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BOUNDED VARIATION AND THE STRENGTH OF HELLY'S SELECTION THEOREM

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Abstract. We analyze the strength of Helly's selection theorem (HST), which is the most important compactness theorem on the space of functions of bounded variation (BV). For this we utilize a new representation of this space intermediate between L_1 and the Sobolev space $W^{1,1}$, compatible with the—so called—weak* topology on BV. We obtain that HST is instance-wise equivalent to the Bolzano-Weierstraß principle over RCA_0. With this HST is equivalent to ACA_0 over RCA_0. A similar classification is obtained in the Weihrauch lattice.

In this paper we investigate the space of functions of bounded variation (BV) and Helly's selection theorem (HST) from the viewpoint of reverse mathematics and computable analysis. Helly's selection theorem is the most important compactness principle on BV. It is used in analysis and optimization, see for instance [1, 3].

This continues our work in [10] and [12] where (instances of) the Bolzano-Weierstraß principle and the Arzelà-Ascoli theorem were analyzed. There we showed, among others, that an instance of the Arzelà-Ascoli theorem is equivalent to a suitable single instance of the Bolzano-Weierstraß principle (for the unit interval $[0, 1]$), which, in turn, is equivalent to an instance of WKL for Σ^0_1-trees. Here, we will show that an instance of Helly's selection theorem is equivalent to a single instance of the Bolzano-Weierstraß principle (and with this to an instance of the other principles mentioned above). It is a priori not clear that this is possible since the proof of HST uses seemingly iterated application of the Arzelà-Ascoli theorem and since there are compactness principles, which are instance-wise strictly stronger than Bolzano-Weierstraß for $[0, 1]$. (For instance the Bolzano-Weierstraß principle for weak compactness on ℓ_2 has this property, see [11].) A fortori this shows that HST is equivalent to ACA_0 over RCA_0.

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We represent BV as a weak derivative space in the style of Sobolev spaces. Our representation differs from all previous treatments in computable analysis or constructive mathematics known to the author. Previously functions of bounded variation were regarded as actual functions, whereas we only regard them as L_1-functions. With this, they can be characterized by the integral of absolute value of their weak derivative. This has the advantage that it is closer to modern applications. Moreover, this allows one to easily define functions of bounded variation not only on the real line but also on $\mathbb{R}^n$, which is not possible with the classical definition of bounded variation. We therefore believe that our representation has also other applications in computable analysis.

This paper is organized as follows. In Section 1 we define the space BV, in Section 2 we compare BV to other spaces and to other possible representations of functions of bounded variation, and in Section 3 we analyze Helly's selection theorem.

1. The space of functions of bounded variation

A *countable vector space* A over a countable field K consists of a set $|A| \subseteq \mathbb{N}$ and mappings $+\colon |A| \times |A| \longrightarrow |A|$, $\cdot\colon K \times |A| \longrightarrow |A|$, and a distinguished element $0 \in |A|$, such that $A, +, \cdot, 0$ satisfies the usual vector space axioms.

A (code for a) *separable Banach space* B consists of a countable vector space A over $\mathbb{Q}$ together with a function $\|\cdot\|\colon A \longrightarrow \mathbb{R}$ satisfying $\|q \cdot a\| = |q| \cdot \|a\|$ and $\|a + b\| \leq \|a\| + \|b\|$ for all $q \in \mathbb{Q}$, $a, b \in A$. A point in B is defined to be a sequence of elements $(a_k)_k$ in A such that $\|a_k - a_{k+1}\| \leq 2^{-k}$. Addition and multiplication on B are defined to be the continuous extensions of $+, \cdot$ from A to B.

The space $L_1 := L_1([0,1])$ will be represented by the $\mathbb{Q}$-vector space of rational polynomials $\mathbb{Q}[x]$ together with the norm $\|p\|_1 := \int_0^1 |p(x)|\, dx$. Since the rational polynomials are dense in the usual space L_1, this defines (a space isomorphic to) the usually used space (provably in suitable higher-order system where the textbook definition of L_1 can be formalized). See Example II.10.4, Exercise IV.2.15 and Chapter X.1 in [18].

1.1. Bounded variation.

The *variation* of a function $f\colon [0,1] \longrightarrow \mathbb{R}$ is defined to be

$$V(f) := \sup_{0 \leq t_1 < \cdots < t_n \leq 1} \sum_{i=1}^{n-1} |f(t_i) - f(t_{i+1})| \,. \tag{1.1}$$

For an L_1-equivalence classes of functions $f \in L_1$ the variation is defined to be the infimum over all elements, i.e.,

$$V_{L_1}(f) := \inf \{ V(g) \mid g\colon [0,1] \to \mathbb{R} \text{ and } g = f \text{ almost everywhere} \} \,. \tag{1.2}$$

The subspace of all L_1-functions of bounded variation form a subspace of L_1 with the following norm

$$\|f\|_{BV} := \|f\|_1 + V_{L_1}(f).$$

However, it is not possible to code this space as a separable Banach space, as we did for L_1, since the variation V is difficult to compute (see Proposition 17 below) and since this space is not separable in this norm. (To see this take for instance the characteristic functions $\chi_{[0,u]}(x)$ of the intervals $[0,u]$. It is clear that these functions belong to BV. For $u, w \in [0,1]$ with $u \neq w$ the function $\chi_{[0,u]} - \chi_{[0,w]}$ contains a bump of height 1, therefore

$\|\chi_{[0,u]} - \chi_{[0,w]}\|_{BV} \geq 2$. Thus, these functions form a set of the size of the continuum which cannot be approximated by countably many functions.)

We will define the space BV to be a subspace of L_1.

Definition 1 (BV, RCA_0)**.** The space $BV := BV([0,1])$ is defined like the space $L_1([0,1])$ with the following exception. A point in BV is a sequence $(p_k)_k \subseteq \mathbb{Q}[x]$ together with a rational number $v \in \mathbb{Q}$, such that

- $\|p_k - p_{k+1}\|_1 \leq 2^{-k}$, and
- $\int_0^1 |p_k'(x)|\, dx \leq v$.

The vector space operations are defined pointwise for p_k and v. (For scalar multiplication one chooses a suitable rational upper bound for the new v.)

The parameter v will be called the *bound on the variation of* f.

This definition is justified by Propositions 7 and 9 below. For later use we will collect the following lemma.

Lemma 2 (RCA_0)**.** *Let $(f_n)_n \subseteq BV$ be a sequence converging in L_1 at a fixed rate to a function $f \in L_1$, i.e., $\|f_n - f\|_1 \leq 2^{-n}$. If the bounds of variations v_n for f_n are uniformly bounded by a v, then $f \in BV$.*

Proof. Let $(p_{n,k})_k$ be the rational polynomials coding f_n. One has $\|p_{k+1,k+1} - f\|_1 \leq \|p_{k+1,k+1} - f_{k+1}\|_1 + \|f_{k+1} - f\|_1 \leq 2^{-k}$. Thus, $(p_{k+1,k+1})_k, v$ is a code for f in the sense of Definition 1. □

For working with functions of BV it will be handy to use mollifiers as defined below, since one can use them to smoothly approximate characteristic functions without increasing the variation.

Definition 3 (Mollifier, RCA_0)**.** Let

$$\eta(x) := \begin{cases} c \cdot \exp\left(\frac{1}{x^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{where } c := \left(\int_{-1}^1 \exp\left(\frac{1}{x^2-1}\right)\, dx\right)^{-1}.$$

The function η is called a *mollifier*. It is easy to see that η is infinitely often differentiable provably in RCA_0. By definition $\int_{-1}^1 \eta\, dx = 1$.

Define $\eta_\epsilon(x) := \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right)$. We have that the support of η_ϵ is contained in $B(0,\epsilon) = \{x \in \mathbb{R} \mid |x| < \epsilon\}$ and that $\int_{-1}^1 \eta_\epsilon\, dx = 1$.

The integral of this mollifier can be used to smoothly approximate characteristic functions of intervals. For instance

$$x \longmapsto \int_{-1}^x \eta_\epsilon\left(y - \tfrac{1}{4}\right) - \eta_\epsilon\left(y - \tfrac{3}{4}\right)\, dy \tag{1.3}$$

approximates $\chi_{\left[\frac{1}{4},\frac{3}{4}\right]}$ in L_1, see Figure 1. Since the approximating function does not oscillate, the variation of it is not bigger that the variation of the approximated function.

The integral of such a mollifier $x \longmapsto \int_0^x \eta_\epsilon(y-z)\, dy$ is contained in BV. To see this let $(q_k)_k \subseteq \mathbb{Q}[x]$ be a sequence approximating $\eta_\epsilon(x-z)$ in L_1, i.e.

$$\|q_k - \eta_\epsilon(x-z)\|_1 \leq 2^{-k}.$$

Since $\|\eta_\epsilon(x-z)\|_1 \leq 1$ we have that $\|q_k\| \leq 2$. Integrating q_k we obtain a sequence of again rational polynomials $p_k(x) = \int_0^x q_k(y)\, dy$. By definition $\|p_k - \int_0^x \eta_\epsilon(y-z)\, dy\|_1 \leq 2^{-k}$. Thus $(p_k)_k, v = 2$ is a code for the integral of the mollifier.

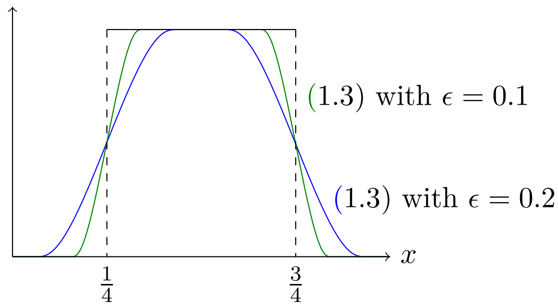

Figure 1: Approximation of $\chi_{[\frac{1}{4},\frac{3}{4}]}$.

Proposition 4 (WWKL_0)**.** *Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be a continuous function. If the variation of f is bounded, that means that there exists a $v \in \mathbb{Q}$ such that all sums in* (1.1) *are bounded by v, then (the L_1-equivalence class of) f belongs to BV.*

For the proof of this proposition we will need the following notation and theorem from [17]. A partition of $[0,1]$ is a finite set $\Delta = \{0 = x_0 \le \xi_1 \le x_1 \le \dots \le \xi_n \le x_n = 1\}$. The mesh of Δ is $|\Delta| := \max\{x_k - x_{k-1} \mid 1 \le k \le n\}$. The Riemann sum for Δ is $S_\Delta(f) := \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1})$. The limit $\lim_{|\Delta|\to 0} S_\Delta(f) = \int_0^1 f(x)\,dx$ is the Riemann integral.

Definition 5. A function f is effectively integrable if there exists a $h\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that for any partitions Δ_1, Δ_2 and $n \in \mathbb{N}$,

$$|\Delta_1| < 2^{-h(n)} \wedge |\Delta_2| < 2^{-h(n)} \to |S_{\Delta_1(f)} - S_{\Delta_2(f)}| < 2^{-n+1}.$$

The function h is called modulus of integrability for f.

Theorem 6 (RCA_0, [17])**.** *The following are equivalent:*

(1) WWKL_0,

(2) *Every bounded, continuous function on* $[0,1]$ *is effectively integrable.*

Proof of Proposition 4. Since the variation of f is bounded, f is bounded. Therefore by Theorem 6 the function f is effectively integrable. In particular, there exists a modulus of integrability h.

Let f_n be the following sequence of step functions approximating f.

$$f_n(x) := \sum\nolimits_k \chi_{[k\cdot 2^{-h(n)},(k+1)\cdot 2^{-h(n)})} \cdot f(k \cdot 2^{-h(n)}) \quad \text{where } k \text{ is such that } x \in \left[\tfrac{k}{2^{h(n)}}, \tfrac{k+1}{2^{h(n)}}\right)$$

Since f_n is a finite sum of characteristic functions of intervals, it belongs to BV. The variation of f_n is obviously bounded by v. By definition $\|f_n - f_{n+1}\|_1 < 2^{-n+1}$, thus $(f_n)_n$ converges in L_1-norm to an $f \in L_1$. By Lemma 2, $f \in BV$. $\square$

In the following we will use right continuous functions. Such a function $f\colon [0,1] \longrightarrow \mathbb{R}$ will be coded by a sequence of real numbers $(x_q)_{q\in\mathbb{Q}}$ index by rational numbers such that the limit from the right

$$\lim_{q\searrow x, q\in\mathbb{Q}} x_q =: f(x)$$

exists. This definition makes sense in ACA_0.

Proposition 7 (ACA_0)**.** *Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be a right continuous function. If the variation of f is bounded as in Proposition 4 then (the L_1-equivalence class of) f belongs to BV.*

Proof. We approximate f using the functions f_n given by

$$f_n(x) := \sum_k \chi_{[k\cdot 2^{-n},(k+1)\cdot 2^{-n})} \cdot f(k \cdot 2^{-n}) \quad \text{where } k \text{ is such that } x \in \left[\tfrac{k}{2^n}, \tfrac{k+1}{2^n}\right)$$

Like in the proof of Proposition 4 the variation of f_n is bounded by the variation v of f. The values of $f_n(x)$ are included in $[f(0) - v, f(0) + v]$. The functions $f_n(x)$ converge to f on all points of continuity of f. We claim that the points of discontinuity of f have measure 0. Indeed, consider the measurable set (in the sense of [18, Defintion X.1.12])

$$A := \bigcup_{n\in\mathbb{N}} \underbrace{\bigcap_{k\in\mathbb{N}} \left\{ x \;\middle|\; \max\left(\left|f(x - 2^{-k}) - f(x)\right|, \left|f(x + 2^{-k}) - f(x)\right|\right) > 2^{-n} \right\}}_{=:A_n}.$$

This formula describes the points of discontinuity of f. Consider the set A_n from above. If for any n the set A_n would have positive measure then there exists $2^n \cdot v$ many points in A_n which would contradict the boundedness of the variation. Thus each A_n has measure 0 and with this A. Therefore, we can apply the dominated convergence theorem (see [2, Theorem 4.3]) and obtain that $(f_n)_n$ converges in L_1 to (the L_1-equivalence class of) f and by Lemma 2 then $f \in BV$. □

Lemma 8 (RCA_0)**.** *For a continuous function $f\colon [0,1] \longrightarrow \mathbb{R}$, such that $|f'(x)|$ is effectively integrable, the variation $V(f)$ is bounded by $\int_0^1 |f'(x)|\, dx$.*

Proof. For two points $t_1, t_2 \in [0,1]$ we can estimate

$$|f(t_1) - f(t_2)| = \left|\int_{t_1}^{t_2} f'(x)\, dx\right| \le \int_{t_1}^{t_2} |f'(x)|\, dx.$$

Therefore,

$$\begin{aligned} V(f) &= \sup_{0\le t_1<\dots<t_n\le 1} \sum_{i=1}^{n-1} |f(t_i) - f(t_{i+1})| \\ &\le \sup_{0\le t_1<\dots<t_n\le 1} \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} |f'(x)|\, dx \le \int_0^1 |f'(x)|\, dx. \end{aligned}$$ □

Proposition 9 (ACA_0)**.** *For each $f \in BV$ there exists a right-continuous function g which is almost everywhere equal to f and with $V(g) < \infty$, or in other words the infimum in* (1.2) *is bounded.*

Proof. Let $(p_k)_k$, v be a code for f. By the previous lemma $V(p_k) \le v$.

By [18, Remark X.1.11] the polynomials $(p_k)_k$ converge to a function g almost everywhere. To be precise there exists an ascending sequence of closed sets $(C_n^f)_n$ with measure $1 - 2^{-n}$ such that $(p_k(x))_k$ converges uniformly on C_n^f for each n. Let $M := \bigcup_n C_n^f$. It is clear that $(p_k)_k$ converges to g also in L_1-norm.

The variation of g with t_i in (1.1) restricted to be in M is, as the pointwise limit of p_k, also bounded by v.

To obtain the proposition the only thing left to show is how to extend g to a proper function on the full unit interval. We claim that there exists a subsequence of $\left(p_{k_j}\right)_j$ such that $\left(p_{k_j}(x)\right)_j$ converges for all $x \in \mathbb{Q} \cap [0,1]$. To obtain this subsequence note that $|p_k(x)| \le \|f\|_1 + v =: v'$. Let q_i be an enumeration of $\mathbb{Q} \cap [0,1]$ and consider for each k the point $(p_k(q_i))_i \in [-v', v']^{\mathbb{N}}$. Now $[-v', v']^{\mathbb{N}}$ is compact and $((p_k(q_i))_i)_k$ contains, by the Bolzano-Weierstraß principle, a convergent subsequence, which also satisfies the claim. See Lemma III.2.5 and Theorem III.2.7 of [18].

Thus, we may assume that $\mathbb{Q} \cap [0,1] \subseteq M$ by passing to a subsequence of (p_k). Then let g_+ be the right continuous extension of g, i.e.

$$g_+(x) := \begin{cases} g(x) & \text{if } x \in M, \\ \lim_{y \searrow x,\, y \in \mathbb{Q}} g(y) & \text{otherwise.} \end{cases}$$

The limit in the second case exists by the boundedness of the variation of g. Suppose that it does not exist then there would be an ϵ and an infinite sequence in M oscillating at least ϵ at each step and, with this, the variation of g would be infinite.

The almost everywhere converging subsequence of $(p_k)_k$ follows by Remark X.1.11 [18] from WWKL. The set M is arithmetic and thus exists provably in ACA_0. Also the extension g_+ of g can be build in using a routine application of the Bolzano-Weierstraß principle again provable in ACA_0. □

Corollary 10 (Jordan decomposition, ACA_0)**.** *For each function $f \in BV$ coded by $(p_k)_k, v$ there exists a measurable set C such that f restricted to C is non-decreasing, that is, $\liminf p_k'(x) \geq 0$ for almost all $x \in C$, and f restricted to the complement of C is non-increasing, that is, $\limsup p_k'(x) \leq 0$.*

Proof. Let g be the right-continuous function as in Proposition 9 and let C be the following measurable set

$$C := \bigcap_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \bigcap_{k > n} \{x \mid g(x) < g(x + 2^{-k}) + 2^{-m}\}.$$

Since g has bounded variation the complement of C is almost everywhere equal to

$$[0,1] \setminus C = \bigcap_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \bigcap_{k > n} \{x \mid g(x) > g(x + 2^{-k}) - 2^{-m}\}.$$

The result follows. □

Independently, the Jordan decomposition was investigated by Nies, Yokoama et al. in [14].

2. Comparison to other spaces

2.1. **Sobolev space $W^{1,1}$.** Our motivation for representing the space BV in the way we did in Definition 1 is that in this way BV lies between L_1 and the Sobolev space $W^{1,1}$. We believe that this is the right way to represent this space since BV is in practice almost always used as an intermediate space between L_1 and $W^{1,1}$.

Recall that the Sobolev space $W^{1,1} := W^{1,1}([0,1])$ is the coded separable Banach space over the rational polynomials $\mathbb{Q}[x]$ together with the following norm

$$\|p\|_{W^{1,1}} := \|p\|_1 + \|p'\|_1.$$

From this definition it is obvious that $W^{1,1}$ is a subspace of BV.

Proposition 11 (RCA_0)**.** *$W^{1,1} \subseteq BV \subseteq L^1$ and all of these inclusions are strict.*

Proof. The inclusions are clear. We show only the strictness. The function

$$f(x) := \begin{cases} x \cdot \sin(1/x \cdot 2\pi) & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

is continuous on $[0,1]$ and therefore contained in L^1. However, it has unbounded variation and therefore $f \notin BV$. A characteristic function of a nontrivial interval, say $\chi_{[\frac{1}{2},1]}$, is contained in BV. It is not contained in $W^{1,1}$, because the derivative of $\chi_{[\frac{1}{2},1]}$ would be almost everywhere 0 and infinite at $\frac{1}{2}$, which is impossible. □

2.2. BV **as dual space.** It is well known that the space BV is isomorphic to the dual space of $C([0,1])$, i.e. the space of uniformly continuous and linear functionals defined on the continuous functions on $[0,1]$ with $\|\cdot\|_\infty$-norm. Before we can show this we will need some more properties of mollifiers.

Definition 12 (Mollification of a function, RCA_0)**.** Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be a continuous, effectively integrable function. We extend f to $[-1,2]$ by setting $f(x) = f(1-x)$ for $x > 1$ and $f(x) = f(-x)$ for $x < 0$. We define the *mollification of* f to be

$$f^\epsilon(x) := (f * \eta_\epsilon)(x) := \int_{x-\epsilon}^{x+\epsilon} \eta_\epsilon(x-y) f(y)\, dy = \int_{-\epsilon}^{\epsilon} \eta_\epsilon(y) f(x-y)\, dy \tag{2.1}$$

for $x \in [0,1]$ and $0 < \epsilon \leq 1$.

For a function $f \in L_1$ the mollification is defined in the same way. (The extension of f can be defined pointwise for each $(p_k)_k$ coding f.)

Proposition 13 (RCA_0)**.** *Let f be as above.*

(i) *f^ϵ is infinitely often differentiable.*
(ii) *If f is uniformly continuous, then $f^\epsilon \xrightarrow{\epsilon\to 0} f$ uniformly. If f has additionally a modulus of uniform continuity then there exists a modulus of convergence for $f^\epsilon \xrightarrow{\epsilon\to 0} f$.*

Proof. (i): We show only that f^ϵ differentiable.

$$\frac{f^\epsilon(x+h) - f^\epsilon(x)}{h} = \frac{1}{\epsilon}\int_0^1 \frac{1}{h}\left(\eta\left(\frac{x+h-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) f(y)\, dy$$

Now for $h \to 0$ we have that $\frac{1}{h}\left(\eta(\frac{x+h-y}{\epsilon}) - \eta(\frac{x-y}{\epsilon})\right)$ converges uniformly in y to $\frac{d}{dx}\eta\left(\frac{x-y}{\epsilon}\right)$. Therefore one can exchange integration and taking the limit of h and obtains that

$$\frac{d}{dx} f^\epsilon(x) = \frac{1}{\epsilon}\int_0^1 \frac{d}{dx}(\eta(x-y)) \cdot f(y)\, dy \tag{2.2}$$

exists.

(ii):

$$\begin{aligned} |f^\epsilon(x) - f(x)| &= \left|\int_{x-\epsilon}^{x+\epsilon} \eta_\epsilon(x-y)(f(y)-f(x))\, dy\right| \\ &\leq \sup_{y\in[x-\epsilon,x+\epsilon]} |f(y)-f(x)| \xrightarrow{\epsilon\to 0} 0 \qquad \text{by uniform continuity.} \end{aligned}$$

Thus from a modulus of uniform continuity one can define a uniform modulus of convergence of $f^\epsilon(x) \xrightarrow{\epsilon\to 0} f(x)$. □

For a code $(p_k)_k, v$ for an $f \in BV$ let T be the following linear functional defined on all $h \in C([0,1])$.

$$T(h) := \lim_{k\to\infty} \int_0^1 h \cdot p_k' \, dx \tag{2.3}$$

Note that T will depend not only on the L_1-class of f but also on the specific sequence of rational polynomials. See Proposition 16 below. We can estimate

$$T(h) \leq \|h\|_\infty \cdot v.$$

Thus T is continuous and therefore in the dual $C^*([0,1])$. It is clear that this is provable in ACA_0. (For a formal definition of bounded functionals and the dual space, see Definitions II.10.5 and X.2.3 in [18].)

For the other direction let $T\colon C([0,1]) \longrightarrow \mathbb{R}$ be a linear, continuous functional with $\|T\| \leq v$ for some $v \in \mathbb{R}$. We can continuously extend T to functions of the form $\chi_{(y,1]}$ (and linear combinations thereof) by approximating this function using the mollifier, cf. (1.3). We claim that the function

$$m(y) := T(\chi_{(y,1]})$$

has bounded variation. Indeed for $0 \leq t_1 < \cdots < t_n \leq 1$ we have

$$\begin{aligned}
\sum_{i=1}^{n-1} |m(t_{i+1}) - m(t_i)| &= \sum_{i=1}^{n-1} e_i \left(m(t_{i+1}) - m(t_i)\right) && \text{for suitable } e_i \in \{-1,1\} \\
&= T\left(\sum_{i=1}^{n-1} e_i \, \chi_{(t_i,t_{i+1}]}\right) \\
&\leq v && \text{since the sum is bounded by 1.}
\end{aligned}$$

It is clear that m is right continuous. Thus, by Proposition 7 we have $m \in BV$. Now let $h \in C([0,1])$ be a uniformly continuous function. The function h can be approximated in $\|\cdot\|_\infty$ by functions of the form

$$h_n(x) := h\left(\frac{i}{2^n}\right) \quad \text{if } x \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right).$$

(A modulus of convergence can be defined from a modulus of uniform continuity of h.) Then

$$\begin{aligned}
T(h) &= T\left(\lim_{n\to\infty} h_n\right) = \lim_{n\to\infty} T(h_n) \\
&= \lim_{n\to\infty} \sum_i \left[h\left(\frac{i}{2^n}\right) \cdot \left(m\left(\frac{i+1}{2^n}\right) - m\left(\frac{i}{2^n}\right)\right)\right]
\end{aligned}$$

(for a suitable choice of $(p_k)_k$ converging pointwise at all $q \in [0,1] \cap \mathbb{Q}$, see proof of Proposition 9)

$$= \lim_{n\to\infty} \lim_{k\to\infty} \sum_i \left[h\left(\frac{i}{2^n}\right) \cdot \left(p_k\left(\frac{i+1}{2^n}\right) - p_k\left(\frac{i}{2^n}\right)\right)\right]$$

(by uniform convergence in n)

$$\begin{aligned}
&= \lim_{k\to\infty} \lim_{n\to\infty} \sum_i \left[h\left(\frac{i}{2^n}\right) \cdot \left(p_k\left(\frac{i+1}{2^n}\right) - p_k\left(\frac{i}{2^n}\right)\right)\right] \\
&= \lim_{k\to\infty} \int_0^1 h \cdot p_k' \, dx.
\end{aligned}$$

These observations give rise to the following propositions.

Proposition 14 (ACA_0)**.** *Each (code of an) $f \in BV$ induces a bounded linear functional $T \in C^*([0,1])$ given by* (2.3)*.*

Proposition 15 (ACA_0)**.** *Each $T \in C^*([0,1])$ is of the form* (2.3) *for a suitable (code of an) $f \in BV$.*

We just note that since h can be approximated by infinitely often differentiable functions we may assume that it is differentiable. Then one can use integration by parts on (2.3) and obtain that

$$T(h) = \lim_{k\to\infty} \left(h(1)p_k(1) - h(0)p_k(0) - \int_0^1 h' \cdot p_k \, dx \right).$$

Under the assumption that $h(0) = h(1) = 0$—this is given for instance if $h \in C_0((0,1))$, that is the space of all uniformly continuous functions with compact support included in $(0,1)$—we get

$$T(h) = -\lim_{k\to\infty} \int_0^1 h' \cdot p_k \, dx.$$

This value can be computed from $\|h'\|_\infty$ since $\|p_k - p_{k+1}\|_1 \le 2^{-k}$. Thus one obtains the following.

Proposition 16 (RCA_0)**.** *The functional $T(h)$ as in* (2.3) *restricted to $h \in C_0((0,1))$ exists and does only depend on the L_1-equivalence class of f (and not on its code).*

Or in other words, in this restricted case one does not need ACA_0 to get Proposition 14. The proposition below shows that ACA_0 is in general necessary.

Proposition 17 (RCA_0)**.** *The statement of Proposition 14 is equivalent to* ACA_0*.*

In fact, it suffices to know for each $f \in BV$ the value $\|T\|$ or $V_{L_1}(f)$ for T as in (2.3) *to obtain* ACA_0*.*

Proof. The right-to-left direction is Proposition 14. For the other direction consider the Π^0_1-statement (indexed by n)

$$\forall i \, \phi(n,i).$$

We show that we can build a set X with $n \in X \leftrightarrow \forall i \, \phi(n,i)$.

Let

$$f_{n,k}(x) := \begin{cases} 1 - 2\int_0^x \eta_{2^{-i'-1}}(y)\,dy & \text{if } \exists i \le k \, \phi(n,i) \text{ and } i' \text{ is minimal with } \phi(n,i'), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\left\|1 - 2\int_0^x \eta_{2^{-i'-1}}(y)\,dy\right\|_1 < 2^{-i'-1}$ the sequence $(f_{n,k})_k$ forms a Cauchy-sequence with rate 2^{-k} for each n and the variation is bounded by 1. By Lemma 2 the limit of f_n of $(f_{n,k})_k$ is contained in BV.

Let T_n be the functional corresponding to f_n as in (2.3). Since the function f_n is the constant 0 function if $\forall i \, \phi(n,i)$ is true and otherwise $\lambda x.1 - 2\int_0^x \eta_{2^{-i'-1}}(y)\,dy$ for an i' we get that

$$T_n(\lambda x.1) = 0 \leftrightarrow \forall i \, \phi(n,i)$$
$$T_n(\lambda x.1) = -1 \leftrightarrow \neg\forall i \, \phi(n,i)$$

Thus, one can read off the real number $T_n(\lambda x.1)$ whether $\forall i \, \phi(n,i)$ is true. To obtain the second statement of the proposition for this particular n note that since T_n is non-increasing $\|T_n\| = V_{L_1}(f) = -T_n(\lambda x.1)$.

To obtain the full result we use a standard Cantor-middle third set construction to embed the Cantor space into the unit interval. See for instance the proof of Theorem IV.1.2 in [18]. Thus, let $f(x) := \sum_{n=0}^{\infty} \frac{2f_n(x)}{3^n}$ and let T be the corresponding functional. Then $\forall i\, \phi(0,i)$ if true if $-T(\lambda x.1) \in [0,1/3]$ and false if it is in $[2/3,1]$. The statement for $n=1$ is true if $-T(\lambda x.1) \in [0,1/9] \cup [2/3,7/9]$ and false if it is in $[2/9,1/3] \cup [8/9,1]$ and so on. From this one can easily construct the set X. $\square$

Remark 18 (Weak* topology). The space $C^*([0,1])$ is a dual space and, with this, one can define the weak* topology on it in the usual way. We say a sequence $(T_n)_n \subseteq C^*([0,1])$ converges to T in the *weak* topology* iff

$$T_n(h) \xrightarrow{n\to\infty} T(h) \quad \text{for all } h \in C([0,1]).$$

Since BV is isomorphic to $C^*([0,1])$ this induces a topology on BV. However, in most cases the following combination with the L_1-topology is used. We say that a sequence of functions $(f_n) \subseteq BV$ converges in the *weak* topology* to f iff $f_n \xrightarrow{n\to\infty} f$ in L_1 and the functionals corresponding to f_n converge in weak* topology of $C^*([0,1])$. See Definition 3.11 of [1].

One can show that for a sequence $(f_n)_n$ and f in BV that if

- $f_n \xrightarrow{n\to\infty} f$ in L_1 and
- the variation of $(f_n)_n$ is uniformly bounded

then there exists a subsequence $(f_{g(n)})_n$ converging in the weak* topology to f. See Proposition 3.13 in [1].[1]

This leads to the following. The representation of BV as given in Definition 1 is *consistent* with the weak* topology in the sense that if a sequence of representations $(r_i)_i \subseteq \mathbb{N}^{\mathbb{N}}$ converges in the Baire space then the sequence of represented elements f_{r_i} contains a weak*-converging subsequence. See also Lemma 2.

2.3. **Other representations.** In [4] Brattka proposes two different ways to represent elements of non-separable spaces. The first representation essentially codes an element f of a space X as a sequence of countable objects plus the norm $\|f\|_X$. Whereas the second representation just consists of the countable objects plus an upper bound v on the norm. See also [8].

In the case of Definition 1 the countable objects are rational polynomials. The representation we defined in Definition 1 is intermediate between those two representations proposed by Brattka because we have an upper bound of the norm of an element $f \in BV$, i.e. $\|f\|_{BV} \leq v$, and thus the second representation is reducible to our representation. However, we have f as full L_1 object including its norm, thus our representation is stronger.

Alternatively, we could have added the value of the variation instead of merely an upper bound to the representation of an element of BV. Since by Proposition 17 going from an upper bound to right value of V_{L_1} requires ACA_0, this representation is too strong in general.

Other ways to represent functions of bounded variation are to take computable functions with a computable variation, see [15], or as a computable function defined on a countable, dense subset of $[0,1]$, see [13, 9]. The first approach is too restricted since very few functions

[1]Note that the theorem there is stated in a misleading way. The statement should actually read "**Proposition 3.13** Let $(u_h) \subset [BV(\omega)]^m$. Then there exists a subsequence $(u_{k(h)})$ converging to u in $[BV(\omega)]^m$ if (u_h) is bounded in $[BV(\omega)]^m$ and u_h converges to u in $[L_1(\omega)]^m$. If (u_h) converges to u in $[BV(\omega)]^m$ then u_h converges to u in $[L_1(\omega)]^m$ and is bounded in $[BV(\omega)]^m$."

of bounded variation are computable. The second approach is orthogonal to ours since it defines points of functions, whereas we define the function in the L_1-sense. This representation has been successfully used in algorithmic randomness, see [7, 16]. However we believe that our approach is more natural since it fits nicely into the Sobolev spaces and easily generalizes to functions defined in $\mathbb{R}^n$, which is not the case for the pointwise definition.

3. Helly's selection theorem

Theorem 19 (Helly's selection theorem, HST, ACA_0)**.** *Let $(f_n)_n \subseteq BV$ be a sequence of functions with bounds for variations v_n. If*

(i) $\|f_n\|_1 \leq u$ *for a* $u \in \mathbb{Q}$*,*
(ii) $v_n \leq v$ *for a* $v \in \mathbb{Q}$*,*

then there exists an $f \in BV$ and a subsequence $f_{g(n)}$ such that $f_{g(n)} \xrightarrow{n\to\infty} f$ in L_1 and the variation of f is bounded by v.

The statement of this theorem will be abbreviated by HST.

Originally Helly's selection theorem was formulated for usual functions and not L_1-functions. There usually (i) is replaced by the statement that $|f_n(x)| \leq u'$ for an $x \in [0,1]$ and a bound u'. Note that this implies (i) since by (ii) with the bound u' we have $\|f_n\|_\infty \leq u' + v$ and with this also $\|f_n\|_1 \leq u' + v =: u$.

For the proof of HST we will need the following lemma.

Lemma 20 (RCA_0)**.** *Let $f \in BV$ and let v be the bound of variation of f. The system RCA_0 proves that for each $\epsilon > 0$ that*

(i) $f^\epsilon \in L_1$ *exists, and that*
(ii) $\|f^\epsilon - f\|_1 \leq 2\epsilon v$.

Proof. Let $(p_k)_k$ be the sequence of rational polynomials coding f. We have

$$\begin{aligned}
\|f^\epsilon - (p_k)^\epsilon\|_1 &= \int_0^1 \int_{-\epsilon}^{\epsilon} \eta_\epsilon(y) \left(f(x-y) - p_k(x-y)\right) dy\, dx \\
&= \int_{-\epsilon}^{\epsilon} \eta_\epsilon(y) \int_0^1 \left(f(x-y) - p_k(x-y)\right) dx\, dy \qquad \text{by Fubini} \\
&\leq 2\|f - p_k\|_1 \int_{-\epsilon}^{\epsilon} \eta_\epsilon(y) = 2\|f - p_k\|_1.
\end{aligned}$$

(The 2 in the above inequality comes from the possible reflection of f in the mollification as we defined it.) It follows that (a 2^{-k+1}-good approximation with rational polynomials of) $(p_{k+2})^\epsilon$ is a code for $f^\epsilon \in L_1$.

For (ii) we have for any k

$$\|f^\epsilon - f\|_1 \leq \|(p_k)^\epsilon - p_k\|_1 + 2^{-k+2}$$

since $\|f - p_k\|_1 < 2^{-k}$ and $\|f^\epsilon - p_k^\epsilon\|_1 < 2^{-k+1}$ by the above estimate. Further,

$$\begin{aligned}
\|(p_k)^\epsilon - p_k\|_1 &= \int_0^1 \int_{-\epsilon}^{\epsilon} \eta_\epsilon(y) \cdot p_k(x-y)\, dy - p_k(x)\, dx \\
&= \int_0^1 \int_{-\epsilon}^{\epsilon} \eta_\epsilon(y) \cdot (p_k(x-y) - p_k(x))\ dy\, dx \qquad \text{since } \int \eta_\epsilon = 1 \\
&= \int_0^1 \int_{-1}^{1} \eta(y) \cdot (p_k(x-\epsilon y) - p_k(x))\ dy\, dx \qquad \text{substituting } y \mapsto \epsilon y
\end{aligned}$$

since $|p_k(x-\epsilon y) - p_k(x)| = \left|\int_0^y \frac{d}{dy} p_k(x-\epsilon y)\, dy\right| = |\epsilon \int_0^y p_k'(x-\epsilon y)\, dy| \leq 2\epsilon \|p_k'\|_1$ for $y \in [-1,1]$

$$\leq \int_0^1 \int_{-1}^{1} \eta(y) \cdot 2\epsilon \|p_k'\|_1\, dy\, dx = 2\epsilon\|p_k'\|_1 \leq 2\epsilon v. \qquad \square$$

Proof of Theorem 19. For the mollifications f_n^ϵ of f_n we have by definition (2.1) that

$$\|f_n^\epsilon\|_\infty \leq \|f_n\|_1 \, \|\eta_\epsilon\|_\infty \leq \frac{u}{\epsilon},$$

and by (2.2) that

$$\left\|(f_n^\epsilon)'\right\|_\infty \leq \|f_n\|_1 \, \|\eta_\epsilon'\|_\infty \leq u \, \|\eta_\epsilon'\|_\infty.$$

Thus, for each fixed ϵ the sequence $(f_n^\epsilon)_n$ is uniformly bounded and—by the uniform bound on the derivative—equicontinuous. We instantiate ϵ with 2^{-i} and obtain a sequence of sequences of bounded, equicontinuous functions $(f_n^{(2^{-i})})_{n,i}$. By the previous lemma this sequence is contained in L_1 and converges as $i \to \infty$ to f_n.

By Proposition 21 below, a variant of the Arzelà-Ascoli theorem, there exists a subsequence $g(n)$, such that for each k

$$\forall j \leq k \, \forall n, n' \geq k \left\| f_{g(n)}^{(2^{-j})} - f_{g(n')}^{(2^{-j})} \right\|_\infty \leq 2^{-k}.$$

Now for $n, n' \geq k$

$$\begin{aligned}
\left\| f_{g(n)} - f_{g(n')} \right\|_1 &\leq \left\| f_{g(n)}^{(2^{-k})} - f_{g(n')}^{(2^{-k})} \right\|_1 + \left\| f_{g(n)} - f_{g(n)}^{(2^{-k})} \right\|_1 + \left\| f_{g(n')} - f_{g(n')}^{(2^{-k})} \right\|_1 \\
&\leq 2^{-k} + 2 \cdot 2 \cdot 2^{-k} v.
\end{aligned}$$

Thus, $f_{g(n)}$ forms a L_1-converging sequence with rate of convergence $2^{-k} + 2^{-k+2}v$. Thus $\lim f_{g(n)} = f \in L_1$. By Lemma 2 we have that $f \in BV$. $\square$

The previous proof was inspired by [1, Theorem 3.23].

Proposition 21 (Diagonalized Arzelà-Ascoli, ACA_0)**.** *Let $f_{n,j} \colon [0,1] \longrightarrow \mathbb{R}$ be a sequence of sequences of functions. If for each j*

(1) *the sequence $(f_{n,j})_n$ is bounded by $u_j \in \mathbb{Q}$, and*
(2) *$(f_{n,j})_n$ is uniformly equicontinuous, i.e., there exists a modulus of uniform equicontinuity $\phi_j(l)$, such that $\forall l \, \forall n \, \forall x, y \in [0,1] \left(|x-y| < 2^{-\phi_j(l)} \to |f_{n,j}(x) - f_{n,j}(y)| < 2^{-l} \right)$,*

then there exists a subsequence $g(n)$ such that for all j the sequence $f_{g(n),j}$ converges uniformly in the sense that

$$\forall k \, \forall j \leq k \, \forall n, n' \geq k \left\| f_{g(n),j} - f_{g(n'),j} \right\|_\infty < 2^{-k}. \tag{3.1}$$

Proof. By replacing $f_{n,k}$ with $\frac{f_{n,k}}{2u_n} + \frac{1}{2}$ we may assume that the image of $f_{n,k}$ is contained in the unit interval $[0,1]$.

In [12, Lemma 3, Corollary 4] we showed that an equicontinuous sequence of functions $h_n \colon [0,1] \longrightarrow [0,1]$ converges uniformly iff h_n converges pointwise on $\mathbb{Q} \cap [0,1]$, i.e., for an enumeration q of $\mathbb{Q} \cap [0,1]$ the sequence $\left((h_n(q(i)))_i\right)_n \subseteq [0,1]^{\mathbb{N}}$ converges in $[0,1]^{\mathbb{N}}$ with the product norm $d((x_i),(y_i)) = \sum_i 2^{-i} d(x_i, y_i)$. Moreover, from a rate of convergence and the modulus of uniform equicontinuity one can calculate a rate of convergence of h_n in $\|\cdot\|_\infty$.

With this the Arzelà-Ascoli theorem follows directly from an application of the Bolzano-Weierstraß principle for the space $[0,1]^{\mathbb{N}}$. For details see [12].

We can parallelize this process for $f_{n,j}$ by applying the Bolzano-Weierstraß principle to the sequence $\left((f_{n,j}(q(i)))_{\langle i,j\rangle}\right)_n \subseteq [0,1]^{\mathbb{N}}$. With this we obtain a subsequence $g(n)$ such that for each j we have $\left(f_{g(n),j}(q(i))\right)_i \in [0,1]^{\mathbb{N}}$ converges at a given rate for $n \to \infty$. By the above considerations we get that $f_{g(n),j} \in C([0,1])$ converges uniformly at a given rate (depending in ϕ_j). By thinning out the sequence $g(n)$ we get (3.1).

This proposition is provable in ACA_0 since Bolzano-Weierstraß principle for the space $[0,1]^{\mathbb{N}}$ is instance-wise equivalent to the Bolzano-Weierstraß principle for $[0,1]$ which is provable in ACA_0, see e.g. [12, 18]. □

We now come the reversal.

Theorem 22. *Over* RCA_0*,* HST *is equivalent to* ACA_0*.*

Proof. The right to left direction is simply Theorem 19. For the left-to-right direction we will show that HST implies the Bolzano-Weierstraß principle (for $[0,1]$) which is by [18, Theorem III.2.2] equivalent to ACA_0. Let $(x_n)_n \subseteq [0,1]$ be any sequence in the unit interval. Let $f_n(x) := x_n$ be the sequence of corresponding constant functions. It is clear that $f_n \in BV$ and that $\|f_n\|_1 = x_n$. One easily verifies that for any limit f as given by HST the value $\|f\|_1$ is a limit point of x_n and thus a solution to BW. □

The proofs of Theorem 19 and Theorem 22 actually give more information on the strength of HST. It shows that for each instance of HST, that is for each sequence of functions $(f_n)_n \subseteq BV$ with a uniform bound of variation, one can compute uniformly a sequence $(x_n)_n \subseteq [0,1]$, such that from any limit point of this sequence one can compute a solution to HST for f_n. By the proof of Theorem 22 the backward direction also holds. This is summarized in the following corollary.

Corollary 23. *The principles* HST *and* BW *are instance-wise equivalent, i.e., writing* $\mathsf{HST}((f_n))$ *for* HST *restricted to* (f_n) *and* $\mathsf{BW}((x_n))$ *for* BW *restricted to* (x_n)*, then we have the following. There are codes for Turing machines* e_1, e_2*, such that*

$$\mathsf{RCA}_0 \vdash \forall X \left(\mathsf{BW}(\{e_1\}^X) \to \mathsf{HST}(X)\right),$$

$$\mathsf{RCA}_0 \vdash \forall X \left(\mathsf{HST}(\{e_2\}^X) \to \mathsf{BW}(X)\right).$$

This corollary should be compared with Theorem 3.1 of [10], where it is shown that BW is instance-wise equivalent to WKL for Σ^0_1-trees, and Theorem 9 of [12], where it is shown that BW is instance-wise equivalent to the Arzelà-Ascoli theorem.

Remark 24 ($\mathsf{HST}_{\mathsf{weak}}$). In [10] we also analyzed the following weaker variant $\mathsf{BW}_{\mathsf{weak}}$ of the Bolzano-Weierstrass principle, which states that for each sequence $(x_n) \subseteq [0,1]$ there is a subsequence that converges but possibly without any computable rate of convergence. Since points are coded as sequences converging at the rate 2^{-k} the existence of the limit point of the sequence might not be provable. This principle is considerably weaker than BW. For instance $\mathsf{BW}_{\mathsf{weak}}$ it does not imply ACA_0 nor WKL_0.

Replacing BW in the above proof immediately yields that $\mathsf{BW}_{\mathsf{weak}}$ is instance-wise equivalent to the variant of HST which only states the existence of a converging subsequence.

3.1. **HST in the Weihrauch lattice.** Helly's selection theorem can be formulated in the Weihrauch lattice. The above proof yields also a classification in these terms. We refer the reader to [5, 6] for an introduction to the Weihrauch lattice.

The functions of the space L_1 can be represented by the rational polynomials closed under the $\|\cdot\|_1$-norm. We will call this representation δ_{L_1}. With this Helly's selection theorem is then a partial multifunction of the following type.

$$\mathsf{HST} :\subseteq (L_1([0,1]), \delta_{L_1})^{\mathbb{N}} \rightrightarrows (L_1([0,1]), \delta_{L_1})$$

where $\mathrm{dom}(\mathsf{HST}) = \left\{ (f_n) \;\middle|\; \int_0^1 |f_n'| \, dx \text{ is uniformly bounded} \right\}$. The derivative of f_n' here is taken in the sense of distributions. We chose this representation since it is customary in the Weihrauch lattice not to add any additional information—like the uniform bound on the variation—to the representation. However, we can easily recover this uniform bound by searching for it. This can be done using the limit lim. Since v is not needed to build the sequence of equicontinuous functions ($f_n^{(2^{-i})}$ in the proof of Theorem 19) the bound of the variation can be computed in parallel to the application of the diagonalized Arzelà-Ascoli theorem (which follows from $\mathsf{BWT}_{[0,1]^{\mathbb{N}}}$) . Thus,we get

$$\mathsf{HST} \leq_{\mathrm{W}} \lim \times \mathsf{BWT}_{[0,1]^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{BWT}_{[0,1]^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}.$$

Using the reversal we obtain in total $\mathsf{HST} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$.

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