that $\vdash_{\mathsf{HG}} (\neg \neg F \to G) \leftrightarrow (G \lor \neg F)$. Hence if $\vdash_{\mathsf{HGK}_{\Box}} (A \to B) \lor \neg C$, then $\vdash_{\mathsf{HGK}_{\Box}} \neg \neg C \to (A \to B)$. But then by (NEC), $\vdash_{\mathsf{HGK}_{\Box}} \Box (\neg \neg C \to (A \to B))$ and using (K_{\Box}) , $\vdash_{\mathsf{HGK}_{\Box}} \Box \neg \neg C \to (\Box A \to \Box B)$. Hence, using (Z_{\Box}) , $\vdash_{\mathsf{HGK}_{\Box}} \neg \neg \Box C \to (\Box A \to \Box B)$, and finally, $\vdash_{\mathsf{HGK}_{\Box}} (\Box A \to \Box B) \lor \neg \Box C$ as required.

Our completeness proof for SGK_{\square} can also be exploited to obtain a precise bound for the complexity of the GK -validity problem for the box fragment, namely the problem of checking $\models_{\mathsf{GK}} A$ for any $A \in \mathrm{Fm}_{\mathcal{L}_{\square}}$.

Theorem 4.9. The validity problem for the box fragment of GK is PSPACE-complete.

Proof. First we show that checking GK-validity for the box fragment is PSPACE-hard. We recall that the modal logic K is PSPACE-complete (see, e.g., [10]). Consider the translation * sending each propositional variable p to its double negation $\neg \neg p$. We can easily show that $\models_{\mathsf{K}} A$ iff $\models_{\mathsf{GK}} A^*$ which establishes that the validity problem for the box fragment of GK must also be PSPACE-hard. For the non-trivial direction consider any Kripke model $\langle W, R, V \rangle$ for GK and define a standard Kripke model $\langle W, R, V' \rangle$ by stipulating: $V'(p, x) = V(\neg \neg p, x)$. Then by a simple induction, $V'(C, x) = V(C^*, x) \in \{0, 1\}$ for any $C \in \mathrm{Fm}_{\mathcal{L}_{\square}}$. Hence if $\not\models_{\mathsf{GK}} A^*$, then $\not\models_{\mathsf{GK}} A$.

For PSPACE-inclusion, we consider derivations in the sequent of relations calculus SGK_{\square} . Given a formula $A \in \mathrm{Fm}_{\mathcal{L}_{\square}}$, let Sub(A) be the set of subformulas of A together with \top, \bot , and consider the set

$$\Phi_A = \{C \triangleleft D \mid C, D \in Sub(A), A \in \{<, \leq\}\}.$$

The cardinality of Φ_A is $O(|A|^2)$. Since any sequent of relations appearing in a derivation of $T \leq A$ is a subset of Φ_A , its size is also $O(|A|^2)$.

We now consider the length of branches in the search for a derivation of $T \leq A$ in SGK_{\square} . Using the GK-invertibility of the logical rules we assume that any branch of a derivation is expanded by applying iteratively the rules upwards in the following order:

- (1) Apply the logical rules, (WL), (WR), (CS), and (COM) in order to obtain a saturated sequent and check the axioms.
- (2) Apply (\Box) and restart from (1) with the premise of this rule.

The length of the branch built in (1) is $O(|A|^2)$ since each logical rule replaces one relation with one or two relations involving formulas of smaller complexity, and each application of (WL), (WR), (CS), and (COM) add exactly one relation at a time, with the total number of different relations possible being $O(|A|^2)$. The sequent obtained in (2) by applying (\square) has a smaller or equal number of relations and a strictly smaller modal degree. The entire length of a proof branch is hence bounded by $O(|A|^2 \times m) = O(|A|^3)$, where m is the modal degree of A.

Thus storing a branch of a proof requires only polynomial space. Moreover, the branching is at most binary. As usual, we search for a proof in a depth-first manner: we store one branch at a time together with some information (requiring a small amount of space, say logarithmic space) to reconstruct branching points and backtracking points, the latter determined by alternative applications of (\Box) . Hence the total amount of space needed for carrying out proof search is polynomial in |A|, and so deciding validity for the box fragment of GK is in PSPACE.

Note that the validity problem for modal finite-valued Gödel logics (in the full language) is also PSPACE-complete. This result was established in [6] using a reduction to the classical case for the equivalent problem for description logics based on finite-valued Gödel logics.

4.3. A Hypersequent Calculus. A more elegant analytic calculus for the box fragment – lacking, however, the invertible logical rules of SG – can be presented in the framework of hypersequents. In particular, let us define the hypersequent calculus GGK_{\square} as the extension of the calculus GG given in Figure 3 (for \mathcal{L}_{\square}) with the modal rule:

$$\frac{\Pi \Rightarrow \mid \Gamma \Rightarrow A}{\Box \Pi \Rightarrow \mid \Box \Gamma \Rightarrow \Box A} \ (\Box)$$

(\square) is a version of the ordinary Gentzen rule for the (classical) modal logic K, obtained by adding the extra sequent $\Pi \Rightarrow$ in the premise and $\square\Pi \Rightarrow$ in the conclusion. This extra component reflects the fact that \bot is interpreted as the bottom element 0 in each world.

Example 4.10. All the axioms of HGK_{\square} are derivable in GGK_{\square} ; e.g., for (Z_{\square}) :

$$\frac{\overline{p} \Rightarrow \overline{p} \text{ (ID)}}{\underline{p}, \neg p \Rightarrow (\neg \Rightarrow)} \xrightarrow{\overline{p} \Rightarrow \overline{p}} \xrightarrow{\text{(ID)}}$$

$$\frac{p, p \Rightarrow |\neg p, \neg p \Rightarrow}{p, p \Rightarrow |\neg p \Rightarrow} \xrightarrow{\text{(COM)}}$$

$$\frac{p, p \Rightarrow |\neg p \Rightarrow}{p, p \Rightarrow |\neg p \Rightarrow} \xrightarrow{\text{(CL)}}$$

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It will be helpful for proving cut-elimination to consider the following generalizations of the rule (\Box) , derivable using (COM), (\Box) , (CL), and (WL):

$$\frac{\Pi_1 \Rightarrow |\dots| \Pi_n \Rightarrow |\Gamma \Rightarrow A}{\square \Pi_1 \Rightarrow |\dots| \square \Pi_n \Rightarrow |\square \Gamma \Rightarrow \square A} (\square)^n \qquad (n \in \mathbb{N})$$

For example, in the case of n=2, we have the derivation:

$$\frac{\Pi_{1} \Rightarrow \mid \Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A}{\vdots} \text{ (WL)} \qquad \frac{\Pi_{1} \Rightarrow \mid \Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A}{\vdots} \text{ (WL)}$$

$$\frac{\Pi_{1}, \Pi_{2} \Rightarrow \mid \Pi_{1}, \Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A}{\vdots} \text{ (WL)}$$

$$\frac{\Pi_{1}, \Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A}{\square\Pi_{1}, \Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A} \text{ (UL)}$$

$$\frac{\Pi_{1}, \Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A}{\square\Pi_{1}, \square\Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A} \text{ (UL)}$$

$$\frac{\Pi_{1}, \Pi_{2} \Rightarrow \mid \Pi_{1}, \Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A}{\square\Pi_{1}, \square\Pi_{2} \Rightarrow \mid \Gamma \Rightarrow A} \text{ (CL)}$$

$$\frac{\square\Pi_{1}, \square\Pi_{1} \Rightarrow \mid \square\Pi_{2}, \square\Pi_{2} \Rightarrow \mid \square\Gamma \Rightarrow \square A}{\vdots} \text{ (CL)}$$

Theorem 4.11. $\vdash_{\mathsf{GGK}_{\square}} \mathcal{G} \text{ iff } \models_{\mathsf{GK}} I_H(\mathcal{G}).$

Proof. The left-to-right direction (soundness) is proved as usual by induction on the height of a derivation in GGK_{\square} . For the right-to-left direction (completeness), we make use of the completeness of the axiom system HGK_{\square} established in Theorem 4.8. I.e., $\models_{\mathsf{GK}} I_H(\mathcal{G})$ implies $\vdash_{\mathsf{HGK}_{\square}} I_H(\mathcal{G})$. But now, since all the axioms of HGK_{\square} are derivable in GGK_{\square} and the rules (MP) and (NEC) $_{\square}$ are also derivable, we have that $\vdash_{\mathsf{HGK}_{\square}} I_H(\mathcal{G})$ implies $\vdash_{\mathsf{GGK}_{\square}} I_H(\mathcal{G})$. Finally, it is straightforward to show (following [25], Proposition 4.61) that $\vdash_{\mathsf{GGK}_{\square}} I_H(\mathcal{G})$ implies $\vdash_{\mathsf{GGK}_{\square}} \mathcal{G}$ as required. \square

Let us show now that cut-elimination holds for GGK_{\square} , i.e., that there is a constructive procedure for transforming a derivation of a hypersequent $\mathcal G$ in this calculus into a derivation of $\mathcal G$ with no applications of (CUT). We write $d \vdash_{\mathsf S} X$ to denote that d is a derivation of X in a calculus $\mathsf S$ and |d| for the height of the derivation considered as a tree. We also recall that the principal formula of an application of a rule is the distinguished formula in the conclusion and that the cut-formula of an application of (CUT) is the formula appearing in the premises but not the conclusion.

Theorem 4.12. Cut-elimination holds for GGK_{\square} .

Proof. Let $\mathsf{GGK}^{\circ}_{\square}$ be GGK_{\square} with (CUT) removed. Then to establish cut-elimination for GGK_{\square} it is sufficient to give a constructive proof of the following:

$$\begin{array}{ll} \textit{Claim.} & \text{If } d_1 \vdash_{\mathsf{GGK}_\square^\circ} [\Gamma_i, [A]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \text{ and } d_2 \vdash_{\mathsf{GGK}_\square^\circ} \mathcal{H} \mid [\Pi_j \Rightarrow A]_{j=1}^m, \\ & \text{then } \vdash_{\mathsf{GGK}_\square^\circ} \mathcal{H} \mid [\Gamma_i, \Pi_j^{\lambda_i} \Rightarrow \Delta_i]_{i=1...n}^{j=1...m}. \end{array}$$

We proceed by induction on the lexicographically ordered pair $\langle |A|, |d_1| + |d_2| \rangle$. If the last step in d_1 or d_2 is an axiom, then the result follows almost immediately. Also, if the last step in either derivation is not (\Box) or does not have the cut-formula A as the principal formula, then the result follows by applications of the induction hypothesis to the premises and applications of the same rule and structural rules. Suppose for example that one of the derivations ends with an application of (COM) (the other derivation may end with (\Box)):

$$\frac{\mathcal{G} \mid \Gamma_1', \Gamma_2', [A]^{\lambda_1' + \lambda_2'} \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_1'', \Gamma_2'', [A]^{\lambda_1'' + \lambda_2''} \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1', \Gamma_1'', [A]^{\lambda_1' + \lambda_1''} \Rightarrow \Delta_1 \mid \Gamma_2', \Gamma_2'', [A]^{\lambda_2' + \lambda_2''} \Rightarrow \Delta_2}$$

where $\mathcal{G} = [\Gamma_i, [A]^{\lambda_i} \Rightarrow \Delta_i]_{i=3}^n$. Then by the induction hypothesis twice:

$$\vdash_{\mathsf{GGK}_{\square}^{\circ}} \mathcal{H}' \mid [\Gamma_1', \Gamma_2', \Pi_j^{\lambda_1' + \lambda_2'} \Rightarrow \Delta_1]_{j=1}^m \quad \text{and} \quad \vdash_{\mathsf{GGK}_{\square}^{\circ}} \mathcal{H}' \mid [\Gamma_1'', \Gamma_2'', \Pi_j^{\lambda_1'' + \lambda_2''} \Rightarrow \Delta_2]_{j=1}^m$$

where $\mathcal{H}' = \mathcal{H} \mid [\Gamma_i, \Pi_i^{\lambda_i} \Rightarrow \Delta_i]_{i=3...n}^{j=1...m}$. The required hypersequent:

$$\mathcal{H}' \mid [\Gamma_1', \Pi_i^{\lambda_1' + \lambda_1''}, \Gamma_1'' \Rightarrow \Delta_1]_{j=1}^m \mid [\Gamma_2', \Pi_i^{\lambda_2' + \lambda_2''}, \Gamma_2'' \Rightarrow \Delta_2]_{j=1}^m$$

is derivable by repeated applications of (COM), (EC), and (EW).

If a distinguished occurrence of A is the principal formula in both derivations and of the form $B \wedge C$, $B \vee C$, or $B \to C$, then we can first use the induction hypothesis applied to the premises in one derivation and the conclusion in the other, and then apply the induction hypothesis again with cut-formulas B and C of smaller complexity. The result follows using applications of (EC) and/or (EW) as required. Consider then the hardest case where both

derivations d_1 and d_2 end as follows with an application of (\Box) and A is of the form $\Box B$:

$$\frac{\vdots}{\Gamma_{1}, [B]^{\lambda_{1}} \Rightarrow |\Gamma_{2}, [B]^{\lambda_{2}} \Rightarrow C} \qquad \qquad \frac{\vdots}{\Sigma \Rightarrow |\Pi \Rightarrow B}$$

$$\square\Gamma_{1}, [\square B]^{\lambda_{1}} \Rightarrow |\square\Gamma_{2}, [\square B]^{\lambda_{2}} \Rightarrow \square C} \qquad \square\Sigma \Rightarrow |\square\Pi \Rightarrow \square B \qquad (\square)$$

Then since $|B| < |\Box B|$, we can apply the induction hypothesis to the $\mathsf{GGK}_{\Box}^{\circ}$ -derivable hypersequents $(\Gamma_1, [B]^{\lambda_1} \Rightarrow |\Gamma_2, [B]^{\lambda_2} \Rightarrow C)$ and $(\Sigma \Rightarrow |\Pi \Rightarrow B)$ to obtain a $\mathsf{GGK}_{\Box}^{\circ}$ -derivation of $(\Sigma \Rightarrow |\Gamma_1, \Pi^{\lambda_1} \Rightarrow |\Gamma_2, \Pi^{\lambda_2} \Rightarrow C)$. Hence by an application of the derived rule $(\Box)^2$, we obtain a $\mathsf{GGK}_{\Box}^{\circ}$ -derivation of $\Box \Sigma \Rightarrow |\Box \Gamma_1, \Box \Pi^{\lambda_1} \Rightarrow |\Box \Gamma_2, \Box \Pi^{\lambda_2} \Rightarrow \Box C$ as required. \Box

We note finally that a related hypersequent calculus GGK^r was defined (along with many other such calculi) in [11] by extending GG with the rule:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A}{\mathcal{G} \mid \Box \Gamma \Rightarrow \Box A}$$

It was shown in [11] that GGK^r is complete with respect to the standard semantics for Gödel logic extended with a unary function on [0,1] that can be interpreted in fuzzy logic as a "truth stresser" modality such as "very true".

5. The Diamond Fragments

- 5.1. Sequent of Relations Calculi. Let us turn our attention now to the (distinct) diamond fragments of GK and GK^F , and for convenience assume for the rest of this section that all notions (formulas, rules, etc.) refer exclusively to the sublanguage \mathcal{L}_{\Diamond} . We introduce the following systems:
- SGK $_{\Diamond}$ consists of SG (for \mathcal{L}_{\Diamond}) extended with:

$$\frac{A \le \Delta \mid \bot < \Sigma \mid \top \le \Theta}{\mathcal{S} \mid \lozenge A \le \lozenge \Delta \mid \bot < \lozenge \Sigma \mid \top \le \lozenge \Theta} \ (\lozenge)$$

• $\mathsf{SGK}_{\Diamond}^{\mathsf{F}}$ consists of SG (for \mathcal{L}_{\Diamond}) extended with:

$$\frac{A \le \Delta \mid \bot < \Sigma}{\mathcal{S} \mid \Diamond A \le \Diamond \Delta \mid \bot < \Diamond \Sigma} \ (\Diamond)^*$$

Example 5.1. Axioms from (Z_{\Diamond}) are derivable in both SGK_{\Diamond} and $SGK_{\Diamond}^{\mathsf{F}}$:

$$\frac{p \leq \bot \mid \bot$$

However, the opposite direction is derivable only in SGK_{\Diamond} :

Ever, the opposite direction is derivable only in SGR
$$_{\Diamond}$$
:
$$\frac{\bot$$

Theorem 5.2. Let $L \in \{GK, GK^F\}$. If $\vdash_{SL_{\wedge}} S$, then $\models_{L} S$.

Proof. Again, the proof is a straightforward induction on the height of a derivation, and it suffices to check that (\lozenge) and $(\lozenge)^*$ preserve $\mathsf{GK}\text{-validity}$ and $\mathsf{GK}^\mathsf{F}\text{-validity}$, respectively. Moreover, using the distributivity of \Diamond over \vee , we can assume that Δ , Σ , and Θ each contain exactly one formula. Hence for (\lozenge) suppose contrapositively that for some K= $\langle W, R, V \rangle$ and $x \in W$: $V(\Diamond A, x) > V(\Diamond B, x)$, $V(\Diamond C, x) = 0$, and $V(\Diamond D, x) < 1$. Then there is a world y such that Rxy and V(A,y) > V(B,y), V(C,y) = 0, and V(D,y) < 1 as required. Now for $(\lozenge)^*$ suppose contrapositively that for some $K = \langle W, R, V \rangle$ and $x \in W$: $V(\Diamond A, x) > V(\Diamond B, x)$ and $V(\Diamond C, x) = 0$. Then for some world y: $\min(V(A, y), Rxy) > 0$ $\min(V(B,y),Rxy)$, which implies V(A,y) > V(B,y), and $V(\Diamond C,x) = 0$.

Completeness proofs mostly follow the same pattern as the proof for SGK_{\square} ; however, there exist a couple of significant differences in the details.

Lemma 5.3. Suppose that S is saturated and L-valid for $L \in \{GK, GK^F\}$. Then either S is propositionally L-valid or the modal part of S is L-valid.

Proof. Proceeding contrapositively, let S_{\Diamond} be the modal part of S and suppose that S_{\Diamond} is not L-valid and S is not propositionally L-valid. Then for some model $K = \langle W, R, V \rangle$ for L and $x \in W$: $V(A,x) \not A V(B,x)$ for all $(A \triangleleft B) \in \mathcal{S}^{\Diamond}$. For each $\Diamond A$ occurring in \mathcal{S} , let us add a constant c_A to the language so that $r_{c_A} = V(\Diamond A, x)$. Let \mathcal{S}^P be \mathcal{S} with each $\Diamond A$ occurring in S replaced by c_A . Exactly as in the proof of Lemma 4.3, it follows that S^P is not L-valid. But now let $v: \operatorname{Fm}_{\mathcal{L}_{\mathsf{G}}} \to [0,1]$ be the propositional counter-valuation for \mathcal{S}^P . Define $K' = \langle W \cup \{x_0\}, R', V' \rangle$ where:

$$(1) R'yz = \begin{cases} Ryz & y, z \in W \\ Rxz & y = x_0, z \in W \\ 0 & z = x_0. \end{cases}$$

(2) V' is V extended with $V'(p, x_0) = v(p)$ for all $p \in Var$.

Then for each $\Diamond A$ occurring in \mathcal{S} :

$$V'(\Diamond A, x_0) = \sup_{y \in W} (\min(V'(A, y), Rx_0 y)) = \sup_{y \in W} (\min(V(A, y), Rxy)) = V(\Diamond A, x) = r_{c_A}.$$

So, since v is a counter-valuation for \mathcal{S}^P , we have that \mathcal{S} is not L-valid as required. **Lemma 5.4.** Let $S \mid S'$ be saturated, purely modal, and L-valid for $L \in \{GK, GK^F\}$, where \mathcal{S}' consists only of relations of the form $\Diamond A < B$. Then \mathcal{S} is L-valid.

Proof. Recall that (COM)' is defined as (COM) restricted to instances where \triangleleft_1 is \leq , and that an atomic sequent of relations S is semi-saturated if whenever S occurs as the conclusion of (COM)', (CS), (WL), or (WR), then S also occurs as one of the premises. It is then sufficient to prove the following:

Claim. If $S \mid \Diamond A < B$ is semi-saturated, purely modal, and L-valid for $L \in \{GK, GK^F\}$, then \mathcal{S} is L-valid.

Proof of claim. Proceeding contrapositively, suppose that S is not L-valid. Then there is a Kripke model $K = \langle W, R, V \rangle$ for L and $x \in W$ such that $V(C, x) \not A V(D, x)$ for all $(C \triangleleft D) \in \mathcal{S}$. Moreover, if $V(\Diamond A, x) \geq V(B, x)$, then $\mathcal{S} \mid \Diamond A < B$ is not L-valid as required, so assume that:

$$(\star)$$
 $V(\Diamond A, x) < V(B, x).$

Since $S \mid \Diamond A < B$ is semi-saturated, for each $(C < D) \in S$:

either
$$(C \le B) \in \mathcal{S}$$
 and so $V(C, x) > V(B, x)$
or $(\lozenge A \le D) \in \mathcal{S}$ and so $V(\lozenge A, x) > V(D, x)$.

In particular:

$$(\star\star)$$
 $V(\Diamond A, x) \leq V(D, x) < V(C, x) \leq V(B, x)$ is not possible.

Suppose that $V(\lozenge A, x) = 0$. Using (CS) and (COM), either $(\lozenge A \leq \bot) \in \mathcal{S}$ or $(\bot < B) \in \mathcal{S}$. In the first case, $V(\Diamond A, w) > 0$, a contradiction. In the second, V(B, x) = 0, also a contradiction. So let us assume $V(\Diamond A, x) > 0$. Then using Lemma 3.1, we define for each $i \in \mathbb{Z}^+$, a Kripke model $K_i = \langle W_i, R_i, V_i \rangle$ for L such that:

- (1) $\langle W_i, R_i \rangle$ is a copy of $\langle W, R \rangle$ with distinct worlds for each $i \in \mathbb{Z}^+$ where x_i is the corresponding copy of x.
- (2) For all formulas E satisfying $V_i(\Diamond A, x_i) \leq V_i(E, x_i) < V_i(B, x_i)$:

$$V_i(B, x_i) - 1/i < V_i(E, x_i) < V(B, x_i).$$

Now we define a model $K' = \langle W', R', V' \rangle$ where:

(1)
$$W' = \{x_0\} \cup \bigcup_{i \in \mathbb{Z}^+} W_i$$

(2) $R'yz = \begin{cases} R_i yz & y, z \in W_i \\ R_i x_i z & y = x_0, z \in W_i \\ 0 & z = x_0 \end{cases}$

But then:

$$V'(\lozenge A, x_0) = \sup_{y \in W} (\min(V(A, y), R'x_0 y))$$

$$= \sup \{\sup_{y \in W_i} (\min(V(A, y), R_i x_i y) \mid i \in \mathbb{Z}^+\}\}$$

$$= \sup \{V_i(\lozenge A, x_i) \mid i \in \mathbb{Z}^+\}$$

$$= V'(B, x_0).$$

Now consider $(C \triangleleft D) \in \mathcal{S}$, recalling that C and D are diamond formulas, \bot , or \top . Clearly, if \triangleleft is <, then $V'(C,x_0) \geq V'(D,x_0)$. If \triangleleft is \leq , then using $(\star\star)$, it follows that $V'(C,x_0) >$ $V'(D, x_0)$. So $S \mid \Diamond A < B$ is not L-valid as required.

The next lemma is particular to the diamond fragment of GK^F and is necessary for the extra step, not required in the case of GK, of removing relations of the form $\top \leq \lozenge A$.

Lemma 5.5. If $(S \mid T \leq \Diamond A)$ is modal, saturated, and $\mathsf{GK}^\mathsf{F}\text{-}valid$, then S is $\mathsf{GK}^\mathsf{F}\text{-}valid$.

Proof. We argue by contraposition. Suppose that S is not GK^F -valid. Then there is a Kripke model $K = \langle W, R, V \rangle$ for GK^F and $x \in W$ such that $V(C, x) \not \land V(D, x)$ for all $(C \triangleleft D) \in \mathcal{S}$. Fix a value $\lambda < 1$ such that whenever $V(\Diamond B, x) < 1$ for some $\Diamond B$ occurring in \mathcal{S} , also $V(\Diamond B, x) < \lambda$. We define a Kripke model $K' = \langle W', R', V' \rangle$ for GK^F where:

(1)
$$W = \{x_0\} \cup W$$

(2) $R'yz = \begin{cases} Ryz & y, z \in W \\ \min(\lambda, Rxz) & y = x_0, z \in W \\ 0 & z = x_0 \end{cases}$
(3) $V'(p, y) = V(p, y)$ for all $y \in W$ and $V'(p, x_0) = 0$.

But then:

$$V'(\lozenge A, x_0) = \sup_{y \in W} (\min(V'(A, y), R'x_0 y))$$

$$= \sup_{y \in W} (\min(V(A, y), \min(\lambda, Rxy)))$$

$$= \begin{cases} V(\lozenge A, x) & \text{if } V(\lozenge A, x) < 1\\ \lambda & \text{otherwise.} \end{cases}$$

So $(S \mid T \leq \Diamond A)$ is not GK^{F} -valid as required.

Note that the following lemma includes relations of the form $\top \leq \Diamond A$ and holds even in the case of GK^F where they are not needed.

Lemma 5.6. If $\models_{\mathsf{L}} \{ \Diamond A_i \leq \Diamond B_i \}_{i=1}^n \mid \bot < \Diamond C \mid \top \leq \Diamond D \text{ for } \mathsf{L} \in \{\mathsf{GK}, \mathsf{GK}^\mathsf{F}\}, \text{ then}$

$$\models_{\mathsf{L}} \bigwedge_{j \in J} A_j \leq \bigwedge_{j \in J} B_j \mid \bot < C \mid \top \leq D$$

for some $\emptyset \subset J \subseteq \{1, \ldots, n\}$.

Proof. We argue by contraposition; i.e., suppose that:

$$\not\models_{\mathsf{L}} \bigwedge_{j \in J} A_j \le \bigwedge_{j \in J} B_j \mid \bot < C \mid \top \le D \quad \text{ for all } \emptyset \subset J \subseteq \{1, \dots, n\}.$$

We obtain a model for each $i \in \{1, ..., n\}$ as follows. By assumption:

$$\not\models_{\mathsf{L}} A_i \wedge \ldots \wedge A_n \leq B_i \wedge \ldots \wedge B_n \mid \bot < C \mid \top \leq D.$$

So we have $K_i = \langle W_i, R_i, V_i \rangle$ and $x_i \in W_i$ (with each W_i distinct) such that:

- (1) $V_i(A_i \wedge \ldots \wedge A_n, x_i) > V_i(B_i \wedge \ldots \wedge B_n, x_i)$
- (2) $V_i(C) = 0$ and $V_i(D) < 1$.

Moreover, without loss of generality we can assume:

$$V_i(B_i, x_i) \le V_i(B_k, x_i)$$
 and so $V_i(A_k, x_i) > V_i(B_i, x_i)$ for $k = i \dots n$.

Now using Lemma 3.1 as in the case of Lemma 4.5, we define iteratively $K'_i = \langle W_i, R'_i, V'_i \rangle$ for $i = n \dots 1$ such that for $j = i \dots n$:

- (1) $V'_k(B_i, x_k) < V_k(A_i, x_i)$ for $k = 1 \dots i 1$
- (2) $V_i^n(B_j, x_j) < V_k'(A_j, x_k)$ for $k = i \dots n$
- (3) $V'_i(C, x_j) = 0$ and $V'_i(D, x_j) < 1$.

Finally, we define a model $K = \langle W, R, V \rangle$ where for a new world x_0 :

$$(i) W = W_1 \cup \ldots \cup W_n \cup \{x_0\}$$

(i)
$$W = W_1 \cup \ldots \cup W_n \cup \{x_0\}$$

(ii) $Rxy = \begin{cases} R_i xy & x, y \in W_i \\ 1 & x = x_0, y = x_i \text{ for } i \in \{1 \dots n\} \\ 0 & y = x_0 \end{cases}$

(iii) $V(p,x) = V'_i(p,x)$ for all $x \in W_i$ and $V(p,x_0) = 0$.

As in Lemma 4.5, we obtain $\not\models_{\mathsf{L}} \{ \lozenge A_i \leq \lozenge B_i \}_{i=1}^n \mid \bot < \lozenge C \mid \top \leq \lozenge D \text{ as required.}$

Our desired result is then obtained following the same pattern as in the completeness proof for the box case.

Theorem 5.7. For
$$L \in \{GK, GK^F\}$$
: $\models_L S \text{ iff } \vdash_{SL_{\Diamond}} S$.

5.2. Consequences. As in the case of the box fragment, we can use our completeness result for sequent of relations calculi to establish completeness also for the axiomatizations of the diamond fragments presented in Section 3.2.

Theorem 5.8. For
$$L \in \{GK, GK^F\}$$
: $\models_L A iff \vdash_{HL_{\Diamond}} A$.

Proof. Let $L \in \{GK, GK^F\}$. Following the proof of Theorem 4.8, it suffices to show that for each rule $S_1, \ldots, S_n / S$ of SL_{\Diamond} , whenever $\vdash_{\mathsf{HL}_{\Diamond}} I(S_i)$ for $i = 1 \ldots n$, also $\vdash_{\mathsf{HL}_{\Diamond}} I(S)$. In particular, let us just consider the case of (\lozenge) for HGK_{\lozenge} , since the case of $(\lozenge)^*$ for $\mathsf{HGK}_{\lozenge}^\mathsf{F}$ follows exactly the same pattern. Using the derivabilities $\vdash_{\mathsf{HGK}_{\Diamond}} ((A \to \Diamond B_1) \lor (A \to \Box B_1))$ $\Diamond B_2)) \leftrightarrow (A \to \Diamond (B_1 \vee B_2))$ and $\vdash_{\mathsf{HGK}_{\Diamond}} (\neg \Diamond A \wedge \neg \Diamond B) \leftrightarrow \neg \Diamond (A \vee B)$, it is enough to show that $\vdash_{\mathsf{HGK}_{\Diamond}} \neg C \to ((A \to B) \lor D)$ implies $\vdash_{\mathsf{HGK}_{\Diamond}} \neg \Diamond C \to ((\Diamond A \to \Diamond B) \lor \Diamond D)$. Suppose that $\vdash_{\mathsf{HGK}_{\Diamond}} \neg C \to ((A \to B) \lor D)$. Then using the derivability $\vdash_{\mathsf{HG}} (\neg \neg F \to G) \leftrightarrow (G \lor \neg F)$, we have $\vdash_{\mathsf{HGK}_{\Diamond}} \neg \neg C \lor ((A \to B) \lor D)$. Applying (NEC) $_{\Diamond}$, we get $\vdash_{\mathsf{HGK}_{\Diamond}} (\Diamond A \to \Diamond B) \lor (\Diamond \neg \neg C \lor D)$, and using (K_{\Diamond}) , we have $\vdash_{\mathsf{HGK}_{\Diamond}} (\Diamond A \to \Diamond B) \lor (\Diamond \neg \neg C \lor D)$. Using (Z_{\Diamond}) , we obtain $\vdash_{\mathsf{HGK}_{\Diamond}} (\Diamond A \to \Diamond B) \lor (\neg \neg \Diamond C \lor \Diamond D)$, and finally $\vdash_{\mathsf{HGK}_{\Diamond}} \neg \Diamond C \to ((\Diamond A \to \Diamond B) \lor \Diamond D)$ as required.

The proof of the following complexity results follows exactly the same pattern as in the proof of Theorem 4.9 for the box fragment of GK.

Theorem 5.9. The validity problems for the diamond fragments of GK and GK^F are PSPACE-complete.

Note finally that it is not so easy to extend the hypersequent calculus GG to calculi for the diamond fragments, since there is no natural way to interpret strict inequality relations of the form $\perp < A$ as sequents. One option would be to add decomposition rules for dealing with formulas $A \to \bot$ on the left of sequents occurring in a hypersequent. Completeness of the cut-free calculus could then be proved via a translation into sequents of relations. However, establishing cut elimination would be complicated and it is difficult to see how such a calculus could be easily extended to the first-order level or adapted to other logics.

6. Discussion

In this paper we have presented proof systems for the diamond and box fragments of two minimal normal modal logics based on a Gödel fuzzy logic GK, where the accessibility relation is classical (crisp), and GK^F , where the accessibility relation is fuzzy. In particular, we have introduced a sequent of relations calculus and a hypersequent calculus (admitting cut-elimination) for the box fragment of GK , which coincides with the same fragment of GK^F , and sequent of relations calculi for the (distinct) diamond fragments of GK and GK^F . We have used the calculi to establish completeness for corresponding axiomatizations of the fragments (a new result in the case of the diamond fragment of GK) and to establish new decidability and PSPACE complexity bounds. Finally, in this section, we discuss some related work and avenues for further research.

6.1. Related Proof Systems. As observed already in the introduction, numerous proof systems for Gödel logic may be found in the literature, encompassing sequent calculi [29, 13], hypersequent calculi [2, 3], sequent of relations calculi [4], decomposition systems [3], graphbased methods [23], and goal-directed systems [25]. Our choice of the sequent of relations and hypersequent frameworks for Gödel modal logics was guided by a number of factors. First, like the sequent, decomposition, graph-based, and goal-directed systems, and also the hypersequent calculus GLC* of [3], the sequent of relations calculus for Gödel logic has invertible logical rules. Unlike most of these other systems, however, the rules deal directly with the top-level connective of formulas and may be described as reasonably "natural" or at least relatively easy to understand. Most significantly, perhaps, the framework facilitates a relatively straightforward addition of modalities and easier completeness proofs than would be obtained in, e.g., a sequent calculus framework. Moreover, it seems (without working through all the details) that the rules obtained and completeness proofs for other calculi with invertible rules would be very similar and indeed could be obtained by translating our results into these frameworks. For example, a multiple conclusion sequent $A_1, \ldots, A_n \Rightarrow$ B_1, \ldots, B_m interpreted as $(A_1 \wedge \ldots \wedge A_n) \to (B_1 \vee \ldots \vee B_m)$ is G-valid iff the sequent of relations $A_1 < \top \mid \ldots \mid A_m < \top \mid \top \leq B_1 \mid \ldots \mid \top \leq B_m$ is G-valid. Conversely, a sequent of relations $A_1 < B_1 \mid \ldots \mid A_n < B_n \mid C_1 \leq D_1 \mid \ldots \mid C_m \leq D_m$ is G-valid iff the sequent $B_1 \to A_1, \dots, B_n \to A_n \Rightarrow C_1 \to D_1, \dots, C_m \to D_m$ is G-valid.

We do not of course mean to suggest that the other mentioned frameworks cannot be useful tools for investigating Gödel modal logics. Indeed, while our sequent of relations calculi provide ready-to-implement decision procedures for the box and diamond fragments of GK of optimal complexity and avoiding loop checking, it may be that other calculi provide a more suitable basis for developing automated reasoning methods. In particular, there exist fast graph-based methods (see [23]) for more details) for the P-time problem of checking whether an atomic sequent of relations is G-valid (required in our algorithm before applying the modal rules). Other gains in efficiency may be obtained by avoiding duplication of subformulas using graph-based representations of sequents of relations, and by implementing some form of "goal-directed" proof search. We mention also that our calculi provide a starting point for developing algorithms for fuzzy description logics based on Gödel logic (as opposed to the systems based on finite-valued or witnessed models [21, 6]), although since our results are so far restricted to the box and diamond fragments, this development would depend on the language under consideration.

The hypersequent calculus GG does not have invertible logical rules and is useful, not as a basis for automated reasoning methods, but rather as a suitable tool for tackling theoretical problems for Gödel logic. Unlike the other mentioned calculi, it has been extended to the first-order level (preserving cut-elimination) and used to establish properties such as Herbrand's theorem, Skolemization, and standard completeness for Gödel logic [5, 25]. Similarly, the hypersequent calculus GGK_\square for the box fragments of GK and GK^F can be extended to a first-order system admitting cut-elimination, although establishing completeness with respect to a semantic based on Kripke frames could be a challenging problem. Let us remark also that since hypersequents provide a general uniform framework for fuzzy and other substructural logics (see [25]), this may hold also for modal fuzzy logics. Certainly, structural rules can be removed from GGK_{\square} to obtain hypersequent calculi admitting cut-elimination. The challenge then, as in the first-order case, is to relate such calculi to a semantics based on Kripke frames. A further limitation currently is also that we have a hypersequent calculus only for the box fragment. Calculi for the diamond fragments can be obtained by translation from the sequent of relations systems but uniformity and extensions to the first-order level or other logics are lost.

6.2. Related Logics. Gödel logic is renowned not just as a fuzzy logic but also as a well-known intermediate logic. It therefore makes sense to ask what relationship GK and GK^F bear to modal intuitionistic and intermediate logics found in the literature. From a semantic perspective, the approaches are significantly different. Kripke models for the most popular intuitionistic modal logic IK (see, e.g., [28]) and other modal intermediate logics (see, e.g., [31]) make use of two accessibility relations, one for the modal operator and another for the intuitionistic connectives. Since Kripke models for Gödel logic are linearly ordered, the resulting Kripke models for Gödel modal logics developed in this way are also linear, which is not the case in general for Kripke models for GK or GK^F. A closer comparison semantically could be made with an "intuitionistic modal logic" defined by considering standard modal Kripke frames equipped with an intuitionistic Kripke frame at each node. Nevertheless, from a syntactic perspective, GK and GK^F may be viewed as intermediate modal logics in the sense that each GK-valid formula (which includes each GK^F-valid formula) is valid in K and also every IK-valid formula is GK^F -valid (and therefore also GK -valid). The latter follows easily from the fact that (see, e.g., [28]) an axiomatization for IK is given by an axiomatization for intuitionistic logic extended with (K_{\square}) , (K_{\Diamond}) , (F_{\Diamond}) , $(NEC)_{\square}$, and the GK^F -valid "connecting axioms" $\square(A \to B) \to (\lozenge A \to \lozenge B)$ and $(\lozenge A \to \square B) \to \square(A \to B)$.

We may also consider the situation regarding modal finite-valued Gödel logics, obtained from our definitions by restricting valuations to 0, 1, and a particular finite number of truth values in [0,1]. We expect that sequent of relations calculi can be obtained for the same fragments of modal finite-valued Gödel logics using the modal rules provided in this paper but adding axioms for the propositional case. However, developing these calculi is not so interesting from a theoretical perspective, since the logics are already known to be PSPACE-complete [6] (proved in the context of Gödel description logics by mapping to the classical case). Similarly, we can consider witnessed Gödel modal logics where the values of box formulas $\Box A$ and diamond formulas $\Diamond A$ are attained by A at some world. Calculi for these logics can (we expect) be developed in our framework, but it also seems possible to obtain PSPACE-completeness results using the methods of [6] by providing a bound (based on the formula to be proved) to reduce validity to validity in a modal finite-valued Gödel logic.

6.3. Further Work. As noted in the introduction, this paper provides only a starting point for investigating Gödel modal logics. The most pressing concern is to extend our proof-theoretic treatment to the full language of GK and GK^F, and thereby obtain axiomatization, decidability, and complexity results for these logics. It has been conjectured by Caicedo and Rodríguez that an axiomatization for GK^F is provided by the axioms and rules for the box and diamond fragments extended with the connection axioms of IK (see the previous subsection), but a proof using their algebraic approach is still lacking. From our more proof-theoretic perspective, the difficulty is that appropriate sequent of relations rules should deal with several combinations of box and diamond fragments occurring in relations, i.e., $\Box A \triangleleft \Box B, \ \Box A \triangleleft \Diamond B, \ \Diamond A \triangleleft \Box B, \ \Diamond A \triangleleft \Diamond B$ where \triangleleft is \leq or <, plus relations involving \bot or \top . The combinatorial nature of the required rules leads to an explosion in the number of cases that should be considered in the completeness proof, and it is not clear that the ideas developed in this paper of squeezing or pushing up or down valuations suffice to cope with all possibilities. It may therefore be preferable to consider a more general framework than sequent of relations, perhaps using labels to encode semantic information.

We also intend to consider stronger modal logics such as the box and diamond fragments axiomatized in [9] based on fuzzy Kripke frames satisfying further properties of reflexivity, transitivity, seriality, and symmetry. Our conjecture is that we can extend the sequent of relation calculus to handle all these conditions except possibly symmetry. Again, however, a more general (labelled) framework might be useful. Finally, a further direction of research is the extension of our proof systems to a richer language comprising multiple modalities and truth constants, interesting for applications to fuzzy description logics (see, e.g., [30, 21, 6]).

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